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Notes on "Finite Groups with Nilpotent Local Subgroups"

LI Yang Ming
(Department of Mathematics, Guangdong College of Education, Guangdong 510310, China)
(E-mail: liyangming@gdei.edu.cn)

Abstract A finite group G is called PN-group if G is not nilpotent and for every p-subgroup P of G, there holds that either P is normal in G or $P \subseteq Z_\infty(G)$ or $N_G(P)$ is nilpotent, $\forall p \in \pi(G)$. In this paper, we prove that PN-group is meta-nilpotent, especially, PN-group is solvable. In addition, we give an elementary, intuitionistic, compact proof of the structure theorem of PN-group.

Keywords PN-group; meta-nilpotent group; structure theorem.

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All groups considered in this paper will be finite. We use conventional notions and notation, as in [1]. G always denotes a finite group, $\pi(G)$ denotes the set of all primes dividing the order of G, G_p is a Sylow p-subgroup of G for some $p \in \pi(G)$, $Z_\infty(G)$ is the supercenter of G, $F(G)$ is the Fitting subgroup of G and $\mathcal{N}$ is the class of nilpotent groups.

It is well-known that G is nilpotent if and only if every Sylow-subgroup of G is normal in G and G is nilpotent if and only if $G = Z_\infty(G)$. By the results in [2], we know that G is nilpotent if and only if the normalizer of each Sylow-subgroup of G is nilpotent. Hence, the following question arises: Suppose for $\forall p \in \pi(G)$ and for any p-subgoup P of G, there holds that either P is normal in G or $P \subseteq Z_\infty(G)$ or $N_G(P)$ is niloptent. Is that G nilpotent? The answer is negative, and S_3 is a counterexample. Therefore, Guo introduced PN-group in [3]: G is called a PN-group if G is not a nilpotent group, and for any p-subgroup P of G, there holds that either P is a normal subgroup of G, or $P \subseteq Z_\infty(G)$ or $N_G(P)$ is a nilpotent group, $\forall p \in \pi(G)$. Guo also gave the structure theorem of solvable PN-group[3, Theorems 4.1,4.2]

In this paper, first we prove that PN-group is a meta-nilpotent group. In particular, PN-group is solvable. This implies that the structure theorem of solvable PN-group given in [3] is actually the structure theorem of arbitrary PN-group. Secondly, we give an elementary, intuitionistic, compact proof of the structure theorem of PN-group.

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Lemma 1[3, Proposition 2.1] *If G is a meta-nilpotent group, then the following sets coincide:*

(1) the set of Carter subgroups of G;

(2) the set of $\mathcal{N}$- covering subgroups of G.

Lemma 2[2, Theorem 2] *If $N_G(G_p)$ is nilpotent for any Sylow p-subgroup G_p of G, $\forall p \in \pi(G)$, then G is a nilpotent group.*

Theorem 1 *Suppose that G is a PN-group. Then $G = [H]K$, where H, K are nilpotent Hall-subgroups of G, therefore PN-group is a meta-nilpotent group. Especially, PN-group is a solvable group.*

Proof Suppose that $\pi(G) = \{p_1, \ldots, p_s\}$ and $\mathcal{S} = \{G_{p_1}, \ldots, G_{p_s}\}$ is a set of Sylow-subgroups of G. For any i, there holds that either G_{p_i} is normal in G or $G_{p_i} \subseteq Z_\infty(G)$, or $N_G(G_{p_i})$ is nilpotent by hypothesis. If $G_{p_i} \subseteq Z_\infty(G)$, then $G_{p_i} \lhd G$ by the fact that G_{p_i} Char $Z_\infty(G) \lhd G$. Hence, for any i, there holds that either $G_{p_i} \lhd G$ or $N_G(G_{p_i})$ is nilpotent. Without loss of generality, suppose that $G_{p_1}, \ldots, G_{p_k}$ are all normal Sylow-subgroups of G in $\mathcal{S}$. Since G is not nilpotent, we have that $1 \le k < s$. Furthermore, denote $\pi_1 = \{p_1, \ldots, p_k\}$ and $\pi_2 = \{p_{k+1}, \ldots, p_s\}$. Take $H = G_{p_1} \cdots G_{p_k}$. Consider the factor group $\overline{G} = G/H$. Then $\{\overline{G}_{p_{k+1}}, \ldots, \overline{G}_{p_s}\}$ is a set of Sylow subgroups of $\overline{G}$. For $\forall j \in \pi_2$, since $N_{\overline{G}}(\overline{G}_{p_j}) = N_G(G_{p_j})H/H$, the nilpotency of $N_G(G_{p_j})$ implies that the nilpotency of $N_{\overline{G}}(\overline{G}_{p_j})$. By Lemma 2, we have that $\overline{G}$ is nilpotent, hence, G is a meta-nilpotent group.

By the above, we can denote $G = [H]K$, where H is the Hall π_1-subgroup of G and K is a Hall π_2-subgroup of G. Obviously, H is nilpotent. Without loss of generality, suppose that $\{G_{p_{k+1}}, \ldots, G_{p_s}\}$ is a set of Sylow-subgroups of K. For $\forall j \in \pi_2$, since $N_G(G_{p_j})$ is nilpotent, we have $N_K(G_{p_j})$ is nilpotent. Applying Lemma 2 again, K is nilpotent. Thus the proof is completed. $\square$

Remark In [3], Guo gave the structure theorem of solvable PN-group. By the theorem above, we know that Guo actually got the structure theorem of arbitrary PN-group, not merely the solvable case.

Theorem 2 *Suppose that G is a non-trivial PN-group and $Z_\infty(G) = 1$. Then G is a Frobenius group and $G^{\mathcal{N}} = F(G)$ is a Frobenius kernel of G and $\mathcal{N}$-injector of G, and a Frobenius complement K of G is a Carter subgroup of G. Furthermore, both $G^{\mathcal{N}}$ and K are Hall subgroups of G.*

Proof Using the notations in the proof of Theorem 1, we have $G = [H]K$, where H is the product of all normal Sylow-subgroups of G, and both H are K are nilpotent Hall-subgroups of G. We prove the theorem by the following steps.

(1) $H = F(G)$.

Obviously $H \le F(G)$. If $F(G) > H$, then denote $F(G) = H \times K_1$, where $K_1 \ne 1$ is the Hall π_2-subgroup of $F(G)$.

Obviously $K_1 \lhd K$. Since K is nilpotent, $K_1 \cap Z(K) \neq 1$. On the other hand,

$$K_1 \cap Z(K) \leq Z(G) = 1,$$

a contradiction.

(2) K is a Carter subgroup of G.

We only need to prove $N_G(K) = K$. If $N_G(K) > K$, then denote $N_G(K) = KH_1$, where H_1 is a Hall π_1-subgroup of $N_G(K)$. Noticing that $[K, H_1] \leq K \cap H = 1$, we have $N_G(K) = K \times H_1$. Without loss of generality, suppose that $H_1 \leq G_{p_1}$. If $H_1 \lhd G_{p_1}$, then

$$1 \neq H_1 \cap Z(G_{p_1}) \leq Z(G) = 1,$$

a contradiction. If $H_1 \ntriangleleft G_{p_1}$, then $H_1 \ntriangleleft G$. By hypothesis, $N_G(H_1)$ is nilpotent. Hence we can write

$$N_G(H_1) = H_2 \times L \times K,$$

where L is a Hall $\pi_1 \backslash \{p_1\}$-subgroup of $N_G(H_1)$, H_2 is a Sylow p_1-subgroup of $N_G(H_1)$ and $H_1 < H_2 \leq G_{p_1}$.

If $H_2 \lhd G_{p_1}$, then $H_2 \cap Z(G_{p_1}) \neq 1$. Therefore,

$$H_2 \cap Z(G_{p_1}) \leq Z(G) = 1,$$

a contradiction. If $H_2 \ntriangleleft G_{p_1}$, then $H_2 \ntriangleleft G$. Thus $N_G(H_2)$ is nilpotent by the hypothesis. Repeating this procedure, we finally have a subgroup H_l such that $H_l \lhd G_{p_1}$. Hence,

$$1 \neq H_l \cap Z(G_{p_1}) \leq Z(G) = 1,$$

a contradiction.

(3) $G^{\mathcal{N}} = F(G)$ is an $\mathcal{N}$-injector of G.

Since $G/F(G) \cong K$ is nilpotent, $G^{\mathcal{N}} \leq F(G)$. By (2), Theorem 1 and Lemma 1, we have that K is an $\mathcal{N}$-covering subgroup of G, so $G = G^{\mathcal{N}}K$. Since $G = F(G)K$, we have that $G^{\mathcal{N}} = F(G)$ by comparing the orders of $F(G)$ and $G^{\mathcal{N}}$.

Noticing that $G/F(G)$ is nilpotent, we get that $F(G)$ is an $\mathcal{N}$-injector of G by [1, Lemma 2.5.6].

(4) For any $K_1 \leq K$, $N_G(K_1)$ still is a π_2-group.

Since both K and $F(G)$ are Hall subgroups of G, $p \in \pi_2$, any p-subgroup P is a non-normal subgroup of G. Hence $N_G(P)$ is nilpotent by hypothesis.

Suppose there exists a subgroup $K_1 \leq K$ such that $N_G(K_1)$ is not a π_2-group, i.e., $N_G(K_1)$ has a non-trivial Hall π_1-subgroup. Without loss of generality, suppose that $K_1 \leq G_{p_{k+1}}$. Denote

$$N_G(K_1) = H_1 \times K_2 \times L_1,$$

where $H_1 \neq 1$ is a Hall π_1-subgroup of $N_G(K_1)$, L_1 is a Hall $\pi_2 \backslash \{p_{k+1}\}$-subgroup of $N_G(K_1)$, K_2 is a Sylow p_{k+1}-subgroup of $N_G(K_1)$ and $K_1 < K_2$. If K_2 is not a Sylow p_{k+1}-subgroup of G, then consider $N_G(K_2)$. Since $N_G(K_2)$ is nilpotent, we can write

$$N_G(K_2) = H_2 \times K_3 \times L_2,$$

where $H_2 \neq 1$ is a Hall π_1-subgroup of $N_G(K_2)$, L_2 is a Hall $\pi_2\backslash\{p_{k+1}\}$-subgroup of $N_G(K_2)$, K_3 is a Sylow p_{k+1}-subgroup of $N_G(K_2)$ and $K_2 < K_3$. Repeating this procedure, we finally have a subgroup p_{k+1}-subgroup K_m such that

$$N_G(K_m) = H_m \times G_{p_{k+1}} \times L_m,$$

where $H_m \neq 1$ is a Hall π_1-subgroup of $N_G(K_m)$ and L_m is a Hall $\pi_2\backslash\{p_{k+1}\}$-subgroup of $N_G(K_m)$. Therefore,

$$N_G(G_{p_{k+1}}) = K \times H_{m+1},$$

where $H_{m+1} \neq 1$ is a Hall π_1-subgroup of $N_G(G_{p_{k+1}})$. Hence $H_{m+1} \leq N_G(K) = K$, a contradiction.

(5) G is a Frobenius group.

We only need to prove that $K \cap K^x = 1$, $\forall x \in G\backslash K$. If there exists $x \in G\backslash K$ such that $K \cap K^x \neq 1$, then let $K_0 = K \cap K^a$ be the maximal element in the set $\{K \cap K^x \mid x \in G\backslash K\}$. By (2), $K \neq K^a$. Since G is solvable, there exists $y \in G$ such that $N_G(K_0) \subseteq K^y$ by (4). Since K is nilpotent, $K_0 < N_K(K_0) \leq N_G(K_0) \leq K^y$. Then we have

$$K_0 < N_K(K_0) \leq K \cap K^y.$$

Since K_0 is maximal, $y \in K$ and $K_0 < N_{K^a}(K_0) \leq N_G(K_0) \leq K^y = K$, we have

$$K_0 < N_{K^a}(K_0) \leq K^a \cap K = K_0,$$

a contradiction.

The proof of the theorem is completed. □

Combining the proof of the sufficiency of [3, Theorem 4.2], we have the structure of PN-group.

Theorem 3 *Suppose that G is a non-trivial group and $\overline{G} = G/Z_\infty(G)$. Then G is a PN-group if and only if there hold:*

(1) $\overline{G}$ is a Frobenius group, and $\overline{G}^{\mathcal{N}} = F(\overline{G})$ is a Frobenius kernel and $\mathcal{N}$-injector. A Frobenius complement $\overline{K}$ of $\overline{G}$ is a Carter subgroup of $\overline{G}$, and both $\overline{G}^{\mathcal{N}}$ and $\overline{K}$ are Hall subgroups of $\overline{G}$.

(2) If a p-subgroup P of G is non-normal in G and $P \subseteq F(G)$ and $P \not\subseteq Z_\infty(G)$, then $N_G(P) \subseteq F(G)$.

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