

The One-Dimensional Distribution and Construction of Semi-Markov Processes

TANG Rong¹, GUO Xian Ping¹, LIU Zai Ming²

(1. School of Mathematics and Computational Science, Sun Yat-Sen University,
Guangdong 510275, China;

2. School of Mathematical Science and Computing Technology, Central South University,
Hunan 410075, China)

(E-mail: Tanyou01@163.com)

Abstract In this paper, we obtain the transition probability of jump chain of semi-Markov process, the distribution of sojourn time and one-dimensional distribution of semi-Markov process. Furthermore, the semi-Markov process $X(t, \omega)$ is constructed from the semi-Markov matrix and it is proved that two definitions of semi-Markov process are equivalent.

Keywords semi-Markov process; Markov skeleton process; sojourn time; one-dimensional distribution.

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1. Introduction and preliminaries

The concept of semi-Markov process was first put forward and discussed by Lévy and Smith in 1954. The theories and applications of semi-Markov process have extensively developed and gradually been perfected in this half a century. Nowadays, semi-Markov has already become an independent research direction of stochastic process.

Definition 1.1 The matrix $Q(t) = (Q_{ij}(t); i, j \in E)$ is called semi-Markov matrix if it satisfies the following conditions:

- (1) $Q_{ij}(t) \equiv 0$ for every $i, j \in E$ and $t < 0$;
- (2) $Q_{ij}(t)$ is non-decreasing and right continuous for every $t \geq 0$;
- (3) $\sum_{j \in E} Q_{ij}(t) \triangleq p_i(t) \leq 1$ for every $i \in E$ and $t \geq 0$.

Let $p_{ij} = Q_{ij}(\infty) = \lim_{t \rightarrow \infty} Q_{ij}(t)$; $G_{ij}(t) = \begin{cases} \frac{Q_{ij}(t)}{Q_{ij}(\infty)} & \text{if } Q_{ij}(\infty) > 0, \\ 0 & \text{if } Q_{ij}(\infty) = 0. \end{cases}$

Semi-Markov process $X(t, \omega)$ is constructed in many ways in [1, §3–§7]. But they have many shortages. For example, the expression of semi-Markov process is not obvious; and they are multidimensional stochastic process. Actually, semi-Markov process has not been constructed in the strict sense. Recently semi-Markov process is defined to be a special Markov skeleton

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process by Hou^[2] etc, that is, the following Definition 1.2. And the shortcoming of construction of semi-Markov process in reference [1, §3–§7] is overcome.

Definition 1.2 Let $X(t, \omega) \triangleq (x_t(\omega); t \geq 0)$ be a Markov skeleton process defined on probability space (Ω, \mathcal{F}, P) and valued in countable set E . $X(t, \omega)$ is called semi-Markov process, if it satisfies the following conditions:

- (1) There exists a sequence of stopping times $\{\tau_n, n = 0, 1, 2, \dots\}$ such that $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots$, $\lim_{n \rightarrow \infty} \tau_n = \tau$, P -a.e.;
- (2) For every $A \in \mathcal{B}(E)$, $P(x_{\tau_{n+1}} \in A | \mathcal{F}_{\tau_n}) = P(x_{\tau_{n+1}} \in A | x_{\tau_n}) = P_{x_{\tau_n}}(x_{\tau_1} \in A)$, $P_{\mathcal{F}_{\tau_n}}$ -a.e.;
- (3) If there exists $n \geq 0$ such that $\tau_{n+1}(\omega) > \tau_n(\omega)$, then $x_t(\omega) = x_{\tau_n}(\omega)$ for all $\tau_n \leq t < \tau_{n+1}$, P -a.e.

Here \mathcal{F}_{τ_n} is a σ -algebra generated by $X(t, \omega)$ prior to τ_n ; $\mathcal{B}(E)$ is a Borel σ -algebra generated by E .

Semi-Markov process is intuitively described as follows: If we give the embedded Markov chain $X_{\tau_n} \triangleq (x_{\tau_n}(\omega); n \geq 0) \triangleq X_n$ of $X(t, \omega)$, then X_n is a homogeneous Markov chain. And if $\tau_{n+1}(\omega) > \tau_n(\omega)$, then $X(t, \omega)$ is a constant in the interval $[\tau_n(\omega), \tau_{n+1}(\omega))$. Let $\theta_n(\omega) = \tau_n(\omega) - \tau_{n-1}(\omega)$ for every $n \geq 1$. Then the sequence of random variables $\{\theta_n(\omega); n \geq 1\}$ are conditionally independent and have the same conditional distributions.

Using the semi-Markov matrix, we will solve the following problems in this paper: (1) We will obtain the transition probability of jump chain and the sojourn time at state i ($i \in E$) of semi-Markov process; (2) We will deduce the one-dimensional distribution of semi Markov process, and construct the semi-Markov process by one-dimensional distribution matrix and initial distribution; (3) Finally, we will prove that the Definitions 1.2 and 3.1, decided by semi-Markov matrix and initial distribution, are equivalent.

2. The transition probability of jump chain and the distribution of sojourn time

For convenience, the semi-Markov process decided by semi-markov matrix and initial distribution is expressed by $X(t, \omega) \triangleq (x_t(\omega); t \geq 0)$.

The two-dimensional Markov chain (ξ_n, θ_n, ζ) , $n = 0, 1, 2, \dots$ defined on the probability space (Ω, \mathcal{F}, P) and valued in $E \times [0, +\infty)$ is introduced in [1, §2], where the initial distribution of first component of (ξ_n, θ_n, ζ) is given by vector $\Pi = (\pi_i; i \in E)$, that is, $P(\xi_0 = i) = \pi_i$; $Q(t)$ is decided by (ξ_n, θ_n, ζ) as follows:

$$\begin{aligned} P(\xi_{n+1} = j, \theta_n \leq t | \xi_0, \dots, \xi_n, \theta_0, \dots, \theta_{n-1}, \zeta > n) \\ = P(\xi_{n+1} = j, \theta_n \leq t | \xi_n, \zeta > n) = Q_{\xi_n j}(t). \end{aligned}$$

The process (ξ_n, θ_n, ζ) is a special two-dimensional Markov process. The transition probability only depends on the first discrete component $\{\xi_n, n \geq 0\}$. ζ means the times that the process may occur to transfer. $(\xi_n, n \geq 0)$ is a Markov chain, whose transition probability is

$$P(\xi_{n+1} = j | \xi_n, \zeta > n) = Q_{\xi_n j}(\infty).$$

It was proved in [1, §2, Lemma 1.1] that the sequence $\{\theta_n; n \geq 0\}$ of second component of (ξ_n, θ_n, ζ) are conditionally independent. Namely, we have

$$P(\theta_0 \leq x_0, \theta_1 \leq x_1, \dots, \theta_{n-1} \leq x_{n-1} | \xi_0 = i_0, \dots, \xi_n = i_n) = \prod_{k=0}^{n-1} G_{i_k i_{k+1}}(x_k).$$

In [1, §5] was introduced the embedded Markov chain ξ_n , whose transition probability is

$$p_{ij} = P(\xi_n = j | \xi_{n-1} = i) = Q_{ij}(\infty),$$

and the sojourn time ζ_{ij} of that $X(t, \omega)$ jump to j from state i , whose distribution is

$$P(\zeta_{ij} \leq t) = P(\theta_{n-1} \leq t | \xi_{n-1} = i, \xi_n = j) = G_{ij}(t).$$

We continue to use the symbol of [1, §5]. The semi-Markov process $X(t, \omega)$ evolves in the following manner: $X(t, \omega)$ chooses next state j for transition probability $p_{ij} = Q_{ij}(\infty)$ at state i , and spends a period of time ζ_{ij} whose distribution function is $G_{ij}(t)$, whereafter transfer to state j ; $X(t, \omega)$ again chooses next state k for transition probability $p_{jk} = Q_{jk}(\infty)$ at state j , and spends a period of time ζ_{jk} whose distribution function is $G_{jk}(t)$, whereafter transfer to state k ; $X(t, \omega)$ goes on evolving in this way.

Since there may exist false jump case, that is, there exists $i \in E$ such that $p_{ii} \triangleq Q_{ii}(\infty) > 0$, ξ_n may not be the jump chain of $X(t, \omega)$. Furthermore, ζ_{ij} may also not be the sojourn time at state i . Obviously, ξ_n become to be jump chain if and only if $p_{ii} = Q_{ii}(\infty) = 0$ for all $i \in E$.

Theorem 2.1 (1) Let $0 = \eta_0 \leq \eta_1 \leq \dots \leq \eta_n \leq \dots$ be sequence of jump points of $X(t, \omega)$, that is, they satisfy $x_{\eta_{n-0}}(\omega) \triangleq \lim_{s \downarrow 0} x_{\eta_{n-s}}(\omega) \neq x_{\eta_n}(\omega)$ for every $n \geq 1$. Then we have

$$P(x_{\eta_n} = k | x_{\eta_{n-1}} = i) = P_i(x_{\eta_1} = k) = \begin{cases} \frac{p_{ik}}{1-p_{ii}} & \text{if } i \neq k, \\ 0 & \text{if } i = k. \end{cases} \quad (2.1)$$

(2) Let $\alpha_{ik}(\omega)$ be the sojourn time at state i of that $X(t, \omega)$ jump to k from i . Set, for every $i, k \in E$,

$$F_{ik}(t) = P(\alpha_{ik} \leq t); \quad \bar{F}_{ik}(\lambda) = \int_0^{+\infty} e^{-\lambda t} dF_{ik}(t); \quad \bar{G}_{ik}(\lambda) = \int_0^{+\infty} e^{-\lambda t} dG_{ik}(t).$$

Then we have

$$F_{ik}(t) = \sum_{n=0}^{\infty} p_{ii}^n (1 - p_{ii}) (G_{ii}^{(n)} * G_{ik})(t); \quad (2.2)$$

$$\bar{F}_{ik}(\lambda) = \sum_{n=0}^{\infty} p_{ii}^n (1 - p_{ii}) \bar{G}_{ii}^n(\lambda) \bar{G}_{ik}(\lambda) = \frac{1 - p_{ii}}{1 - p_{ii} \bar{G}_{ii}(\lambda)} \bar{G}_{ik}(\lambda), \quad (2.3)$$

where $G_{ii}^{(n)}(t)$ is the n -fold convolution of $G_{ii}(t)$; $(G_{ii}^{(n)} * G_{ik})(t)$ is the convolution of $G_{ii}^{(n)}(t)$ with $G_{ik}(t)$.

(3) Let $\alpha_i(\omega)$ be the sojourn time at state i of $X(t, \omega)$. Set, for every $i, j \in E$,

$$F_i(t) = P(\alpha_i \leq t); \quad \bar{F}_i(\lambda) = \int_0^{+\infty} e^{-\lambda t} dF_i(t); \quad \bar{G}_{ij}(\lambda) = \int_0^{+\infty} e^{-\lambda t} dG_{ij}(t).$$

Then we have

$$F_i(t) = \sum_{j \in E - \{i\}} \frac{p_{ij}}{1 - p_{ii}} F_{ij}(t); \quad \bar{F}_i(\lambda) = \sum_{j \in E - \{i\}} \frac{p_{ij}}{1 - p_{ii} \bar{G}_{ii}(\lambda)} \bar{G}_{ij}(\lambda).$$

Proof (1) Since ξ_n is a homogeneous Markov chain, $X_n \triangleq (x_{\eta_n}; n \geq 0)$ satisfy homogeneity. So we have, if $i \neq k$,

$$\begin{aligned} P(x_{\eta_n} = k | x_{\eta_{n-1}} = i) &= P(x_{\eta_1} = k | x_0 = i) = P_i(X_1 = k) \\ &= \sum_{n=0}^{\infty} P_i(\xi_{n+1} = k, \xi_1 = i, \dots, \xi_n = i) \\ &= \sum_{n=0}^{\infty} P_i(\xi_1 = i, \dots, \xi_n = i) \cdot P_i(\xi_{n+1} = k | \xi_1 = i, \dots, \xi_n = i) \\ &= \sum_{n=0}^{\infty} p_{ii}^n p_{ik} = \frac{p_{ik}}{1 - p_{ii}}. \end{aligned}$$

If $i = k$, obviously, $P(x_{\eta_n} = i | x_{\eta_{n-1}} = i) = 0$ holds by the definition of η_n .

(2) Let v_i be the false jump times when $X(t, \omega)$ stays at state i . Obviously, it is a random variable with geometric distribution, whose distribution is

$$P(v_i = n) = p_{ii}^n (1 - p_{ii}).$$

Here we suppose without loss of generality that $p_{ii} < 1$ (Otherwise, $X(t, \omega)$ have not jump point starting from i for almost all ω). Obviously,

$$\{\omega : \alpha_{ik}(\omega) \leq t\} = \{\omega : \sum_{m=0}^{v_i(\omega)} \zeta_{ii}(\omega) + \zeta_{ik}(\omega) \leq t\}, \text{ P-a.e.}$$

Therefore, using total probability formula in the following formulae, we obtain

$$\begin{aligned} F_{ik}(t) &= P(\alpha_{ik} \leq t) = P\left(\sum_{m=0}^{v_i(\omega)} \zeta_{ii}(\omega) + \zeta_{ik}(\omega) \leq t\right) \\ &= \sum_{n=0}^{\infty} P(v_i = n) P\left(\sum_{m=0}^n \zeta_{ii}(\omega) + \zeta_{ik}(\omega) \leq t\right) \\ &= \sum_{n=0}^{\infty} p_{ii}^n (1 - p_{ii}) P\left(\sum_{m=0}^n \zeta_{ii}(\omega) + \zeta_{ik}(\omega) \leq t\right). \end{aligned}$$

Again $P(\sum_{m=0}^n \zeta_{ii}(\omega) \leq t)$ is equal to the n -fold convolution, denoted by $G_{ii}^{(n)}(t)$, of $G_{ii}(t)$ from $P(\zeta_{ii} \leq t) = G_{ii}(t)$ (Convention: $G_{ii}^{(0)}(t) = 1$), and $P(\sum_{m=0}^n \zeta_{ii}(\omega) + \zeta_{ik}(\omega) \leq t)$ is equal to the convolution, denoted by $(G_{ii}^{(n)} * G_{ik})(t)$, of $G_{ii}^{(n)}(t)$ with $G_{ik}(t)$. Hence,

$$F_{ik}(t) = \sum_{n=0}^{\infty} p_{ii}^n (1 - p_{ii}) (G_{ii}^{(n)} * G_{ik})(t),$$

which is (2.2). Taking Laplace transform we have

$$\bar{F}_{ik}(\lambda) = \sum_{n=0}^{\infty} p_{ii}^n (1 - p_{ii}) \bar{G}_{ii}^n(\lambda) \bar{G}_{ik}(\lambda) = \frac{1 - p_{ii}}{1 - p_{ii} \bar{G}_{ii}(\lambda)} \bar{G}_{ik}(\lambda),$$

which is (2.3).

(3) If the jump chain $\{x_{\eta_n}; n \geq 0\}$ of semi-Markov process is honest, that is, $\sum_{j \in E} P(x_{\eta_n} = j) = 1$ for every $n \geq 1$, by total probability formula, we have

$$F_i(t) = P(\alpha_i \leq t) = \sum_{j \in E - \{i\}} P(\alpha_i \leq t, x_{\eta_1} = j) = \sum_{j \in E - \{i\}} \frac{p_{ij}}{1 - p_{ii}} F_{ij}(t).$$

If the jump chain $\{x_{\eta_n}; n \geq 0\}$ of semi-Markov process is not honest, by the above description about the sample path of $X(t, \omega)$ we know that $X(t, \omega)$ moves in E . So we may take an interrupt state d . Set the time when $X(t, \omega)$ jumps to d starting from any state $i \in E$ is equal to ∞ , that is, we define $G_{id}(t) = 0$ for all $t \geq 0$ and $i \in E$, and the transition probability, from i to d , of the jump chain $\{x_{\eta_n}; n \geq 0\}$ of $X(t, \omega)$ is defined by $P(x_{\eta_n} = d | x_{\eta_{n-1}} = i) = \frac{1 - \sum_{j \in E} p_{ij}}{1 - p_{ii}}$. In this case, by an analogous proof of the case that jump chain is honest we know that 3 is true. \square

3. One-dimensional distribution and the construction of semi-Markov process

One-dimensional distribution of semi-Markov process was not obtained in [1]. That one-dimensional distribution of semi-Markov is the smallest nonnegative solution of some equation is obtained in [2, §4.13.2], but the solution was not computed. We will derive one-dimensional distribution in the following.

Let $\tau(\omega) = \lim_{n \rightarrow \infty} \eta_n(\omega)$. $\tau(\omega)$ is called leap point of $X(t, \omega)$. By the definition of semi-Markov process we know that $X(t, \omega)$ is actually the process prior to leap point. Namely, it may be written as $(X(t, \omega), t < \tau(\omega))$ (that is, $X(t, \omega) = (X(t, \omega), t < \tau(\omega))$).

Lemma 3.1 *Let $\mathcal{F}(\eta_1)$ be the smallest σ -algebra generated by η_1 . For every fixed $B \in \mathcal{F}(\eta_1)$, set $\mathcal{B} = \{B, \bar{B}, \emptyset\}$. Let $\mathcal{F}(B, x_{\eta_1}, x_{\eta_0})$ be the smallest σ -algebra generated by all sets of form $\{B, x_{\eta_1} = j, x_{\eta_0} = l\}$ for any $j, l \in E$, and every $B \in \mathcal{B}$. Then we have, for every $A \in \mathcal{F}(x_t(\omega), t \geq \eta_1)$,*

$$P(A | \mathcal{F}(B, x_{\eta_1}, x_{\eta_0})) = P(A | \mathcal{F}(B, x_{\eta_1})), \text{ P-a.e.}$$

In particular, we obtain

$$P(A | B, x_{\eta_1} = k, x_{\eta_0} = i) = P(A | B, x_{\eta_1} = k), \text{ P-a.e.}$$

Proof For every $G \in \mathcal{F}(B, x_{\eta_1}, x_{\eta_0})$, we will prove the following formula holds.

$$P(AG) = \int_G P(A | \mathcal{F}(B, x_{\eta_1})) P(d\omega) \text{ for every } G \in \mathcal{F}(B, x_{\eta_1}, x_{\eta_0}). \quad (3.1)$$

First suppose that G has the form $G = \{B, x_{\eta_1} = j, x_{\eta_0} = l\}$.

If $P(B) > 0$, $P(x_{\eta_1} = j, x_{\eta_0} = l) > 0$ and $P(B, x_{\eta_1} = j, x_{\eta_0} = l) > 0$, let P_B be the conditional probability measure relative to B . Then

$$\begin{aligned} P(AG) &= P(B) P_B(A \cap \{x_{\eta_1} = j, x_{\eta_0} = l\}) \\ &= P(B) \int_{\{x_{\eta_1} = j, x_{\eta_0} = l\}} P_B(A | \mathcal{F}(x_{\eta_1}, x_{\eta_0})) P_B(d\omega). \end{aligned} \quad (3.2)$$

Since $\{x_{\eta_1} = j, x_{\eta_0} = l\}$ is an atom of $\mathcal{F}(x_{\eta_1}, x_{\eta_0})$, by the property of conditional expectation [4, Chapter 5, §2.3, Theorem 5] we know that $P_B(A|\mathcal{F}(x_{\eta_1}, x_{\eta_0}))(\omega)$ is equal to some constant K for all $\omega \in \{\omega : x_{\eta_1} = j, x_{\eta_0} = l\}$. So we have

$$K = P_B(A|x_{\eta_1} = j, x_{\eta_0} = l) = \frac{P(AB, x_{\eta_1} = j, x_{\eta_0} = l)}{P(B, x_{\eta_1} = j, x_{\eta_0} = l)} = \frac{P(AB|x_{\eta_1} = j, x_{\eta_0} = l)}{P(B|x_{\eta_1} = j, x_{\eta_0} = l)}.$$

Again $B \in \mathcal{F}(x_t(\omega), t \geq \eta_1)$ since η_1 is a stopping time. Therefore, using the Markov property at jump point, the formula above is changed into

$$K = \frac{P(AB|x_{\eta_1} = j)}{P(B|x_{\eta_1} = j)} = \frac{P(AB, x_{\eta_1} = j)}{P(B, x_{\eta_1} = j)} = P(A|B, x_{\eta_1} = j). \quad (3.3)$$

From which and (3.2) it follows that

$$\begin{aligned} P(AG) &= P(B) \int_{\{x_{\eta_1}=j, x_{\eta_0}=l\}} P(A|B, x_{\eta_1} = j) P_B(d\omega) \\ &= \int_{\{x_{\eta_1}=j, x_{\eta_0}=l\}} P(B) P(A|B, x_{\eta_1} = j) P_B(d\omega) = \int_G P(A|B, x_{\eta_1} = j) P(d\omega), \end{aligned} \quad (3.4)$$

where the last equality follows from theorem of integral transformation.

If $P(B, x_{\eta_1} = j, x_{\eta_0} = l) = 0$, we define

$$P_B(A|x_{\eta_1} = j, x_{\eta_0} = l) = \begin{cases} P(A|B, x_{\eta_1} = j) & \text{if } P(B, x_{\eta_1} = j) \neq 0, \\ \text{any fixed constant } k \leq 1 & \text{if } P(B, x_{\eta_1} = j) = 0. \end{cases}$$

In this case (3.4) also holds.

By (3.4) we know that (3.1) holds for G of the form $\{B, x_{\eta_1} = j, x_{\eta_0} = l\}$. So (3.4) holds for G of the form

$$G = \{C, x_{\eta_1} \in A_1, x_{\eta_0} \in A_0\}, \text{ where } A_0, A_1 \subseteq E, C \text{ is the union of elements in } \mathcal{B}. \quad (3.5)$$

All G which satisfy (3.1) form a λ -system Λ . All G shaped as (3.5) form a π -system Π . Hence by λ - π -system method we obtain $\Lambda \supset \mathcal{F}(\Pi) = \mathcal{F}(B, x_{\eta_1}, x_{\eta_0})$. Therefore (3.1) holds. Again from the definition of conditional probability we complete proof of this lemma. \square

Theorem 3.1 (1) Set $p_{ij}(t) = P_i(x_t = j)$. Then $p_{ij}(t)$ satisfies the following equations:

$$p_{ij}(t) = \delta_{ij}[1 - F_i(t)] + \sum_{k \neq i} \frac{p_{ik}}{1 - p_{ii}} \cdot \int_0^t p_{kj}(t-s) dF_{ik}(s), \text{ for any } i, j \in E.$$

$$(2) \text{ Set } h_i(\lambda) = \int_0^{+\infty} e^{-\lambda t} (1 - F_i(t)) dt; \quad \varphi_{ij}(\lambda) = \int_0^{+\infty} e^{-\lambda t} p_{ij}(t) dt;$$

$$\pi_{ij}(\lambda) = \begin{cases} \bar{F}_{ij}(\lambda) \cdot \frac{p_{ij}}{1 - p_{ii}} & \text{if } i \neq j, \\ 0 & \text{if } i = j; \end{cases} \quad \Pi(\lambda) = (\pi_{ij}(\lambda); i, j \in E);$$

$$H(\lambda) = \text{diag}(h_i(\lambda); i \in E); \quad \Phi(\lambda) = (\varphi_{ij}(\lambda); i, j \in E).$$

Then $\Phi(\lambda) = \sum_{k=0}^{\infty} \Pi^k(\lambda) H(\lambda)$.

Proof Set ${}_n p_{ij}(t) = P_i(x_t = j, t < \eta_n)$. Let \mathcal{F}_{η_1} be the σ -algebra prior to η_1 . Obviously,

$$\eta_1 = \alpha_{ik} \quad P_{\{x_{\eta_0}=i, x_{\eta_1}=k\}} - a.e; \quad \lim_{n \rightarrow \infty} {}_n p_{ij}(t) = P_i(x_t = j, t < \tau) = p_{ij}(t).$$

Again

$$\begin{aligned}
{}_n p_{ij}(t) &= P_i(x_t = j, t < \eta_n) = P_i(x_t = j, t < \eta_n, t < \eta_1) + P_i(x_t = j, t < \eta_n, \eta_1 \leq t) \\
&= P_i(x_t = j, t < \eta_1) + P_i[P_i(x_t = j, t < \eta_n, \eta_1 \leq t | \mathcal{F}_{\eta_1})] \\
&= \delta_{ij} P_i(t < \eta_1) + \sum_{k \neq i} P_i(x_{\eta_1} = k) P_i(x_t = j, t < \eta_n, \eta_1 \leq t | x_{\eta_1} = k) \\
&= \delta_{ij} [1 - F_i(t)] + \sum_{k \neq i} \frac{p_{ik}}{1 - p_{ii}} P(x_t = j, t - \eta_1 < \eta_n - \eta_1, 0 \leq t - \eta_1 | x_{\eta_0} = i, x_{\eta_1} = k) \\
&\stackrel{5}{=} \delta_{ij} [1 - F_i(t)] + \sum_{k \neq i} \frac{p_{ik}}{1 - p_{ii}} \int_0^t P_k(x_{t-s} = j, t - s < \eta_{n-1}) dF_{ik}(s) \\
&= \delta_{ij} [1 - F_i(t)] + \sum_{k \neq i} \frac{p_{ik}}{1 - p_{ii}} \int_0^t {}_{n-1} p_{kj}(t - s) dF_{ik}(s). \tag{3.6}
\end{aligned}$$

Let $n \rightarrow +\infty$, using monotone convergence theorem we know (1) holds.

(2) Let ${}_n \varphi_{ij}(\lambda) = \int_0^{+\infty} e^{-\lambda t} {}_n p_{ij}(t) dt$; ${}_n \Phi(\lambda) = ({}_n \varphi_{ij}(\lambda); i, j \in E)$. Taking Laplace transform at both sides of (3.6), we have

$${}_n \varphi_{ij}(\lambda) = \delta_{ij} h_i(\lambda) + \sum_{k \neq i} \bar{F}_{ik}(\lambda) \frac{p_{ik}}{1 - p_{ii}} {}_{n-1} \varphi_{kj}(\lambda); i, j \in E.$$

Obviously, ${}_1 \varphi_{ij}(\lambda) = \delta_{ij} h_i(\lambda)$, ${}_0 \varphi_{ij}(\lambda) = 0$. Above equalities may be written as the following matrix equation:

$$\begin{cases} {}_n \Phi(\lambda) = H(\lambda) + \Pi(\lambda) {}_{n-1} \Phi(\lambda), & n = 1, 2, \dots, \\ {}_1 \Phi(\lambda) = H(\lambda). \end{cases}$$

Solving equations above gives

$${}_n \Phi(\lambda) = \sum_{k=0}^{n-1} \Pi^k(\lambda) H(\lambda), \tag{3.7}$$

from which and monotone convergence theorem it follows that

$$\lim_{n \rightarrow \infty} {}_n \Phi(\lambda) = \Phi(\lambda) = \sum_{k=0}^{\infty} \Pi^k(\lambda) H(\lambda),$$

which is (2). □

Remark The total probability formula of continuous type random variable and the fact that $x_t(\omega)$ has homogeneity at jump point are used in the fifth equality of (3.6). What is more, we may also deduce it by using the definition of R-S integral, Markov property (that is, Lemma 3.1) and homogeneity. It is deduced as follows:

$$\begin{aligned}
&P(x_t = j, t - \eta_1 < \eta_n - \eta_1, 0 \leq t - \eta_1 | x_{\eta_0} = i, x_{\eta_1} = k) \\
&= \sum_{v=2}^{[2^m t]} P\left(\frac{v-1}{2^m} < \eta_1 \leq \frac{v}{2^m} | x_{\eta_0} = i, x_{\eta_1} = k\right) \times \\
&\quad P\left(x_t = j, t - \eta_1 < \eta_n - \eta_1, 0 \leq t - \eta_1 | x_{\eta_0} = i, x_{\eta_1} = k, \frac{v-1}{2^m} < \eta_1 \leq \frac{v}{2^m}\right) +
\end{aligned}$$

$$\begin{aligned}
& P\left(\frac{[2^m t]}{2^m} < \eta_1 \leq t | x_{\eta_0} = i, x_{\eta_1} = k\right) \times \\
& P(x_t = j, t - \eta_1 < \eta_n - \eta_1, 0 \leq t - \eta_1 | x_{\eta_0} = i, x_{\eta_1} = k, \frac{[2^m t]}{2^m} < \eta_1 \leq t) + \\
& P(0 \leq \eta_1 \leq \frac{1}{2^m} | x_{\eta_0} = i, x_{\eta_1} = k) \times \\
& P(x_t = j, t - \eta_1 < \eta_n - \eta_1, 0 \leq t - \eta_1 | x_{\eta_0} = i, x_{\eta_1} = k, 0 \leq \eta_1 \leq \frac{1}{2^m}) \\
& = \lim_{m \rightarrow \infty} \sum_{v=2}^{[2^m t]} P\left(\frac{v-1}{2^m} < \eta_1 \leq \frac{v}{2^m} | x_{\eta_0} = i, x_{\eta_1} = k\right) \times \\
& P(x_t = j, t - \eta_1 < \eta_n - \eta_1, 0 \leq t - \eta_1 | x_{\eta_0} = i, x_{\eta_1} = k, \frac{v-1}{2^m} < \eta_1 \leq \frac{v}{2^m}) + \\
& \lim_{m \rightarrow \infty} P\left(\frac{[2^m t]}{2^m} < \eta_1 \leq t | x_{\eta_0} = i, x_{\eta_1} = k\right) \times \\
& P(x_t = j, t - \eta_1 < \eta_n - \eta_1, 0 \leq t - \eta_1 | x_{\eta_0} = i, x_{\eta_1} = k, \frac{[2^m t]}{2^m} < \eta_1 \leq t) + \\
& \lim_{m \rightarrow \infty} P(0 \leq \eta_1 \leq \frac{1}{2^m} | x_{\eta_0} = i, x_{\eta_1} = k) \times \\
& P(x_t = j, t - \eta_1 < \eta_n - \eta_1, 0 \leq t - \eta_1 | x_{\eta_0} = i, x_{\eta_1} = k, 0 \leq \eta_1 \leq \frac{1}{2^m}) \\
& = \lim_{m \rightarrow \infty} \sum_{v=2}^{[2^m t]} P\left(\frac{v-1}{2^m} < \eta_1 \leq \frac{v}{2^m} | x_{\eta_0} = i, x_{\eta_1} = k\right) \times \\
& P_k(x_{t-\eta_1} = j, t - \eta_1 < \eta_n - \eta_1, 0 \leq t - \eta_1 | \frac{v-1}{2^m} < \eta_1 \leq \frac{v}{2^m}) + \\
& \lim_{m \rightarrow \infty} P\left(\frac{[2^m t]}{2^m} < \eta_1 \leq t | x_{\eta_0} = i, x_{\eta_1} = k\right) \times \\
& P_k(x_{t-\eta_1} = j, t - \eta_1 < \eta_n - \eta_1, 0 \leq t - \eta_1 | \frac{[2^m t]}{2^m} < \eta_1 \leq t) + \\
& \lim_{m \rightarrow \infty} P(0 \leq \eta_1 \leq \frac{1}{2^m} | x_{\eta_0} = i, x_{\eta_1} = k) \times \\
& P_k(x_t = j, t - \eta_1 < \eta_n - \eta_1, 0 \leq t - \eta_1 | 0 \leq \eta_1 \leq \frac{1}{2^m}) \\
& = \int_0^t P_k(x_{t-s} = j, t - s < \eta_{n-1}) dF_{ik}(s).
\end{aligned}$$

Definition 3.1 $(\pi_i; i \in E)$ is called a distribution defined on E if $\pi_i \geq 0, \sum_{j \in E} \pi_i = 1$.

Theorem 3.2 Suppose that we are given a distribution $(\pi_i; i \in E)$ defined on E and a semi-Markov matrix $Q(t) \triangleq (Q_{ij}(t); i, j \in E)$. Then there exists a probability space (Ω, \mathcal{F}, P) and a stochastic process $X(t, \omega) \triangleq (x_t(\omega); t \geq 0)$, which is defined on the probability space (Ω, \mathcal{F}, P) with initial distribution $\{\pi_i; i \in E\}$ and one-dimensional matrix $P(t) \triangleq (p_{ij}(t); i, j \in E)$. Here $P(t) = (p_{ij}(t); i, j \in E)$ is obtained by Theorem 3.1.

Proof Without loss of generality, we suppose that E is a rational number set, and $P(t)$ is honest, that is, $P(t) \cdot 1 = 1$. Otherwise, we take interrupt state Δ , and let

$$\tilde{p}_{ij}(t) = \begin{cases} p_{ij}(t) & i, j \in E, \\ 1 - \sum_{j \in E} p_{ij}(t) & i \in E, j = \Delta, \\ 1 & i = j = \Delta, \\ 0 & i = \Delta, j \in E. \end{cases}$$

Then $\tilde{P}(t) \triangleq (\tilde{p}_{ij}(t); i, j \in E \cup \{\Delta\})$ is honest, and is one-dimensional distribution matrix. In the same way we may construct stochastic process $\tilde{X}(t, \omega)$ whose one-dimensional distribution matrix is $\tilde{P}(t)$. And then if $\tilde{X}(t, \omega)$ is tabooed on E we obtain $X(t, \omega)$.

We are given a probability space $(\Omega', \mathcal{F}', P')$. For any $t_j \geq 0, j = 1, 2, \dots$, we define a discrete random variable α_{t_j} valued in E with the following distributions:

$$P'(\alpha_{t_j} \leq \lambda_j) = \begin{cases} \sum_{m \leq \lambda_j} \sum_{k \in E} \pi_k p_{km}(t_j) & \text{if } t_j > 0, \\ \sum_{m \leq \lambda_0} \pi_m & \text{if } t_j = 0; \end{cases} \quad (3.8)$$

$$F_{0, t_j}(\lambda_0, \lambda_j) = F_{t_j, 0}(\lambda_j, \lambda_0) = P'(\alpha_0 \leq \lambda_0, \alpha_{t_j} \leq \lambda_j) = \sum_{m \leq \lambda_j} \sum_{k \leq \lambda_0} \pi_k p_{km}(t_j), \quad t_j > 0. \quad (3.9)$$

Let

$$F_{t_1, \dots, t_n}(\lambda_1, \dots, \lambda_n) = P'(\alpha_{t_1} \leq \lambda_1, \dots, \alpha_{t_n} \leq \lambda_n). \quad (3.10)$$

We obtain a multivariate distribution function family

$$F = \{F_{t_1, \dots, t_n}(\lambda_1, \dots, \lambda_n), n = 1, 2, \dots, t_j \geq 0, j = 1, 2, \dots, n\}.$$

From (3.8)–(3.10) it follows that F satisfies consistency conditions, that is, F satisfies the following conditions:

(A) For an arbitrary permutation (i_1, \dots, i_n) of $(1, \dots, n)$ it follows that

$$F_{t_1, \dots, t_n}(\lambda_1, \dots, \lambda_n) = F_{t_{i_1}, \dots, t_{i_n}}(\lambda_{i_1}, \dots, \lambda_{i_n}).$$

(B) If $m < n$, then,

$$F_{t_1, \dots, t_m}(\lambda_1, \dots, \lambda_m) = \lim_{\lambda_{m+1}, \dots, \lambda_n \rightarrow \infty} F_{t_1, \dots, t_n}(\lambda_1, \dots, \lambda_n).$$

So by an analogous proof of [3, §1.1, Theorem 1] we obtain a probability space (Ω, \mathcal{F}, P) and a stochastic process $X(t, \omega) = (x_t(\omega); t \geq 0)$ which is defined on (Ω, \mathcal{F}, P) such that

$$F_{t_1, \dots, t_n}(\lambda_1, \dots, \lambda_n) = P(x_{t_1} \leq \lambda_1, \dots, x_{t_n} \leq \lambda_n) \quad (3.11)$$

for any natural number n and any $\lambda_j \in R_1, t_j \geq 0, j = 1, 2, \dots, n$. By (3.8), (3.11) we obtain $P(x_0 = i) = \pi_i$ whenever $n = 1, t_1 = 0$. By (3.11), (3.9) we obtain $P(x_0 = i, x_t = j) = P'(\alpha_0 = i, \alpha_t = j) = \pi_i p_{ij}(t)$ whenever $n = 2, t_1 = 0, t_2 = t$. Hence, $P(x_t = j | x_0 = i) = p_{ij}(t)$. Here if $\pi_i = 0, P(x_t = j | x_0 = i)$ is undefined, we may stipulate $P(x_t = j | x_0 = i) = p_{ij}(t)$. \square

We know that the processes, constructed by virtue of Kolmogorov's concordant theorem and the same finite dimensional function family, are in fact one class of processes, all of which are equivalent (that is, all of these processes have the same initial distribution and finite dimensional

function family). So we may choose a representative process (that is, if and only if almost all sample path is right continuous) from these processes. Therefore, we suppose that the process obtained by Theorem 3.2 is a representative process.

Definition 3.2 $X(t, \omega)$ obtained by Theorem 3.2 is called semi-Markov process decided by initial distribution $\Pi = (\pi_i; i \in E)$ and semi-Markov matrix $Q(t)$.

4. The equivalence of two definitions

We shall show that Definitions 1.2 and 3.2 are equivalent.

Theorem 4.1 (1) Let $X(t, \omega)$ be the semi-Markov process defined by Definition 3.2. Then we have the following statements:

There exists a sequence of stopping times $\{\tau_n, n = 0, 1, 2, \dots\}$ with $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots$ and $\lim_{n \rightarrow \infty} \tau_n = \tau$, P -a.e, such that:

$$(i) \quad P(x_{\tau_{n+1}} \in A | \mathcal{F}_{\tau_n}) = P(x_{\tau_{n+1}} \in A | x_{\tau_n}) = P_{x_{\tau_n}}(x_{\tau_1} \in A) = \sum_{j \in A} \frac{p_{x_{\tau_n} j}}{1 - p_{x_{\tau_n} x_{\tau_n}}};$$

(ii) If $\tau_{n+1} - \tau_n > 0$ for some $n \geq 0$, then $x_t = x_{\tau_n}$ for all t with $\tau_n \leq t < \tau_{n+1}$, P -a.e.

(2) Let $X(t, \omega)$ be the semi-Markov process defined by Definition 1.2. Then there are one distribution $\Pi = (\pi_i; i \in E)$ on E and one semi-Markov matrix $Q(t)$.

Proof (1) We take the representative process whose sample path is right continuous. Let

$$\begin{aligned} \tau_0(\omega) &\equiv 0, \tau_1(\omega) = \inf(t : x_t(\omega) \neq x_0(\omega), t > 0), \dots, \tau_n(\omega) \\ &= \inf(t : x_t(\omega) \neq x_{\tau_{n-1}}(\omega), t > \tau_{n-1}(\omega)) \dots \end{aligned}$$

Hence $\{\tau_n, n = 0, 1, 2, \dots\}$ are stopping times. Let $\lim_{n \rightarrow \infty} \tau_n = \tau$.

(i) By the Markov property and homogeneity of jump point we know (i) holds.

(ii) The proof of (ii) is obtained by the definition of $\tau_n, n \geq 0$.

(2) Suppose that $X(t, \omega)$ is the semi-Markov process defined by Definition 1.2. Set $p_{ij} \triangleq P(x_{\tau_n} = j, \tau_n < +\infty | x_{\tau_{n-1}} = i) = P(x_{\tau_1} = j, \tau_1 < +\infty | x_0 = i)$, and

$$\begin{aligned} G_{ij}(t) &\triangleq \begin{cases} P(\theta_n \leq t | x_{\tau_{n-1}} = i, x_{\tau_n} = j, \tau_n < +\infty) & \text{if } P(\tau_n < +\infty) > 0, \\ 0 & \text{if } P(\tau_n < +\infty) = 0, \end{cases} \\ &= \begin{cases} P(\theta_1 \leq t | x_0 = i, \tau_1 = j, \tau_1 < +\infty) & \text{if } P(\tau_1 < +\infty) > 0, \\ 0 & \text{if } P(\tau_1 < +\infty) = 0. \end{cases} \end{aligned}$$

Let

$$\begin{aligned} Q_{ij}(t) &\triangleq P(\theta_n \leq t, x_{\tau_n} = j | x_{\tau_{n-1}} = i) \\ &= P(\theta_n \leq t, x_{\tau_n} = j, \tau_n < +\infty | x_{\tau_{n-1}} = i) + P(\theta_n \leq t, x_{\tau_n} = j, \tau_n = +\infty | x_{\tau_{n-1}} = i) \\ &= P(\theta_n \leq t, x_{\tau_n} = j, \tau_n < +\infty | x_{\tau_{n-1}} = i) = p_{ij} G_{ij}(t). \end{aligned}$$

It is easy to know $\lim_{t \rightarrow +\infty} G_{ij}(t) = \begin{cases} 1 & \text{if } P(\tau_1 < +\infty) > 0, \\ 0 & \text{if } P(\tau_1 < +\infty) = 0. \end{cases}$ Set $Q(t) = (Q_{ij}(t); i, j \in E)$.

Then for every $i, j \in E$, $Q(t)$ satisfies the following properties:

$$\left\{ \begin{array}{ll} Q_{ij}(t) \equiv 0 & t < 0, \\ Q_{ij}(t) \text{ is non-decreasing and right continuous} & t \geq 0, \\ \sum_{j \in E} Q_{ij}(t) \triangleq p_i(t) \leq \sum_{j \in E} p_{ij} \leq 1 & i \in E, t \geq 0. \end{array} \right. \quad (4.1)$$

Therefore, $Q(t)$ is a semi-Markov matrix. Let $\pi_i = P(x_0(\omega) = i)$, $\Pi = (\pi_i; i \in E)$. \square

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