Study of Modules over $3 \times 3$ Formal Triangular Matrix Rings

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Abstract In this paper we carry out a study of modules over a $3 \times 3$ formal triangular matrix ring

$$
\Gamma = \begin{pmatrix}
T & 0 & 0 \\
M & U & 0 \\
N \otimes U M & N & V
\end{pmatrix},
$$

where $T$, $U$, $V$ are rings, $M$, $N$ are $U$-$T$, $V$-$U$ bimodules, respectively. Using the alternative description of left $\Gamma$-module as quintuple $(A, B, C; f, g)$ with $A \in \text{mod} T$, $B \in \text{mod} U$ and $C \in \text{mod} V$, $f : M \otimes_T A \rightarrow B \in \text{mod} U$, $g : N \otimes_U B \rightarrow C \in \text{mod} V$, we shall characterize uniform, hollow and finitely embedded modules over $\Gamma$, respectively. Also the radical as well as the socle of $(A \oplus B \oplus C)$ is determined.

Keywords triangular matrix ring; uniform module; hollow module; radical; socle.

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1. Introduction

All the rings are associative with the identity and all modules are unital modules. Unless otherwise mentioned, all modules are left modules. For any ring $R$, we denote by $\text{mod} R$ the category of finitely generated left $R$-module.

Given a $3 \times 3$ formal triangular matrix ring

$$
\Gamma = \begin{pmatrix}
T & 0 & 0 \\
M & U & 0 \\
N \otimes U M & N & V
\end{pmatrix},
$$

Let $\Omega$ be the category whose objects are the quintuple $(A, B, C; f, g)$ with $A \in \text{mod} T$, $B \in \text{mod} U$, $C \in \text{mod} V$ and $f : M \otimes_T A \rightarrow B$ is a $U$-morphism, $g : N \otimes_U B \rightarrow C$ is a $V$-morphism. The morphism between two objects $(A, B, C; f, g)$ and $(A', B', C'; f', g')$ are triples of morphisms $(\alpha, \beta, \gamma)$ where $\alpha : A \rightarrow A'$ is a $T$-morphism, $\beta : B \rightarrow B'$ is a $U$-morphism, and $\gamma : C \rightarrow C'$ is a $V$-morphism such that the following two diagrams commute.

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The composition between two morphisms \((\alpha, \beta, \gamma)\) and \((\alpha', \beta', \gamma')\) is defined by \((\alpha', \beta', \gamma')(\alpha, \beta, \gamma)\) as follows.

The equivalent functor \(F : \Omega \rightarrow \text{mod}\Gamma\) is defined as follows. For \((A, B, C; f, g)\) \(\in\) \(\text{Obj}\Omega\) we define \(F(A, B, C; f, g) := A \oplus B \oplus C\) as an abelian group and the \(\Gamma\)-module structure on it is defined as

\[
\begin{pmatrix}
  t & 0 & 0 \\
m & u & 0 \\
m \odot n & n & v
\end{pmatrix}
\begin{pmatrix}
a, b, c
\end{pmatrix} = \begin{pmatrix}
ta, f(m \odot a) + ub, g(n \odot f(m \odot a)) + g(n \odot b) + vc
\end{pmatrix}
\]

for \(t \in T, u \in U, v \in V, m, n \odot n \in M, n, \pi \in N\). If \((\alpha, \beta, \gamma) : (A, B, C; f, g) \rightarrow (A', B', C', f', g')\) in \(\Omega\), then \(F(\alpha, \beta, \gamma) := \alpha \oplus \beta \oplus \gamma : A \oplus B \oplus C \rightarrow (A' \oplus B' \oplus C')\). Denote by \(r(A \oplus B \oplus C)\) the left \(\Gamma\)-module associated to \((A, B, C; f, g)\). In Section 2, corresponding to submodules and quotient modules of \(r(A \oplus B \oplus C)\), we describe subobjects and quotient objects of \(\Omega\). We give some equivalent conditions for a submodule \(r(A' \oplus B' \oplus C')\) of \(r(A \oplus B \oplus C)\) to be essential (resp. small) in \(r(A \oplus B \oplus C)\). We then use these conditions to characterize uniform and hollow modules over \(\Gamma\) respectively. In Section 3, the radical as well as the socle of \(r(A \oplus B \oplus C)\) is determined. Using the description of the socle, we find necessary and sufficient conditions for \(\Gamma\)-modules to be finitely embedded.

2. Uniform and hollow over \(\Gamma\) respectively

We explain the notations adopted in this paper firstly. Let \((A, B, C; f, g) \in \text{Obj}\Omega\) and \(r(A \oplus B \oplus C)\) be the associated left \(\Gamma\)-module. We will describe subobjects (resp. quotient objects) of \((A, B, C; f, g)\) which correspond to submodules (resp. quotient modules) of \(r(A \oplus B \oplus C)\). Assume \(r(A' \oplus B' \oplus C') \leq r(A \oplus B \oplus C)\), where \(rA' \leq rA\), \(rB' \leq rB\) and \(rC' \leq rC\) satisfying \(f(M \odot_T A') \leq B'\) and \(g(N \odot_U B') \leq C'\). Denoting the inclusions \(A' \hookrightarrow A, B' \hookrightarrow B, C' \hookrightarrow C\) by \(i, j, k\), respectively, and writing \(f' = f \circ (1_M \odot i), g' = g \circ (1_N \odot j)\), we have the following two commutative diagrams
Thus, we see that \((A', B', C'; f', g') \in \text{Obj} \Omega\) and \((i, j, k) : (A', B', C'; f', g') \to (A, B, C; f, g)\) is a map in \(\Omega\) realizing \(\Gamma(A' \oplus B' \oplus C')\) as a \(\Gamma\)-submodule of \(\Gamma(A \oplus B \oplus C)\). Also it is clear that every \(\Gamma\)-submodule of \(\Gamma(A \oplus B \oplus C)\) is obtained in this way. Let \(\tau A'', \nu B''\) and \(\nu C''\) be quotient modules of \(\tau A, \nu B\) and \(\nu C\) respectively with \(\eta_1 : A \to A'', \eta_2 : B \to B''\) and \(\eta_3 : C \to C''\) the canonical quotient maps. Let \(A' = \ker \eta_1, B' = \ker \eta_2\) and \(C' = \ker \eta_3\). Suppose \(f(M \otimes_T A') \leq B'\) and \(g(N \otimes_U B') \leq C'\). Denote the inclusions \(A' \to A, B' \to B\) and \(C' \to C\) by \(j_1, j_2\) and \(j_3\), respectively. We get two maps \(f'' : M \otimes_T A'' \to B''\) and \(g'' : N \otimes_U B'' \to C''\) rendering the following two diagrams commutative.

![Diagram](attachment:image.png)

In the two diagrams \(f' = f \circ (1_M \otimes j_1), g' = g \circ (1_N \otimes j_2)\) and the rows are exact. It is clear that \((\eta_1, \eta_2, \eta_3) : (A, B, C; f, g) \to (A'', B'', C''; f'', g'')\) is a map in \(\Omega\) realizing \(\Gamma(A'' \oplus B'' \oplus C'')\) as a quotient of \(\Gamma(A \oplus B \oplus C)\). When we talk of a submodule \(\Gamma(A' \oplus B' \oplus C')\) of \(\Gamma(A \oplus B \oplus C)\), we have \(\tau A' \leq_T A, \nu B' \leq_U B, \nu C' \leq_V C, f \circ (1_M \otimes j_1)(M \otimes_T A') \leq B', g \circ (1_N \otimes j_2)(N \otimes_U B') \leq C'\), and the maps \(f' : M \otimes_T A' \to B'\) and \(g' : N \otimes_U B' \to C'\) are completely determined, respectively. They have to be \(f \circ (1_M \otimes j_1)\) and \(g \circ (1_N \otimes j_2)\), respectively. Similarly, when we deal with a quotient \(\Gamma(A'' \oplus B'' \oplus C'')\) of \(\Gamma(A \oplus B \oplus C)\), the two maps \(f'' : M \otimes_T A'' \to B''\) and \(g'' : N \otimes_U B'' \to C''\) are completely determined. Because of these facts we will not specifically mention the maps \(f', g', f''\) and \(g''\) in these situations.

Let \((\tau A, U B, V C; f, g) \in \text{Obj} \Omega\). \(L = \{a \in A | f(m \otimes a) = 0, \text{ for all } m \in M\}\) and \(Q = \{b \in B | g(n \otimes b) = 0, \text{ for all } n \in N\}\). Then clearly \(\tau L \leq_T A, \nu Q \leq_U B\) and \(\tau (L \oplus Q \oplus 0) \leq_T (A \oplus B \oplus C)\).

**Theorem 2.1** \(\Gamma(A' \oplus B' \oplus C')\) is essential in \(\Gamma(A \oplus B \oplus C)\) if and only if \(\tau (A' \cap L), \nu (B' \cap Q)\) and \(\nu C'\) are essential in \(\tau L, \nu Q\) and \(\nu C\), respectively.

**Proof** Assume that \(\Gamma(A' \oplus B' \oplus C')\) is essential in \(\Gamma(A \oplus B \oplus C)\).
Let $0 \neq c \in C$. Then $(0, 0, c)$ is non-zero in $\Gamma(A \oplus B \oplus C)$. We can find an element
\[
\begin{pmatrix}
t & 0 & 0 \\
m & u & 0 \\
\pi \otimes \bar{m} & n & v
\end{pmatrix} \in \Gamma
\]
with
\[
(0, 0, 0) \neq \begin{pmatrix}
t & 0 & 0 \\
m & u & 0 \\
\pi \otimes \bar{m} & n & v
\end{pmatrix} (0, 0, c) \in \Gamma (A' \oplus B' \oplus C').
\]
But
\[
\begin{pmatrix}
t & 0 & 0 \\
m & u & 0 \\
\pi \otimes \bar{m} & n & v
\end{pmatrix} (0, 0, c) = (0, 0, vc).
\]
Thus $0 \neq vc \in C'$. This proves that $\nu C'$ is essential in $\nu C$.

Let $0 \neq a \in T L$. Then $(a, 0, 0) \neq (0, 0, 0) \in \Gamma (A \oplus B \oplus C)$. We can find an element
\[
\begin{pmatrix}
t & 0 & 0 \\
m & u & 0 \\
\pi \otimes \bar{m} & n & v
\end{pmatrix} \in \Gamma
\]
with
\[
(0, 0, 0) \neq \begin{pmatrix}
t & 0 & 0 \\
m & u & 0 \\
\pi \otimes \bar{m} & n & v
\end{pmatrix} (a, 0, 0) \in \Gamma (A' \oplus B' \oplus C').
\]
But
\[
\begin{pmatrix}
t & 0 & 0 \\
m & u & 0 \\
\pi \otimes \bar{m} & n & v
\end{pmatrix} (a, 0, 0) = (ta, 0, 0) \text{ (since } f(m \otimes a) = 0).\]

Thus $0 \neq ta \in A'$. Since $ta \in T L$, we see $0 \neq ta \in A' \cap L$, showing that $\tau (A' \cap L)$ is essential in $\tau L$.

Similarly, let $0 \neq b \in U Q$. Then $(0, 0, 0) \neq (0, b, 0) \in \Gamma (A \oplus B \oplus C)$. We can find an element
\[
\begin{pmatrix}
t & 0 & 0 \\
m & u & 0 \\
\pi \otimes \bar{m} & n & v
\end{pmatrix} \in \Gamma
\]
with
\[
(0, 0, 0) \neq \begin{pmatrix}
t & 0 & 0 \\
m & u & 0 \\
\pi \otimes \bar{m} & n & v
\end{pmatrix} (0, b, 0) \in \Gamma (A' \oplus B' \oplus C').
\]
But
\[
\begin{pmatrix}
t & 0 & 0 \\
m & u & 0 \\
\pi \otimes \bar{m} & n & v
\end{pmatrix} (0, b, 0) = (0, ub, 0) \text{ (since } g(n \otimes b) = 0).\]
Thus, $0 \neq ub \in B'$. Since $ub \in U$, we see that $0 \neq ub \in B' \cap Q$, showing that $U(B' \cap Q)$ is essential in $UQ$.

Conversely, assume that $\pi(A' \cap L)$, $U(B' \cap Q)$, and $V C'$ are essential in $\pi L$, $U Q$, and $V C$, respectively. We need to prove that $\pi(A' \oplus B' \oplus C')$ is essential in $\pi(A \oplus B \oplus C)$. In order to do so, let $(0, 0, 0) \neq (a, b, c) \in \Gamma (A \oplus B \oplus C)$. We divide the proof into the following cases.

(1) $c \neq 0$;
(2) $c = 0, b \neq 0$ and $b \notin Q$;
(3) $c = 0, b \neq 0$ and $b \in Q$;
(4) $c = 0, b = 0, a \neq 0$ and $a \notin L$;
(5) $c = 0, b = 0, a \neq 0$ and $a \in L$.

In Case (1), we can find $0 \neq vc \in C'$. It follows that

$$0 \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & v \end{pmatrix} (0, 0, c) = (0, 0, vc) \in \Gamma (A' \oplus B' \oplus C').$$

In Case (2), we can find an $n \in N$ with $0 \neq g(n \otimes b) \in C$. Hence there exists a $v \in V$ with $0 \neq vg(n \otimes b) \in C'$. Thus

$$0 \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & v \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n \end{pmatrix} (0, b, 0) = (0, 0, vg(n \otimes b))$$

is a non-zero element of $\Gamma (A' \oplus B' \oplus C')$.

In Case (3), we have $g(n \otimes b) = 0$ for all $n \in N$. Since $U(B' \cap Q)$ is essential in $UQ$, we can find an element $u \in U$ with $0 \neq ub \in U (B' \cap Q)$. Thus

$$0 \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (0, b, 0) = (0, 0, ub, 0) \in \Gamma (A' \oplus B' \oplus C').$$

In Case (4), we can find an $m \in M$ with $0 \neq f(m \otimes a) \in B$. On the one hand, if $f(m \otimes a) \in Q$, we can find an $n \in N$ with $0 \neq g(n \otimes f(m \otimes a)) \in C$. Hence there exists $v \in V$ with $0 \neq vg(n \otimes f(m \otimes a)) \in C'$. Thus

$$0 \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & v \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ n \otimes m & 0 & 0 \end{pmatrix} (a, 0, 0) = (0, 0, vg(n \otimes f(m \otimes a)))$$

is a non-zero element of $\Gamma (A' \oplus B' \oplus C')$. On the other hand, if $f(m \otimes a) \in Q$, we have $g(n \otimes f(m \otimes a)) = 0$ for all $n \in N$. Since $U(B' \cap Q)$ is essential in $U Q$, we can find an element $u \in U$ with $0 \neq uf(m \otimes a) \in U (B' \cap Q)$. In this case

$$0 \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \end{pmatrix} (a, 0, 0) = (0, 0, uf(m \otimes a), 0)$$
and \((0,0,0) \neq (0, uf(m \otimes a), 0) \in \Gamma (A' \oplus B' \oplus C').\)

In Case (5), we have \(f(m \otimes a) = 0\) for all \(m \in M\). Since \(A' \cap L\) is essential in \(\tau L\), we can find an element \(t \in T\) with \(0 \neq ta \in A' \cap L\). In this case

\[
\begin{pmatrix}
t & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
(a,0,0) = (ta,0,0)
\]

and \((0,0,0) \neq (ta,0,0) \in \Gamma(A' \oplus B' \oplus C').\) This proves that \(\Gamma(A' \oplus B' \oplus C')\) is essential in \(\Gamma(A \oplus B \oplus C)\).

\[\square\]

**Definition 2.1**[2] A module is said to be uniform if and only if every nonzero submodule is essential.

Theorem 2.1 enables us to obtain the following.

**Corollary 2.1** \(\Gamma_3(A \oplus B \oplus C)\) is uniform if and only if one and only one of the following three conditions holds.

(a) \(\nu C = 0\) and \(\tau L = \tau A, \nu Q = \nu B\) are uniform.

(b) \(\tau L = 0\) and \(\nu C, \nu Q = \nu B\) are uniform.

(c) \(\nu Q = 0\) and \(\tau L = \tau A\) are uniform.

If \(V \in \text{mod}R\) and \(W \leq_R V\) we write \(W \ll V\) to indicate that \(W\) is a small (or superfluous) submodule of \(V\). The following proposition gives a necessary and sufficient condition for \(\Gamma_3(A' \oplus B' \oplus C')\) to be small in \(\Gamma_3(A \oplus B \oplus C)\).

**Theorem 2.2** \(\Gamma_3(A' \oplus B' \oplus C')\) is small in \(\Gamma_3(A \oplus B \oplus C)\) if and only if \(\tau A', \beta(B')\) and \(\delta(C')\) are small in \(\tau A, B/f(M \otimes_T A)\) and \(C/g(N \otimes_U B)\), respectively, where \(\beta: B \to B/f(M \otimes_T A)\) and \(\delta: C \to C/g(N \otimes_U B)\) are the canonical quotient maps.

**Proof** Assume that \(\Gamma(A' \oplus B' \oplus C') \ll \Gamma(A \oplus B \oplus C)\).

Let \(\tau H \leq \tau A\) which satisfies \(A' + H = A\). If \(\mu: H \hookrightarrow A\) denotes the inclusion map, then with \(f' = f \circ (1_M \otimes \mu)\) we have \((H, B, C; f', \gamma)\) giving rise to a \(\Gamma\)-submodule \(\Gamma(H \oplus B \oplus C)\) of \(\Gamma(A \oplus B \oplus C)\) which satisfies \(\Gamma(A' \oplus B' \oplus C') + \Gamma(H \oplus B \oplus C) = \Gamma(A \oplus B \oplus C)\). The assumption \(\Gamma(A' \oplus B' \oplus C') \ll \Gamma(A \oplus B \oplus C)\) yields \(\Gamma(H \oplus B \oplus C) = \Gamma(A \oplus B \oplus C)\), hence \(H = A\). Thus \(A' + H = A\) implies \(H = A\). This proves that \(A' \ll A\).

Let \(\nu E \leq \nu (B/f(M \otimes_T A))\) satisfy \(\beta(B') + E = \beta(B) = B/f(M \otimes_T A)\) and \(D = \beta^{-1}(E)\). Then \(f(M \otimes_T A) \subseteq D\). Let \(v: D \hookrightarrow B\) be the inclusion map. Then with \(g' = g \circ (1_N \otimes v)\) we have \((A, D, C; f', g')\) giving rise to a \(\Gamma\)-submodule \(\Gamma(A \oplus D \oplus C)\) of \(\Gamma(A \oplus B \oplus C)\). From \(\beta(B') + E = \beta(B)\), we get \(B' + D = B\). Hence \(\Gamma(A' \oplus B' \oplus C') + \Gamma(A \oplus D \oplus C) = \Gamma(A \oplus B \oplus C)\).

The assumption \(\Gamma(A' \oplus B' \oplus C') \ll \Gamma(A \oplus B \oplus C)\) yields \(D = B\). Since \(\beta^{-1}(E) = B\), we get \(E = \beta(B) = B/f(M \otimes_T A)\). Thus \(\beta(B') + E = \beta(B)\) implies \(E = \beta(B)\). This gives \(\beta(B') \ll \beta(B)\).

Let \(\nu F \leq \nu (C/g(N \otimes_U B))\) satisfy \(\delta(C') + F = \delta(C) = C/g(N \otimes_U B)\) and \(K = \delta^{-1}(F)\). Then \(g(N \otimes_U B) \subseteq K\). Hence \((A, B, K; f, g)\) gives rise to a \(\Gamma\)-submodule \(\Gamma(A \oplus B \oplus K)\) of \(\Gamma(A \oplus B \oplus C)\).

From \(\delta(C') + F = \delta(C)\) we get \(C' + K = C\). Hence \(\Gamma(A' \oplus B' \oplus C') + \Gamma(A \oplus B \oplus K) = \Gamma(A \oplus B \oplus C)\).
The assumption \( r(A' + B' + C') \ll r(A \oplus B \oplus C) \) yields \( K = C \). Since \( \delta^{-1}(F) = C \) we get \( F = \delta(C) = C/\delta(U \otimes B) \). Thus \( \delta(C') + F = \delta(C) \) implies \( F = \delta(C) \). This proves \( \delta(C') \ll \delta(C) \).

Conversely, assume that \( A' \ll A, \beta(B') \ll \beta(B) \) and \( \delta(C') \ll \delta(C) \).

Let \( T X \leq T A, U Y \leq U B \) and \( V Z \leq V C \) satisfy the conditions \( f \circ (1_M \otimes \mu)(M \otimes_T X) \leq Y \) and \( g \circ (1_N \otimes \nu)(N \otimes_U Y) \leq Z \) where \( \mu : X \hookrightarrow A \) and \( \nu : Y \hookrightarrow B \) denote the inclusions. Write \( f' = f \circ (1_M \otimes \mu) \) and \( g' = g \circ (1_N \otimes \nu) \). Suppose \( (X, Y, Z, f', g') \) satisfies \( r(A' + B' + C') + r(X + Y + Z) = r(A \oplus B \oplus C) \). Then \( A' + X = A, B' + Y = B \) and \( C' + Z = C \). From \( A' \ll A \) we get \( X = A \). Hence \( \mu = 1_A \), \( f(M \otimes_T A) \leq Y \), \( B' + Y = B \) and \( \beta(B') \ll \beta(B) \), and the hypothesis \( \beta(B') \ll \beta(B) \) yields \( \beta(Y) = \beta(B) \). Since \( f(M \otimes_T A) \leq Y \), we get \( Y = B \). Thus \( v = 1_B \) and \( g(N \otimes_U B) \leq Z \). From \( C' + Z = C \) we get \( \delta(C') + \delta(Z) = \delta(C) \), the hypothesis \( \delta(C') \ll \delta(C) \) yields \( \delta(Z) = \delta(C) \). Since \( g(N \otimes_U B) \leq Z \), we get \( Z = C \). Thus \( r(A' + B' + C') + r(X + Y + Z) = r(A \oplus B \oplus C) \) implies \( X = A, Y = B \) and \( Z = C \). This proves that \( r(A' + B' + C') \ll r(A \oplus B \oplus C) \).

\[ \square \]

**Definition 2.2** A module is said to be hollow if and only if it is nonzero and in it every proper submodule is small.

From Theorem 2.2 and Definition 2.2 we get the following.

**Corollary 2.2** The left \( \Gamma \)-module \( r(A \oplus B \oplus C) \) determined by \( (A, B, C; f, g) \) is hollow if and only if exact one of the following three conditions holds.

(a) \( \tau A \) is hollow and \( B = f(M \otimes_T A), C = g(N \otimes_U B) \).

(b) \( A = 0, B \) is hollow and \( C = g(N \otimes_U B) \).

(c) \( A = 0, B = 0 \) and \( V C \) is hollow.

**Proof** Assume \( r(A \oplus B \oplus C) \) is hollow.

Suppose \( A \neq 0 \). Then there exists \( \tau A' \leq \tau A \) with \( \tau A' \neq \tau A \). If \( i : A' \hookrightarrow A \) denotes the inclusion and \( f' = f \circ (1_M \otimes i) \), the submodule \( r(A' \oplus B \oplus C) \) corresponding to \( (A', B, C; f', g) \) is a proper submodule of \( r(A \oplus B \oplus C) \). Thus \( r(A' \oplus B \oplus C) \ll r(A \oplus B \oplus C) \). By Theorem 2.2, \( \tau A' \ll \tau A, \beta(B') \ll \beta(B) \) and \( \delta(C') \ll \delta(C) \). But both \( \beta(B) \) and \( \delta(C) \) are zero, i.e., \( B = f(M \otimes_T A) \) and \( C = g(N \otimes_U B) \). Thus if \( A \neq 0, \tau A \) is hollow, \( B = f(M \otimes_T A) \) and \( C = g(N \otimes_U B) \). Therefore, (a) is true when \( A \neq 0 \).

In Case \( A = 0 \) and \( B \neq 0 \), \( r(A \oplus B \oplus C) \) is hollow by assumption. For any \( U B' \leq U B \) with \( U B' \neq U B \), \( r(0 \oplus B' \oplus C) \) is a proper submodule of \( r(0 \oplus B \oplus C) \). From Theorem 2.2, we get \( \beta(B') \ll \beta(B) \) and \( \delta(C') \ll \delta(C) \). Since \( A = 0 \) with \( f : M \otimes_T A \rightarrow B, f = 0 \). Since \( \delta : B \rightarrow B/f(M \otimes_T A) \), we get \( \delta = 1_B \). Thus \( B' \ll B \) and \( U B \) is hollow. Also \( \delta(C) \ll \delta(C) \) means \( \delta(C) = 0 \), i.e., \( C = g(N \otimes_U B) \). Thus if \( A = 0 \) and \( B \neq 0 \), we get \( U B \) is hollow and \( C = g(N \otimes_U B) \). (b) is true in this case.

The last case is \( A = 0 \) and \( B = 0 \). \( r(A \oplus B \oplus C) \) is hollow by assumption. For any \( \nu C' \leq \nu C \) with \( \nu C' \neq \nu C \), \( r(0 \oplus 0 \oplus C') \) is a proper submodule of \( r(0 \oplus 0 \oplus C') \). From Theorem 2.2, we get \( \nu C' \ll \nu C \). This means (c) is true.

Conversely assume (a) or (b) or (c) is valid.
In Case (a), if $\Gamma(A' \oplus B' \oplus C')$ is a proper submodule of $\Gamma(A \oplus B \oplus C)$ we should necessarily have $A' \subseteq A$. Otherwise, assume $A' = A$. Then $B' \supseteq f(M \otimes_T A') = f(M \otimes_T A) = B$ which implies $B = B'$ and $C' \supseteq g(N \otimes_U B') = g(N \otimes_U B) = C$ which implies $C = C'$. Hence $\Gamma(A' \oplus B' \oplus C') = \Gamma(A \oplus B \oplus C)$ which is a contradiction. Now we only need to prove that $\Gamma(A' \oplus B' \oplus C') \ll \Gamma(A \oplus B \oplus C)$. But it follows from Theorem 2.2 immediately since $A' \ll A$, $\beta(B) = 0$ and $\delta(C) = 0$.

In Case (b), for any proper submodule $\Gamma(0 \oplus B' \oplus C')$ of $\Gamma(0 \oplus B \oplus C)$ with $B' \subset B$. Otherwise, assume $B' = B$. Then $C' \supseteq g(N \otimes_U B') = g(N \otimes_U B) = C$ implies $C = C'$. Which is contradict to $\Gamma(0 \oplus B' \oplus C') \neq \Gamma(0 \oplus B \oplus C)$. Thus we need only to prove that $\Gamma(0 \oplus B' \oplus C') \ll \Gamma(0 \oplus B \oplus C)$. This follows from Theorem 2.2 immediately since $B' \ll B$ and $C = g(N \otimes_U B)$.

In Case (c), any proper submodule of $\Gamma(0 \oplus 0 \oplus C)$ is of the form $\Gamma(0 \oplus 0 \oplus C')$ with $C' \subset C$. From (c), $C' \ll C$. By Theorem 2.2 $\Gamma(0 \oplus 0 \oplus C') \ll \Gamma(0 \oplus 0 \oplus C)$ is gotten immediately. $\square$

We now construct an example to show that $\Gamma(A \oplus B \oplus C)$ can be hollow even when $\gamma B$ and $\gamma C$ have infinite dual Goldie dimensions in the sense of [3].

**Example 2.1** Let $K$ be a field. Suppose $T = K, U = K, V = K$ and $M, N$ are infinite dimensional vector spaces over $K$ regarded as $(K-K)$ bimodules in the usual way. Let

$$
\left(\begin{array}{ccc}
T & 0 & 0 \\
M & U & 0 \\
N \otimes_K M & N & V
\end{array}\right) =
\left(\begin{array}{ccc}
K & 0 & 0 \\
M & K & 0 \\
N \otimes_K M & N & K
\end{array}\right).
$$

Let $A = K, B = M$ and $C = N \otimes_K M$. Define $f : M \otimes_T A \rightarrow B$ by $f(m \otimes a) = ma$ and $g : N \otimes_U B \rightarrow C$ by $g(n \otimes m) = n \otimes m$. Then from Theorem 2.2 we see that $\Gamma(A \oplus B \oplus C)$ corresponding to $(A, B, C; f, g)$ is hollow, since $\tau A$ is hollow and $B = f(M \otimes_T A), C = g(N \otimes_U B)$. However the dual Goldie dimensions of $\gamma B$ and $\gamma C$ are infinite.

**3. Determination of Rad $\Gamma(A \oplus B \oplus C)$ and Soc $\Gamma(A \oplus B \oplus C)$**

In this section we will determine the radical $\mathrm{Rad}_\Gamma(A \oplus B \oplus C)$ and $\mathrm{Soc}_\Gamma(A \oplus B \oplus C)$ of $\Gamma(A \oplus B \oplus C)$. For this purpose we will describe the maximal (resp. simple) submodules of $\Gamma(A \oplus B \oplus C)$ firstly. Let $\tau L = \{a \in A | f(m \otimes a) = 0 \text{ for all } m \in M\}$ and $\tau Q = \{b \in B | g(n \otimes b) = 0 \text{ for all } n \in N\}$.

**Theorem 3.1** Let $\mathcal{S}_1 = \{\Gamma(A' \oplus B' \oplus C') | A' \text{ a maximal submodule of } A\}$, $\mathcal{S}_2 = \{\Gamma(A \oplus B' \oplus C') | B' \text{ a maximal submodule of } B \text{ with } f(M \otimes_T A) \subseteq B'\}$ and $\mathcal{S}_3 = \{\Gamma(A \oplus B \oplus C') | C' \text{ a maximal submodule of } C \text{ with } g(N \otimes_U B) \subseteq C'\}$. Let $\Psi_1 = \{(L' \oplus 0 \oplus 0)|L' \text{ a simple submodule of } L\}$, $\Psi_2 = \{(0 \oplus Q' \oplus 0)|Q' \text{ a simple submodule of } Q\}$ and $\Psi_3 = \{(0 \oplus 0 \oplus C')|C' \text{ a simple submodule of } C\}$. Then

(a) The family $\mathcal{S}$ of maximal submodules of $\Gamma(A \oplus B \oplus C)$ is precisely $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$.

(b) The family $\Psi$ of minimal submodules of $\Gamma(A \oplus B \oplus C)$ is precisely $\Psi = \Psi_1 \cup \Psi_2 \cup \Psi_3$.

**Proof** Let $\Gamma(A' \oplus B' \oplus C')$ be any maximal submodule of $\Gamma(A \oplus B \oplus C)$. If $A' \subset A$, since
\((A' \oplus B' \oplus C') \subseteq (A' \oplus B \oplus C) \subseteq (A \oplus B \oplus C)\), we conclude \(B' = B\) and \(C' = C\). Also from \((A' \oplus B \oplus C) \subseteq (A'' \oplus B \oplus C) \subseteq (A \oplus B \oplus C)\) for any \(A' \subseteq A'' \subseteq A\) we see that \(A' = A''\). Hence \(A'\) is a maximal submodule of \(A\), which shows that \((A' \oplus B' \oplus C') = (A' \oplus B \oplus C)\) is in \(\Xi_3\). Now suppose \(A' = A\). Then \(f(M \otimes T A) \subseteq B'\). On the one hand, if \(B' \neq B\), since \((A' \oplus B' \oplus C') \subseteq (A \oplus B \oplus C)\), we have \(C = C'\). Also from \((A' \oplus B' \oplus C) \subseteq (A' \oplus B' \oplus C') \subseteq (A \oplus B \oplus C)\) for any \(B' \subseteq B'' \subseteq B\) we see that \(B' = B''\) immediately. Hence \(B'\) is a maximal submodule of \(B\). Thus \((A' \oplus B' \oplus C') = (A' \oplus B' \oplus C)\) is in \(\Xi_3\). On the other hand, if \(B' = B\), then \(g(N \otimes U B) \subseteq C'\). Since \(\Gamma(A' \oplus B' \oplus C') = \Gamma (A \oplus B \oplus C')\) is a maximal submodule of \(\Gamma(A \oplus B \oplus C)\), we see that \(C'\) is a maximal submodule of \(C\) immediately. Thus \((A' \oplus B' \oplus C') = (A \oplus B \oplus C)\) is in \(\Xi_3\).

Conversely, it is straightforward to see that any submodule of \(\Gamma(A \oplus B \oplus C)\) belonging to \(\Xi_1 \cup \Xi_2 \cup \Xi_3\) is a maximal submodule of \(\Gamma(A \oplus B \oplus C)\). This proves (a).

Let \(\Gamma(A' \oplus B' \oplus C)\) be any simple submodule of \(\Gamma(A \oplus B \oplus C)\). If \(C' \neq 0\), for any \(0 \neq C'' \subseteq C'\), since \((0 \oplus 0 \oplus C'') \subseteq (A' \oplus B' \oplus C') \subseteq (A \oplus B \oplus C)\), we see that \(A' = 0\), \(B' = 0\) and \(C'\) is a simple submodule of \(C\). If \(C' = 0\), we have \(g \circ (1_N \otimes j)(N \otimes U B) = 0\), hence \(j(B') \subseteq_U Q\) where \(j : B' \hookrightarrow B\) denotes the inclusion. Thus \(B' \leq_U Q\). If \(B' \neq 0\), for any \(0 \neq C'' \subseteq B'\), since \((0 \oplus B' \oplus 0) \subseteq (A' \oplus B' \oplus 0) \subseteq (A \oplus B \oplus C)\), we see that \(A' = 0\). Thus \(B'\) is a simple submodule of \(B\). If \(B' = 0\), we have \(f \circ (1_M \otimes i)(M \otimes T A) = 0\) and thus \(i(A') \subseteq_T L\) where \(i : A' \rightarrow A\) denotes the inclusion. Thus \(A' \leq_T L\). If \(0 \neq A'' \subseteq A' \subseteq_T L\), we have \((A'' \oplus 0 \oplus 0) \subseteq (A' \oplus 0 \oplus 0)\). Hence \(A'\) is a simple submodule of \(\Gamma(L)\). Therefore, \(\Psi \subseteq \Psi_1 \cup \Psi_2 \cup \Psi_3\).

For (c), it is straightforward to see that any submodule of \(\Gamma(A \oplus B \oplus C)\) belonging to \(\Psi_1 \cup \Psi_2 \cup \Psi_3\) is a simple submodule of \(\Gamma(A \oplus B \oplus C)\). This proves (c).

**Corollary 3.1** Let \(\beta : B \rightarrow B/f(M \otimes T A)\) and \(\delta : C \rightarrow C/g(N \otimes U B)\) denote the canonical quotient maps. Then

(a) \(\text{Rad}(A \oplus B \oplus C) = \text{Rad}A \oplus \beta^{-1}(\text{Rad}B/f(M \otimes T A)) \oplus \delta^{-1}(\text{Rad}C/g(N \otimes U B))\).

(b) \(\text{Soc}_{\Gamma}(A \oplus B \oplus C) =_{\Gamma} (\text{Soc}_{\Gamma} L \oplus \text{Soc}_{\Gamma} U Q \oplus \text{Soc}_{\Gamma} C)\).

**Proof** These are immediate consequences of Theorem 3.1. □

**Definition 3.1**[4,5] A module \(V\) is said to be finitely embedded (or finitely co-generated) if \(\text{Soc}V\) is finitely generated and essential in \(V\).

Theorem 3.1 enables us to obtain the following.

**Theorem 3.2** (1) \(\text{Soc}_{\Gamma}(A \oplus B \oplus C)\) is finitely generated if and only if \(\text{Soc}_{\Gamma} L, \text{Soc}_{\Gamma} U Q\) and \(\text{Soc}_{\Gamma} C\) are finitely generated.

(2) \(\text{Soc}_{\Gamma}(A \oplus B \oplus C) = \text{essential in }_{\Gamma}(A \oplus B \oplus C)\) if and only if \(\text{Soc}_{\Gamma} L, \text{Soc}_{\Gamma} U Q\) and \(\text{Soc}_{\Gamma} C\) are essential in \(_{\Gamma} L, _{\Gamma} U Q\) and \(_{\Gamma} C\) respectively.

(3) \(\text{finitely embedded if and only if }_{\Gamma} L, _{\Gamma} U Q\) and \(_{\Gamma} C\) are finitely embedded.

**Proof** From Corollary 3.1 (b) we have \(\text{Soc}_{\Gamma}(A \oplus B \oplus C) =_{\Gamma} (\text{Soc}_{\Gamma} L \oplus \text{Soc}_{\Gamma} U Q \oplus \text{Soc}_{\Gamma} C)\). Note that \(_{\Gamma}(\text{Soc}_{\Gamma} L \oplus \text{Soc}_{\Gamma} U Q \oplus \text{Soc}_{\Gamma} C)\) corresponds to the quintuple \((\text{Soc}_{\Gamma} L, \text{Soc}_{\Gamma} U Q, \text{Soc}_{\Gamma} C; 0, 0)\).

From a well-known result (Exercise 1D(b) on p.7 of [6]) we see that \(\text{Soc}_{\Gamma}(A \oplus B \oplus C)\) is finitely
generated ⇔ Soc_T L, Soc_U Q/(M ⊗ Soc_T L), Soc_V C/(N ⊗ Soc_U Q) are finitely generated. This proves (1). (2) is an immediate consequence of Theorem 2.1. (3) is immediate from (1) and (2).

Often one is interested in finding conditions implying \( \text{Rad}(V) \ll V \). In this connection, we have the following.

**Theorem 3.3** \( \text{Rad}^\Gamma(A \oplus B \oplus C) \) is small in \( \Gamma(A \oplus B \oplus C) \) if and only if \( \text{Rad}_T A, \text{Rad}_U (B/f(M \otimes_T A)) \) and \( \text{Rad}_V (C/g(N \otimes_U B)) \) are small in \( \Gamma(A), \Gamma(B/f(M \otimes_T A)) \) and \( \Gamma(C/g(N \otimes_U B)) \), respectively.

**Proof** This is an immediate consequence of Corollary 3.1(a) and Theorem 2.2. \( \square \)

**Definition 3.2** [7] Let \( R \) be a ring and \( V \in \text{Mod} R \). \( V \) is said to be co-hopfian if every injective endomorphism \( f : V \to V \) is automatically an isomorphism.

In [7] we know that any quasi-injective finitely embedded module \( V \) is co-hopfian. Using (3) of Theorem 3.2 which characterizes finitely embedded modules over \( 3 \times 3 \) formal triangular matrix rings, we will construct an example of a finitely embedded module which is not co-hopfian, thus showing that quasi-injectivity of \( V \) cannot be dispensed with for the validity of the above result.

**Example 3.1** Let

\[
\Gamma = \begin{pmatrix} Z & 0 & 0 \\ Z_{p^\infty} & Z & 0 \\ Z_{p^\infty} & Z_{p^\infty} & Z \end{pmatrix},
\]

where \( p \) is a prime. Consider the \( \Gamma \)-module \( \Gamma(Z \oplus Z_{p^\infty} \oplus Z_{p^\infty}) \) associated to the quintuple \((Z, Z_{p^\infty}, Z_{p^\infty}; f, g)\) where \( f : Z_{p^\infty} \otimes Z \to Z_{p^\infty} \) by \( x \otimes k \to xk \) and \( g : Z_{p^\infty} \otimes Z_{p^\infty} \to Z_{p^\infty} \) by \( x \otimes y \to xy \) for all \( k \in Z \) and \( x, y \in Z_{p^\infty} \). In this case \( Z_{p^\infty} \) is finitely embedded in \( \text{Mod} Z \).

Also \( L = \{ k \in Z \mid xk = 0 \text{ for all } x \in Z_{p^\infty} \} \) is finitely embedded in \( \text{Mod} Z \), \( Q = \{ x \in Z_{p^\infty} \mid yx = 0 \text{ for all } y \in Z_{p^\infty} \} \) is finitely embedded in \( \text{Mod} Z \). From (3) of Theorem 3.2 we see that \( \Gamma(Z \oplus Z_{p^\infty} \oplus Z_{p^\infty}) \) is finitely embedded. Let \( n \) be an integer \( \geq 2 \) and relatively prime to \( p \). Let \( \sigma_1 : Z \to Z, \sigma_2 : Z_{p^\infty} \to Z_{p^\infty}, \sigma_3 : Z_{p^\infty} \to Z_{p^\infty} \) be given by multiplication by \( n \). Clearly

\[
\begin{array}{cccc}
Z_{p^\infty} \otimes Z & 1_{Z_{p^\infty}} \otimes \sigma_1 & Z_{p^\infty} \otimes Z & Z_{p^\infty} \otimes Z_{p^\infty} \\
\downarrow f & \downarrow f & \downarrow g & \downarrow g \\
Z_{p^\infty} & \sigma_2 & Z_{p^\infty} & Z_{p^\infty}
\end{array}
\]

are two commutative diagrams. The map \( \sigma = (\sigma_1, \sigma_2, \sigma_3) : \Gamma(Z \oplus Z_{p^\infty} \oplus Z_{p^\infty}) \to \Gamma(Z \oplus Z_{p^\infty} \oplus Z_{p^\infty}) \) is an injective endomorphism. This is because \( \sigma_1 \) is injective, \( \sigma_2 \) and \( \sigma_3 \) are isomorphisms (since \( (p, n) = 1 \)). However \( \sigma \) is not surjective, because \( \sigma_1 \) is not. Thus \( \Gamma(Z \oplus Z_{p^\infty} \oplus Z_{p^\infty}) \) is not co-hopfian.
References


