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Completeness of Complex Exponential System in L^p_α Space

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Abstract A necessary and sufficient condition is obtained for the complex exponential system to be dense in the weighted Banach space $L^p_\alpha=\{f:\int_{-\infty}^{\infty}|f(t)e^{-\alpha(t)}|^p\mathrm{d}t<\infty\}$, where $1\le p<+\infty$ and $\alpha(t)$ is a nonnegative continuous function on $\mathbf{R}$.

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1. Introduction

Suppose $\alpha(t)$ is a nonnegative continuous function on $\mathbf{R}$ satisfying

$$\lim_{t\to\infty}\frac{\alpha(t)}{t}=\infty \quad\text{and}\quad \lim_{t\to-\infty}\frac{\alpha(t)}{|t|}=0, \tag{1}$$

$\alpha(t)$ is henceforth called a weight on $\mathbf{R}$. Given a weight $\alpha(t)$, we take a weighted Banach space C_α consisting of complex continuous functions $f(t)$ defined on $\mathbf{R}$ with $f(t)\exp(-\alpha(t))$ vanishing at infinity, and the norm of f is given by $||f||_\alpha=\sup\{|f(t)e^{-\alpha(t)}|:t\in\mathbf{R}\}$. Suppose $L^p_\alpha=\{f:||f||_\alpha=(\int_{-\infty}^{\infty}|f(t)e^{-\alpha(t)}|^p\mathrm{d}t)^{\frac{1}{p}}<\infty\}$, $1\le p<+\infty$. Then L^p_α is also a Banach space.

Let $\Lambda=\{\lambda_n=|\lambda_n|e^{i\theta_n}:n=1,2,\ldots\}$ be a sequence of complex numbers in the open right half-plane $\mathbf{C}_+=\{z=x+iy:x>0\}$, and $L=\{l_n:n=1,2,\ldots\}$ be a sequence of positive integers. Denote by $M(\Lambda,L)$ the set of complex exponential polynomials which are finite linear combinations of the exponential system $M(\Lambda,L)=\{t^{k-1}e^{\lambda_n t}:k=1,2,\ldots,l_n;n=1,2,\ldots\}$. Our condition (1) guarantees that $M(\Lambda,L)$ is a subspace of C_α and L^p_α. In [1], the author has obtained some results on completeness of $M(\Lambda,L)$ in C_α:

Theorem A[1] *Let nonnegative convex function $\alpha(t)$ on $\mathbf{R}$ satisfy the following conditions:*

(i) $\alpha(t)$ is twice continuously differentiable on $\mathbf{R}$, and $\alpha'(t)$ is strictly increasing on $[1,\infty)$;

(ii) There exists $\varepsilon_0>0$ such that

$$\alpha'(t)\ge\varepsilon_0\alpha(t)\quad t\ge 1; \tag{2}$$

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(iii) There exists $A_1 > 0$ such that

$$\int_1^{\infty} \frac{\alpha'(t-A_1)\alpha''(t)}{(\alpha'(t))^2} \mathrm{d}t < +\infty. \tag{3}$$

Let $\Lambda = \{\lambda_n = |\lambda_n| e^{i\theta_n} : n = 1, 2, \ldots\}$ be a sequence of complex numbers in the right half-plane $\mathbf{C}_+$ and $L = \{l_n : n = 1, 2, \ldots\}$ be a sequence of positive integers satisfying

$$|\lambda_n| \geq 1, \quad n = 1, 2, \ldots, \tag{4}$$

$$\limsup_{r\to\infty} \frac{N_1(r)}{r} < \infty. \tag{5}$$

Then $M(\Lambda, L)$ is complete in C_α if and only if

$$\limsup_{r\to\infty} (2N_2(\alpha'(r)) - r) = \infty, \tag{6}$$

where

$$N_k(r) = \sum_{1<|\lambda_n|\leq r} \frac{l_n \cos\theta_n}{|\lambda_n|^{k-1}}, \quad r > 1, k = 0, 1, 2, 3. \tag{7}$$

A similar result is obtained for L^p_α in this paper as follows:

Theorem 1 *Let $\alpha(t)$ be a nonnegative even convex function on $\mathbf{R}$, satisfying (i), (ii), (iii) of Theorm A, $\Lambda = \{\lambda_n = |\lambda_n| e^{i\theta_n} : n = 1, 2, \ldots\}$ a sequence of complex numbers in the right half-plane $\mathbf{C}_+$ and $L = \{l_n : n = 1, 2, \ldots\}$ a sequence of positive integers satisfying (4) and (5). Then $M(\Lambda, L)$ is complete in L^p_α if and only if (6) holds.*

Remark 1 There exist some functions which satisfy (i), (ii), (iii) of Theorem A, for example: $\alpha(t) = e^{|t|^a} (\log^+ |t|)^b, t \neq 0, a > 1, b \geq 0$.

2. Proof of Theorem 1

Hereafter, we denote a positive constant by A, not necessaryly the same at each occurrence. In order to prove Theorem 1, we need the following technical lemmas:

Lemma 1[1] *Let $\alpha(x)$ be a nonnegative convex function on $\mathbf{R}$ satisfying (1), and assume that*

$$\beta(t) = \sup\{xt - \alpha(x) : x \in \mathbf{R}\}, t \in \mathbf{R} \tag{8}$$

is the Young transform[2] of the function $\alpha(x)$. Suppose that $k(r)$ is an increasing function on $[0, +\infty)$. If there exists $a \in \mathbf{R}$, such that

$$\int_1^{\infty} \frac{\alpha(k(t) - a)}{1 + t^2} \mathrm{d}t < +\infty, \tag{9}$$

then there exists an analytic function $f(z) \not\equiv 0$ in $\mathbf{C}_+$ satisfying

$$|f(z)| \leq \frac{e^{-x}}{1 + |z|^2} \exp\{\beta(x-1) - xk(|z|)\}, z = x + iy \in \mathbf{C}_+, \quad x \geq 1. \tag{10}$$

Remark 2 Lemma 1 is a result of uniqueness about Watson's problem [3–5]. Some similar results have been proved in [4].

Lemma 2[1] *Let $\Lambda = \{\lambda_n = |\lambda_n|e^{i\theta_n} : n = 1, 2, \ldots\}$ be a sequence of complex numbers in $\mathbf{C}_+$ and $L = \{l_n : n = 1, 2, \ldots\}$ a sequence of positive integers satisfying (4) and (5). Let*

$$g_n(z) = (\frac{1 - z|\lambda_n|^{-1}e^{-i\theta_n}}{1 + z|\lambda_n|^{-1}e^{i\theta_n}}) \exp(\frac{2z\cos\theta_n}{|\lambda_n|}), \quad n = 1, 2, \ldots. \tag{11}$$

Then the weighted Blaschke product

$$G(z) = \prod_{n=1}^{\infty} (g_n(z))^{l_n} \tag{12}$$

is analytic in $\mathbf{C}_+$, and for all $\varepsilon \in (0, 1)$, there exists an $A_\varepsilon > 0$, such that

$$|G(z)| \le \exp\{2xN_2(\varepsilon r) + A_\varepsilon x\}, \quad z = x + iy, r = |z| > 1. \tag{13}$$

Sufficiency of Theorem 1 If $M(\Lambda, L)$ is incomplete in L^p_α, then by Hahn-Banach Theorem[6], there exists a bounded linear functional T on L^p_α, such that

$$||T|| = 1,\ T(t^{k-1}e^{\lambda_n t}) = 0, k = 1, 2, \ldots, l_n; n \in \mathbb{N}. \tag{14}$$

By the Riesz representation theorem, there exists a $g \in L^q_{-\alpha}$ such that $||T|| = ||g||_{q,-\alpha}$ and $T(f) = \int_{-\infty}^{\infty} f(t)g(t)\mathrm{d}t$, $f \in L^p_\alpha$, where $\frac{1}{p} + \frac{1}{q} = 1$ ($q = \infty$ if $p = 1$),

$$L^q_{-\alpha} = \{g : ||g||_{q,-\alpha} = (\int_{-\infty}^{\infty} |g(t)e^{\alpha(t)}|^q \mathrm{d}t)^{\frac{1}{q}} < \infty\},$$

$$L^\infty_{-\alpha} = \{g : ||g||_{\infty,-\alpha} = \text{ess sup}\{|g(t)|e^{\alpha(t)} : x \in \mathbf{R}\} < \infty\}.$$

Let $h(z) = T(e^{tz}) = \int_{-\infty}^{\infty} e^{zt}g(t)\mathrm{d}t$. Then by (14), for each $b > 0$ and $x = \mathrm{Re}z \in [0, b]$, since $|e^{zt}g(t)| \le e^{bt}|g(t)|$ for $t \ge 0$ and $|e^{zt}g(t)| \le |g(t)| \in L^1$ for $t < 0$, we see that $h(z)$ is continuous on $\overline{\mathbf{C}}_+$ and analytic in $\mathbf{C}_+$. The number of the zeros of $h(z)$ at $\lambda_n \in \Lambda$ is not less than l_n.

Let $\beta(x)$ be defined by (8), by virtue of Hölder inequality, $|h(z)| \le e^{\beta(x)}$. Furthermore, for $r > 1$, by Carleman formula[7],

$$\varphi(r) = \frac{1}{\pi r}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log|h(re^{i\theta})|\cos\theta \mathrm{d}\theta + \frac{1}{2\pi}\int_1^r (\frac{1}{y^2} - \frac{1}{r^2}) \log|h(iy)h(-iy)|\mathrm{d}y + d(r),$$

where $\varphi(r) = N_2(r) - r^{-2}N_0(r)$ and $d(r)$ is bounded on $(1, \infty)$. Hence there exists $A > 0$ such that

$$N_2(r) \le A + \varphi(r) \le A + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\beta(r)}{r}(\cos\theta)^2 \mathrm{d}\theta = A + \frac{\beta(r)}{2r}. \tag{15}$$

By the definition of $\beta(x)$, there exists $t_0 > 0$ such that, for each $t \ge t_0$, there exists an $x(t) \ge 1$ satisfying

$$t = \alpha'(x(t)), \beta(t) = x(t)t - \alpha(x(t)) = t(x(t) - \frac{\alpha(x(t))}{\alpha'(x(t))}).$$

By (2), $\beta(t)t^{-1} - x(t)$ is bounded on $[t_0, \infty)$. Let $x = \gamma(y)$ be the inverse function of $y = \alpha'(x)$. Then $\beta(t)t^{-1} - \gamma(t)$ is bounded on $[t_0, \infty)$. By (15), (6) does not hold. This completes the proof of sufficiency of Theorem 1. □

Necessity of Theorem 1 If (6) does not hold, there exists $A_2 > 0$ such that $2N_2(\alpha'(t)) - t \le$

A_2 $(t \geq 1)$. Let $x = \gamma(y)(y \geq \alpha'(1))$ be the inverse function of $y = \alpha'(x)(x \geq 1)$. Then $2N_2(r) - \gamma(r) \leq A_2$ $(r \geq \alpha'(1))$. By (2), (3) and (7), we have

$$\int_{\alpha'(1)}^{\infty} \frac{\alpha(2N_2(r) - A_1 - A_2)}{r^2} \mathrm{d}r \leq \int_{\alpha'(1)}^{\infty} \frac{\alpha(\gamma(r) - A_1)}{r^2} \mathrm{d}r \leq \int_{1}^{\infty} \frac{\alpha'(t - A_1)}{\varepsilon_0(\alpha'(t))^2} \alpha''(t) \mathrm{d}t < \infty.$$

By Lemma 1, there exists an analytic function $f(z) \not\equiv 0$ in $\mathbf{C}_+$ satisfying (10) with $k(r)$ replaced by $2N_2(r)$.

Let $g_0(z) = G(z)f(z)(1+z)^{-2}e^{-Az-A}$, where A is a large positive constant and $G(z)$ is defined by (12). By (13), we have

$$|g_0(z)| \leq \frac{1}{1+|z|^2} \exp\{\beta(x-1) - x\} \quad z \in \mathbf{C}_+. \tag{16}$$

The function

$$h_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_0(1+iy) e^{-(1+iy)t} \mathrm{d}y$$

is continuous on $\mathbf{R}$ and by the Cauchy theorem,

$$h_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_0(x+iy) e^{-(x+iy)t} \mathrm{d}y, \quad x > 0. \tag{17}$$

Since $\beta(t)$ is the Young transform of the convex function $\alpha(x)$, we may assume, without loss of generality, that[2] $\alpha(t) = \sup\{xt - \beta(x) : x \geq 0\}$, $t \geq 0$. We obtain from (16) and (17) that

$$|h_0(t)| \leq \exp(-\alpha(t) - |t|), \quad g_0(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h_0(t) e^{tz} \mathrm{d}t, \quad x > 0. \tag{18}$$

Moreover, by the definition of g_0,

$$g_0^{(k)}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h_0(t) t^k e^{tz} \mathrm{d}t, \quad g_0^{(k)}(\lambda_n) = 0, \ k = 0, 1, \ldots, l_n - 1,$$

i.e.,

$$\int_{-\infty}^{\infty} h_0(t) t^{k-1} e^{t\lambda_n} \mathrm{d}t = 0, \quad k = 1, 2, \ldots, l_n, n \in \mathbb{N}. \tag{19}$$

Therefore by (18) and (19), linear functional T defined by $T(h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h_0(t)h(t)\mathrm{d}t$ ($h \in L_\alpha^p$) is a bounded linear functional on L_α^p and satisfies $T(e^{zt}) = g_0(z)$, $T(t^{k-1}e^{\lambda_n t}) = 0, k = 1, 2, \ldots, l_n; n \in \mathbb{N}$. The norm of T is $||T|| = \frac{1}{\sqrt{2\pi}} ||h_0||_{q,-\alpha} > 0$. By the Hahn-Banach Theorem and Riesz representation theorem, the space $M(\Lambda, L)$ is incomplete in L_α^p. This completes the proof of Theorem 1. □

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