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On CWC-Mappings and Metrization Theorem

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Abstract In this paper, applications of CWC-mappings are discussed in metrization theorem, and some new metrization theorems of topological spaces are obtained by Nagata's condition and weak γ-spaces, which generalizes several metrization theorems obtained by R.E. Hodel and J.Nagata.

Keywords g-functions; CWC-mappings; metrizable spaces; Nagata's condition.

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Bases are one of importiant concepts in topologcal spaces and are a good tool to characterize metric spaces. In 1966, Arhangel'skiĭ[1] introduced a concept of weak bases which deepens the concept of bases and plays an important role in the theory of generalized metric spaces. In recent years, some metrization theorems were obtained in [2–5] by using CWC-mappings. In this paper, we discuss some characterizations of metrizability of spaces in terms of CWC-mappings and some new metrization theorems are obtained, which improve some results in [2]–[5] and [10].

All spaces in this paper are regular and T_1. All maps are continuous and surjections. Let N denote the set of all positive integers. Readers may refer to [6] for unstated definitions and terminologies. First let us recall several definitions.

Definition 1 *Let (X,τ) be a topological space and 2^X denote the collection of all subsets of X.*

(1) A map $g: N\times X \to \tau$ is called a g-function on a topological space X[2], if $x \in g(n+1,x) \subset g(n,x)$ for each $x \in X$ and $n \in N$.

(2) A map $g: N \times X \to 2^X$ is called a CWC-mapping on a topological space X[7], if for every $x \in X$ and each $n \in N$, $x \in g(n+1,x) \subset g(n,x)$ and for each subset U of X, U is open set in X if and only if for each $x \in U$, there exists $n \in N$ such that $g(n,x) \subset U$.

Some metrization theorems are characterized in terms of g-functions, which are important make-up of modern metrization theorems[8] . A definition of open subset of spaces by using CWC-map is the same as the one by using weak bases[1]. CWC-mapping is called CWBC-mapping and weak bases g-function in [2] and [3], respectively.

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For convenience, we excerpt the Theorem 2.2 in [4] as follows.

Lemma 2 *A space X is metrizable if and only if X has CWC-mapping g satisfying that:*

(2.1) The sequence $y_n \to x$, if for all $n \in N$, $y_n \in g(n, x_n)$ and $x_n \to x$.

(2.2) The sequence $y_n \to x$, if for all $n \in N$, $x_n \in g(n, y_n)$ and $x_n \to x$.

Given a map $g : N \times X \to 2^X$. For every $n, k \in N$ and any $A \subset X$, let

$$g^1(n,x) = g(n,x), g^k(n,x) = \bigcup\{g(n,y) : y \in g^{k-1}(n,x)\} \quad \text{for} \quad k \geq 2$$

and

$$g^k(n,A) = \bigcup\{g^k(n,x) : x \in A\}, \quad g(n,A) = g^1(n,A).$$

Theorem 3 *A space X is metrizable if and only if X has a CWC-mapping $g : N \times X \to 2^X$ satisfying that*

(3.1) $\bar{A} = \bigcap_{n \in N} g(n, A)$ for any $A \subset X$.

(3.2) If $y \in g(n,x)$, then $g(n,y) \subset g(n-1,x)$ for $n > 2$.

Proof The necessity is clear. We only prove the sufficiency. Let X have a CWC-mapping g satisfying conditions (3.1) and (3.2).

Let $V(n,x) = \{y : x \in g(n,y)\}$ for each $(n,x) \in N \times X$. Then $V(n,x)$ is a neighborhood of x. In fact, $x \in V(n,x)$ and $\overline{X - V(n,x)} \subset g(n, X - V(n,x))$ by condition (3.1). If $y \notin V(n,x)$, then $x \notin g(n,y)$. Thus $x \notin \overline{X - V(n,x)}$ and $x \in X - \overline{X - V(n,x)} \subset V(n,x)$. Hence $V(n,x)$ is a neighborhood of x. Let $h(n,x) = g(n,x) \bigcap V(n,x)$. Then $x \in h(n+1,x) \subset h(n,x)$.

(1) A map h is a CWC-mapping on a topological space X.

We only need prove that for each subset U of X, U is an open set in X if and only if for each $x \in U$, there exists $n \in N$ such that $h(n,x) \subset U$. Let U be an open set in X. Then there exists some $g(n,x) \subset U$ for each $x \in U$, hence $h(n,x) \subset U$. On the other hand, for every $x \in U$ there exists $h(n,x) \subset U$. Because $V(n,x)$ is a neighborhood of x, there exists some $g(m,x) \subset V(n,x)$. Let $k = \max\{m,n\}$. Then $g(k,x) \subset g(n,x) \bigcap V(n,x) = h(n,x) \subset U$. Hence U is an open set in X by a map $g : N \times X \to 2^X$ which is a CWC-mapping on X.

(2) A map $h : N \times X \to 2^X$ satisfies conditions (2.1) and (2.2) in Lemma 2.

Let $y_n \in h(n, x_n)$ for all $n \in N$ and $x_n \to x$. Then

$$y_n \in h(n,x_n) \subset V(n,x_n) = \{y : x_n \in g(n,y)\}.$$

Thus $x_n \in g(n,y_n)$ and $x_n \to x$. This property can also be obtained by h satisfying condition (2.2). We prove that sequence $\{y_n\}$ converges to x by this property .

Let $F = \{y_n : n \in N\}$. For each $m > n \in N$, put $A = \{x_k : k > m\}$. Since sequence $\{x_n\}$ converges to x, it follows from condition (3.1) that $x \in \overline{A} = \bigcap_{i \in N} g(i,A) \subset g(m,A)$, and there exists $k > m$ such that $x \in g(m,x_k)$. Since $x_k \in g(m,y_k)$ and $g(m,x_k) \subset g(m-1,y_k)$ by condition (3.2), we have $x \in g(m,x_k) \subset g(m-1,y_k) \subset g(n,y_k) \subset g(n,F)$. By condition (3.1) again, we have $x \in \bigcap_{n \in N} g(n,F) = \overline{F}$, so x is an accumulation point of set F. Therefore x is an accumulation point of the set which consists of any subsequence of sequence $\{y_n\}$. Thus sequence $\{y_n\}$ converges to x. Hence conditions (2.1) and (2.2) are satisfied.

By means of Lemma 2 we know that X is metrizable. $\square$

Corollary 4[10] *A space X is metrizable if and only if X has a g-function $g : N \times X \to \tau$ satisfying conditions (3.1) and (3.2) of Theorem 3.*

Theorem 5 *A space X is metrizable if and only if X has a CWC-mapping $g : N \times X \to 2^X$ and there is a positive integer $m \geq k$ satisfying:*

(5.1) For any $A \subset X$ and each $n \in N$, $\overline{A} \subset g^k(n, A)$;

(5.2) If for all $n \in N$, $y_n \in g^m(n, x_n)$ and $x_n \to x$, then $y_n \to x$.

Proof We only prove sufficiency.

Assume that X has a CWC-mapping g and $k \in N$ satisfies condition (5.1) in the theorem. For each $x \in X$ and $n \in N$, let $V(n,x) = \{y : x \in g^k(n,y)\}$. We show that $V(n,x)$ is a neighborhood of x. Clearly, $x \in V(n,x)$ and $\overline{X - V(n,x)} \subset g^k(n, X - V(n,x))$ by condition (5.1). If $y \notin V(n,x)$, then $x \notin g^k(n,y)$. Thus $x \notin \overline{X - V(n,x)}$. Hence $V(n,x)$ is a neighborhood of x. Again let $h(n,x) = g(n,x) \bigcap V(n,x)$ for each $(n,x) \in N \times X$. Using the same method as in the proof of (1) in Theorem 3, we have that the map $h : N \times X \to 2^X$ is a CWC-mapping on X. We prove that the map h satisfies condition (2.1) of Lemma 2.

Assume for all $n \in N$, $y_n \in h(n, x_n)$ and $x_n \to x$. Then $y_n \in g(n, x_n) \subset g^m(n, x_n)$ by definition of h. By means of condition (5.2) again, we have $y_n \to x$. Hence condition (2.1) of Lemma 2 is satisfied.

Suppose for all $n \in N$, $x_n \in h(n, y_n)$ and $x_n \to x$. Then $x_n \in V(n, y_n) = \{y : y_n \in g^k(n, y_n)\}$. Thus $y_n \in g^k(n, x_n) \subset g^m(n, x_n)$. By condition (5.2) again, we have $y_n \to x$. Hence condition (2.2) of Lemma 2 is satisfied.

By means of Lemma 2 we know that X is metrizable. $\square$

Corollary 6 *A space X is metrizable if and only if there exists a CWC-mapping $g : N \times X \to 2^X$ such that:*

(6.1) For any $A \subset X$ and each $n \in N$, $\overline{A} \subset g(n, A)$;

(6.2) If for all $n \in N$, $y_n \in g(n, x_n)$ and $x_n \to x$, then $y_n \to x$.

Corollary 7 *A space X is metrizable if and only if there exists a CWC-mapping $g : N \times X \to 2^X$ and some $k \in N$ such that:*

(7.1) For any $A \subset X$ and each $n \in N$, $\overline{A} \subset g^k(n, A)$;

(7.2) If for all $n \in N$, $y_n \in g(n, x_n)$ and $x_n \to x$, then $y_n \to x$.

Proof By the proof of sufficiency of Theorem 5, we note that a CWC-mapping $g : N \times X \to 2^X$ satisfying the condition (7.2) can be got from the condition (5.2) by definition of $g^k(n,x)$. The sufficiency of Theorem 7 is proved. $\square$

A CWC-mapping of Corollary 7 is weaker than a g-function of Corollary 3.5 in [8].

Remark 8 The conditions of above theorems provide plenty characteristics of generalized metric spaces. We use a CWC-mapping in stead of g-function. Then conditions (2.1), (2.2), (3.2), (5.1), (5.2) and (6.1) are characteristics of a γ-space, a k-semistratifiable space, a quasi-metric space, Nagata's condition (N_k), weak γ-space and Nagata's condition, respectively.

Problem 9 Is a space X metrizable, if for the space X there exists a CWC-mapping $g : N \times X \to 2^X$ and some $k \in N$ satisfying the following conditions:

(9.1) For any $A \subset X$ and each $n \in N$, $\overline{A} \subset g(n, A)$;

(9.2) If for all $n \in N$, $p \in g^{k+1}(n, x_n)$ and $x_n \in g^{k+1}(n, p)$, then $x_n \to p$?

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