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Axioms in the Variety of eO-Algebras

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Abstract The variety **eO** of extended Ockham algebras consists of those algebras $(L;\wedge,\vee,f,k,0,1)$ such that $(L;\wedge,\vee,0,1)$ is a bounded distributive lattice together with a dual endomorphism f on L and an endomorphism k on L such that $fk=kf$. In this paper we extend Urquhart's theorem to **eO**-algebras and we are in particular concerned with the subclass $\mathbf{e_2M}$ of **eO**-algebras in which $f^2=id$ and $k^2=id$. We show that there are 19 non-equivalent axioms in $\mathbf{e_2M}$ and then order them by implication.

Keywords Extended Ockham algebra; dual space; subdirectly irreducible algebra; equational basis.

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1. Introduction

In [3], we introduced the notion of extended Ockham algebra. By an extended Ockham algebra $(L;\wedge,\vee,f,k,0,1)$, we mean a bounded distributive lattice L on which two operations f and k are defined such that

(1) f is a dual lattice endomorphism with $f(1)=0$ and $f(0)=1$;

(2) k is a lattice endomorphism with $k(1)=1$ and $k(0)=0$;

(3) f and k commute.

The class of such algebras is denoted by **eO**. In the class of **eO**-algebras, a particular interesting subclass is so called class of symmetric extended de Morgan algebras for which $f^2=id_L$ and $k^2=id_L$. Such an algebra $(L;f,k)$ is usually written as $(L;^-,^+)$, and we denote by $\mathbf{e_2M}$ the class of such algebras.

We recall [1] that if L is a bounded distributive lattice, then the *dual space* of L is $I_p(L)\equiv(I_p(L);\tau,\subseteq)$, where $(I_p(L);\subseteq)$ is the lattice of primes ideals of L and the topology τ has as a base the sets $\{x\in I_p(L)|x\ni a\}$ and $\{x\in I_p(L)|x\not\ni a\}$ for every $a\in L$. By an extended Ockham space we mean a compact totally ordered disconnected topological space $(X;\tau,\leq)$ on which there are defined a continuous antitone mapping g and a continuous isotone mapping h such that $gh=hg$. The set of clopen down-sets of such a space will be denoted by $\mathcal{O}(X)$. The

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following is an extension to **eO** of a fundamental theorem of Urquhart[6].

Theorem 1[3,Theorem 4.1] *If $(X; g, h)$ is an extended Ockham space, then $(\mathcal{O}(X); f, k)$ is an extended Ockham algebra, where*

$$(\forall A \in \mathcal{O}(X)) \quad f(A) = X \setminus g^{-1}(A),\ k(A) = h^{-1}(A).$$

Conversely, if $(L; f, k)$ is an extended Ockham algebra, then $(I_p(L); g, h)$ is an extended Ockham space, where

$$(\forall x \in I_p(L)) \quad g(x) = \{a \in L | f(a) \notin x\},\ h(x) = \{a \in L | k(a) \in x\}.$$

Moreover, these constructions give a dual equivalence.

For more details of Ockham algebras and extended Ockham algebras we refer the reader to [1, 3, 4].

In [6], [7], Urquhart gave a nice theorem on the dual spaces of Ockham algebras showing that any inequality that holds in an Ockham algebra can be translated into simple conditions on its dual space. In 1996, Blyth, Fang and Varlet[2] extended this property to a double MS-algebra. Here we shall extend it to **eO**-algebras, and use it to investigate axioms in class of symmetric extended de Morgan algebras.

Through this paper we shall use the same methods and terminologies as used in [1] for the variety of Ockham algebras and in [3] for the variety of double MS-algebras.

2. Extension of Urquhart's theorem

Here we shall be interested in discussing some connection between axioms in the **eO**-algebras and the universally quantified disjunctions in **eO**-spaces. In what follows we say $A \le B$ is an inequality in an extended Ockham algebra $(L; f, k)$ if A and B are polynomials built from variables $a, b, c, \ldots$ and the constants $0, 1$ by means of the operations $\wedge, \vee, f, k$. We say that two inequalities are equivalent if they determine the same equational class.

The following result is abopted by extending Urquhart's theorem to extended Ockham algebras. The proof is the same as that of [1] or [2].

Theorem 2 *Let $(L; f, k)$ be an extended Ockham algebra and let (X, g, h) be its dual space. Let E be a finite subset of $\mathbf{N}^4$, say*

$$E = \{(p_i, q_i, u_i, v_i) | i = 1, 2, \ldots, n\}$$

and let

$$A = \{f^{p_i} k^{q_i}(a_i) | p_i \text{ even}\} \cup \{f^{u_i} k^{v_i}(a_i) | u_i \text{ odd}\},$$

$$B = \{f^{p_i} k^{q_i}(a_i) | p_i \text{ odd}\} \cup \{f^{u_i} k^{v_i}(a_i) | u_i \text{ even}\}.$$

Consider the inequality

$$\bigwedge A \le \bigvee B \tag{*}$$

and the universally quantified disjunction

$$(\forall x \in X) \quad \bigvee_i \{g^{p_i}h^{q_i} \geq g^{u_i}h^{v_i} | (p_i, q_i, u_i, v_i) \in E\}. \tag{**}$$

Then $(L; f, k)$ *satisfies* $(*)$ *if and only if* $(X; g, h)$ *satisfies* $(**)$.

Proof Observing first that the inequality $(*)$ fails in L if and only if there exist clopen down-sets A_i of X such that

$$\bigcap_{p_i \text{even}} f^{p_i}k^{q_i}(A_i) \cap \bigcap_{u_i \text{odd}} f^{u_i}k^{v_i}(A_i) \nsubseteq \bigcup_{p_i \text{odd}} f^{p_i}k^{q_i}(A_i) \cup \bigcup_{u_i \text{even}} f^{u_i}k^{v_i}(A_i)$$

which is the case if and only if there exists $x \in X$ such that

$$\begin{cases} x \in f^{p_i}k^{q_i}(A_i) \iff p_i \text{ even} \\ x \in f^{u_i}k^{v_i}(A_i) \iff u_i \text{ odd.} \end{cases} \tag{\dagger}$$

Since $f(A_i) = X \setminus g^{-1}(A_i)$, $k(A_i) = h^{-1}(A_i)$, and note that g is antitone and h is isotone with $gh = hg$, we can see

$$x \in f^r k^s(A_i) \iff \begin{cases} g^r h^s(x) \in A_i, & \text{if } r \text{ is even} \\ g^r h^s(x) \notin A_i, & \text{if } r \text{ is odd.} \end{cases}$$

Thus the above condition $(\dagger)$ is equivalent to the existence of down-sets A_i and element x such that for every $i \in \{1, 2, \ldots\}$,

$$g^{p_i}h^{q_i}(x) \in A_i \text{ and } g^{u_i}h^{v_i}(x) \notin A_i.$$

This in turn is equivalent to the existence of an element x with the property $g^{p_i}h^{q_i}(x) \ngeq g^{u_i}h^{v_i}(x)$ for every i, which is the case if and only if $(**)$ fails in X. $\square$

The inequality $(*)$ in above Theorem 2 can be represented by the following tabulation:

$$\begin{vmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ \vdots & & & \\ t_{n1} & t_{n2} & t_{n3} & t_{n4} \end{vmatrix}$$

where $t_{i1} = (p_i, q_i)$ if p_i is even; $t_{i2} = (u_i, v_i)$ if u_i is odd; $t_{i3} = (p_i, q_i)$ if p_i is odd; and $t_{i4} = (u_i, v_i)$ if u_i is even.

This notion was first introduced by Blyth and Varlet in [1] for Ockham algebras and was also introduced in [2] for double MS-algebras.

Example The inequality

$$f^2(a) \wedge fk^3(a) \wedge f^2k^2(c) \leq fk(b) \vee f^2k(b) \vee f^2(c)$$

is represented by the tabulation

$$\begin{vmatrix} (2,0) & (1,3) & - & - \\ - & - & (1,1) & (2,1) \\ (2,2) & - & - & (2,0) \end{vmatrix}.$$

3. Axioms in $\mathbf{e_2}$M-algebras

By way of illustrating the power of the extension of Urquhart's theorem to **eO**-algebras, we now concentrate on the class $\mathbf{e_2M}$ of symmetric extended de Morgan algebras. We say that $(X; g, h)$ is a symmetric extended de Morgan space (shortly, $\mathbf{e_2M}$-space) if it is the duality of a symmetric extended de Morgan algebra. Clearly, for an $\mathbf{e_2M}$-space $(X; g, h)$, $g^2 = id_X$ and $h^2 = id_X$ with $gh = hg$.

In [5], we characterized equational bases for subvarieties of $\mathbf{e_2M}$-algebras. Here we use the dual space to rediscover the determination of all equational bases for subdirectly irreducible $\mathbf{e_2M}$-algebras. In particular, we shall characterize all non-equivalent equational bases in the variety of $\mathbf{e_2M}$-algebras.

For any given element x in an $\mathbf{e_2M}$-space, since h is isotone, we see that either $h(x) = x$ or $h(x)\|x$ (it means that they are not comparable). Thus

$$x \geq h(x) \iff h(x) \geq x \iff h(x) = x.$$

The following property can also be easily seen:

(a) $x \geq g(x) \iff h(x) \geq gh(x)$;

(b) $gh(x) \geq x \iff g(x) \geq h(x)$;

(c) $x \geq gh(x) \iff h(x) \geq g(x)$.

Therefore, we have by the above observations that there are exactly five non-equivalent and non-trivial binary relations to be considered in an $\mathbf{e_2M}$-space:

(1) $x \geq g(x)$;

(2) $g(x) \geq x$;

(3) $x = h(x)$;

(4) $h(x) \geq g(x)$;

(5) $g(x) \geq h(x)$.

The binary relations (1), (2), (3), (4) and (5) are the basic axioms of this work. For later purpose we denote by (0) the trivial case:

(0) $h(x) = g(x) = x$.

This is the case where x satisfies (1), (2) and (3), and so clearly, (0) $\Rightarrow$ (1), (0) $\Rightarrow$ (2), (0) $\Rightarrow$ (3), (0) $\Rightarrow$ (4) and (0) $\Rightarrow$ (5).

If the dual space $(X; g, h)$ of the $\mathbf{e_2M}$-algebra $(L;^-, ^+)$ satisfies $(n_1) \vee (n_2) \vee \cdots \vee (n_r)$, then we say that L satisfies the axiom $(n_1 n_2 \cdots n_r)$. For instance, if every element of X satisfies (2), or (3), or (5), namely, $(2) \vee (3) \vee (5)$, we say that the axiom (235) holds in L. Clearly, an axiom (A) implies an axiom (B) if the set of digits in the label of (A) (or in any equivalent) is contained in the set of digits of that of (B) (or in any equivalent). For instance, (1) $\Rightarrow$ (14) $\Rightarrow$ (145) $\Rightarrow$ (12345).

The following simple property can be easily verified:

Theorem 3 *Let X be an $\mathbf{e_2M}$-space with $x \in X$. If x satisfies*

(i) (1) and (3), then it satisfies (4);

(ii) (3) and (4), then it satisfies (1);

(iii) (2) *and* (3)*, then it satisfies* (5)*;*

(iv) (3) *and* (5)*, then it satisfies* (2)*;*

(v) (1) *and* (5)*, then it satisfies* (0)*;*

(vi) (2) *and* (4)*, then it satisfies* (0)*.*

Proof Suppose that x satisfies (1) and (3). Then $x \geq g(x)$ and $x = h(x)$, which implies $h(x) \geq g(x)$, and so x satisfies (4). By the similar arguments we can also easily prove that the cases (ii)–(vi) are true.

For the purpose of characterizing the implication relationship of axioms in $\mathbf{e_2M}$, we first give the following property.

Theorem 4 *In* $\mathbf{e_2M}$ *one can define at most 19 non-equivalent axioms. The equivalences between the 31 axioms that can be defined are as follows:*

(a) (1) = (2)*;* (4) = (5).

(b) (14) = (25)*;* (15) = (24)*;* (13) = (23)*;* (34) = (35).

(c) (124) = (125)*;* (145) = (245)*;* (135) = (234)*;* (134) = (235).

(d) (1234) = (1235)*;* (1345) = (2345).

Proof This follows by the following observations of substituting $g(x)$ for x, say $\alpha : x \to g(x)$, in each axiom:

(a) (1) $\overset{\alpha}{\Rightarrow}$ (2) $\overset{\alpha}{\Rightarrow}$ (1); (4) $\overset{\alpha}{\Rightarrow}$ (5) $\overset{\alpha}{\Rightarrow}$ (4).

(b) (14) $\overset{\alpha}{\Rightarrow}$ (25) $\overset{\alpha}{\Rightarrow}$ (14); (15) $\overset{\alpha}{\Rightarrow}$ (24) $\overset{\alpha}{\Rightarrow}$ (15); (13) $\overset{\alpha}{\Rightarrow}$ (23) $\overset{\alpha}{\Rightarrow}$ (13); (34) $\overset{\alpha}{\Rightarrow}$ (35) $\overset{\alpha}{\Rightarrow}$ (34).

(c) (124) $\overset{\alpha}{\Rightarrow}$ (215) $\overset{\alpha}{\Rightarrow}$ (124); (145) $\overset{\alpha}{\Rightarrow}$ (254) $\overset{\alpha}{\Rightarrow}$ (145); (135) $\overset{\alpha}{\Rightarrow}$ (234) $\overset{\alpha}{\Rightarrow}$ (135); (134) $\overset{\alpha}{\Rightarrow}$ (235) $\overset{\alpha}{\Rightarrow}$ (134).

(d) (1234) $\overset{\alpha}{\Rightarrow}$ (2135) $\overset{\alpha}{\Rightarrow}$ (1234); (1345) $\overset{\alpha}{\Rightarrow}$ (2354) $\overset{\alpha}{\Rightarrow}$ (1345). □

Theorem 5 *In an* $\mathbf{e_2M}$-*algebra* $(L;^-,^+)$*, we have the following statements:*

(i) (14) $\Rightarrow$ (15)*;*

(ii) (134) $\Rightarrow$ (135).

Proof (i) Suppose that (14) holds in L, but (15) fails to hold. Then there exists $x_o \in X$ such that $x_o \ngeq g(x_o)$ and $g(x_o) \ngeq h(x_o)$. Since (14) holds, we have either $x_o \geq g(x_o)$ or $h(x_o) \geq g(x_o)$, and then we must have that $h(x_o) \geq g(x_o)$. Substituting now $g(x_o)$ for x_o in (14), then we have either $g(x_o) \geq x_o$ or $g(x_o) \geq h(x_o)$. The former gives $h(x_o) \geq g(x_o) \geq x_o$, which leads to the contradiction that $h(x_o) = g(x_o) = x_o$; the later gives the contradiction again that $g(x_o) = h(x_o)$. Therefore (15) holds in L.

(ii) Suppose that (134) holds in L. Then, for every $x \in X$ we have either $x \geq g(x)$ or $x = h(x)$ or $h(x) \geq g(x)$. We assume now that (135) fails to hold in L. Then there exists $x_o \in X$ such that $x_o \ngeq g(x_o)$, $x_o \neq h(x_o)$ and $g(x_o) \ngeq h(x_o)$. Since x_o satisfies (134), we must have $h(x_o) \geq g(x_o)$. Substituting now $g(x_o)$ for x_o in (134) and noting that $x_o \neq h(x_o)$, we see that either $g(x_o) \geq x_o$ or $g(x_o) \geq h(x_o)$. The former gives $h(x_o) \geq g(x_o) \geq x_o$, which results in the contradiction that $h(x_o) = g(x_o) = x_o$; the later gives the contradiction again that $g(x_o) = h(x_o)$.

Hence (135) holds in L. □

In [3] we showed that there are precisely nine non-isomorphic subdirectly irreducible symmetric extended de Morgan algebras, all of which are simple. In the same paper, we also gave the Hasse diagrams of these subdirectly irreducible algebras. Here we use the same symbols A_i $(i = 1, 2, \ldots, 9)$ as in [3] to denote subdirectly irreducible $\mathbf{e_2M}$-algebras. In order to describe the implications linking the 19 basic axioms and to show that these axioms are indeed non-equivalent, we require the determination of axioms on subdirectly irreducible $\mathbf{e_2M}$-algebras. Here we list them in the table below.

algebra L	dual space	satisfies	not satisfies
A_1	$x = g(x) = h(x)$	(0)	—
A_2	$x = gh(x)$ $g(x) = h(x)$	(4)=(5)	(123)
A_3	$x = g(x)$ $h(x) = gh(x)$	(1)=(2)	(345)
A_4	$x = h(x)$ $g(x) = gh(x)$	(3)	(1245)
A_5	$x = h(x)$ $g(x) = gh(x)$	(3), (12), (15)=(24), (45)	(14)=(25)
A_6	x $h(x)$ $g(x)$ $gh(x)$	(12)	(1345)=(2345)
A_7	x $h(x)$ $gh(x)$ $g(x)$	(45)	(1234)=(1235)
A_8	x $h(x)$ $g(x)$ $gh(x)$	(12), (15)=(24), (45)	(134)=(235)
A_9	x $g(x)$ $h(x)$ $gh(x)$	—	(12345)

Table 1

By the above table we have the following observations:

(i) A_2 satisfies (4) but not (123). This shows that (4) $\succ$ (0), (14) $\succ$ (1), (34) $\succ$ (3), (124) $\succ$ (12), (134) $\succ$ (13) and (1234) $\succ$ (123).

(ii) A_3 satisfies (1) but not (345). Hence it shows that (1) $\succ$ (0), (13) $\succ$ (3), (14) $\succ$ (4), (134) $\succ$ (34), (145) $\succ$ (45) and (1345) $\succ$ (345).

(iii) A_4 satisfies (3) but not (1245). This shows that (3) $\succ$ (0), (13) $\succ$ (1), (34) $\succ$ (4), (123) $\succ$ (12), (134) $\succ$ (14), (135) $\succ$ (15), (345) $\succ$ (45), (1234) $\succ$ (124), (1345) $\succ$ (145) and (12345) $\succ$ (1245).

(iv) A_5 satisfies (3), (12), (15) and (45) but not (14). Hence this shows that (3) $\succ$ (0), (12) $\succ$ (1), (13) $\succ$ (1), (15) $\succ$ (14), (45) $\succ$ (4) and (134) $\succ$ (14).

(v) A_6 satisfies (12) but not (2345). This shows that (12) $\succ$ (1), (123) $\succ$ (13), (124) $\succ$ (15), (1234) $\succ$ (135), (1245) $\succ$ (145) and (12345) $\succ$ (1345).

(vi) A_7 satisfies (45) but not (1234). This shows that (45) $\succ$ (4), (145) $\succ$ (15), (345) $\succ$ (34), (1245) $\succ$ (124), (1345) $\succ$ (135) and (12345) $\succ$ (1234).

(vii) A_8 satisfies (12), (15) and (45) but not (235). This shows that $(12) \succ (1)$, $(15) = (24) \succ (14)$, $(45) \succ (4)$, $(123) \succ (13)$, $(135) \succ (134)$ and $(345) \succ (34)$.

Using the above observations and Theorems 4 and 5, we now can draw the poset of all non-equivalent axioms in $\mathbf{e_2M}$ as in the following Hasse diagram:

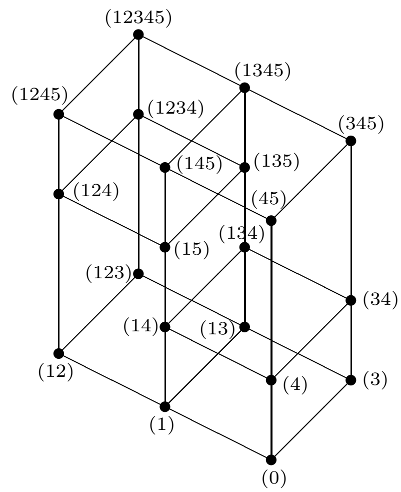

Figure 1

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