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On Odd Arithmetic Graphs

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Abstract The following results are obtained: (1) The graph $C_m^n \cdot P_t$ is odd arithmetic when (i) $m \equiv 0 \pmod 2$ and t=m or $m+1$; (ii) $m \equiv 1 \pmod 2$ and t=$m+1$. (2) The graph C_{2m}^n is odd arithmetic when (i) m=2,4 and n is any positive integer; (ii) m=3 and n is even. (3) The graph C_m^n is odd arithmetic when m=4n and t=2. (4) P_{m+1}^n is odd arithmetic when (i) n is odd; (ii) $m \leq 3$ and n is any positive integer. (5) Windmill graph K_n^t is odd arithmetic if and only if n=2. (6) Cycle C_n is odd arithmetic if and only if $n \equiv 0 \pmod 4$. (7) For any positive integer n and any positive integer m, $K_{m,n}$ is odd arithmetic.

Keywords odd arithmetic graph; complete graph; cycle; graph $C_m^n \cdot P_t$.

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1. Introduction

The theory of graph labeling has attracted many mathematicians mainly because of its aesthetic aspect, as well as its wide range of applications in such areas as radar pulse codes, x-ray crystallography, circuit design, missile guidance, radio astronomy, sonar ranging, and broadcast frequency assignments[6]. Acharya and Hegde[3] introduced the conception of (k,d)-arithmetic graph, and showed the following: $K_{m,n,1}$ is $(d+2r,d)$-arithmetic; C_{4t+1} is $(2dt+2r,d)$-arithmetic; C_{4t+2} is not (k,d)-arithmetic for any values of k and d; C_{4t+3} is $((2t+1)d+2r,d)$-arithmetic; while W_{4t+2} is $(2dt+2r,d)$-arithmetic; and W_{4t} is $((2t+1)d+2r,d)$-arithmetic. They obtained a number of necessary conditions for various kinds of graphs to have a (k,d)-arithmetic labeling. Hegde and Shetty[1] discussed the generalized web $W(t;n)$ to the problem. Shee and Ho[2] have investigated the cordiality of the one-point union of n copies of various graphs. Among them are the one-point union of n copies of C_m for C_m^n and the one-point union of n copies of K_m for K_m^n. Our goal in the paper is to prove that the graphs C_m^n, K_m^n and $C_m^n \cdot P_t$ etc. are odd arithmetic.

Let Z be the ring of integers and $a,b \in Z$. It will be convenient to use the following notations. $[a,b]=\{x \mid x \in Z,\ a \leq x \leq b\}$, $[a,b]_k=\{x \in Z \mid a \leq x \leq b, x \equiv a \pmod k\}$, $\lfloor x \rfloor = \max\{y \mid y \leq x, y \in Z\}$ for any real number x, and $f(S)=\{f(x) | \ x \in S\}$ where S is a set and f is a function.

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A graph that has order p and size q is called a (p,q)-graph. Let $G=(V,E)$ be a finite simple (p,q)-graph, D be a non-negative integer set, and let k and d be positive integers. A labeling f from V to D is said to be (k,d)-arithmetic if the vertex labels are distinct non-negative integers and the edge labels induced by $f^*(xy)=f(x)+f(y)$ for each edge xy are $k, k+d, k+2d, \ldots, k+(q-1)d$. The G is said to be a (k,d)-arithmetic graph. The case where $k=1$, $d=2$ and $D=[0,2q-1]$ is called odd arithmetic labeling. The case where $d=1$ and $D=[0,q-1]$ is called sequential labeling. The case where $k=1$, $d=1$ and $D=[0,q]$ is called strongly harmonious labeling. It is easy to obtain the following results.

Lemma 1.1 *If f is an odd arithmetic labeling of the (p,q)-graph G, then*

(1) Odd arithmetic graphs, sequential graphs and strongly harmonious graphs are (k,d)-arithmetic graphs,

(2) The maximal label of all vertices in an odd arithmetic graph G is at most $2q$-$\delta(G)$, where $\delta(G)$ is the minimum degree of the vertices of G,

(3) Each x in the set $\{0,\ 1\}$ has inverse image in an odd arithmetic graph, and the two inverse images are adjacent.

Lemma 1.2 *(1) If G is an odd arithmetic graph, then G is a bipartite graph.*

(2) Let $(d_1, d_2, \ldots, d_p)$ be a degree sequence of (p,q)-graph G. If the graph G is odd arithmetic, then the equation

$$\sum_{i=1}^{p} d_i x_i = q^2 \tag{I}$$

has non-negative integer solutions $(x_1, x_2, \ldots, x_p)$ satisfying $x_i \neq x_j$ if $i \neq j$ and $x_i \leq 2q-\delta(G)$ for $i \in [1,p]$.

Proof Part (1). First, we show that G has no odd-cycle if G is an odd arithmetic graph with odd arithmetic labelling f. Suppose that $C=v_1v_2\cdots v_{2n-1}v_{2n}v_{2n+1}v_1$ is an odd-cycle of G, and without loss of generality, one may suppose $f(v_1)$ is an odd number. Since $f^*(uv)$ is odd for any $uv \in E(G)$ and $v_1v_2 \in E(G)$, then the $f(v_2)$ is even. In general, $f(v_{2i-1})$ is odd and $f(v_{2i})$ is even. Since $v_1v_{2n+1} \in E(G)$ and $f(v_{2n+1})$ is odd, $f(v_1)$ is even. This is a contradiction. It is well known that a graph is bipartite if and only if it has no odd-cycle. Therefore, the graph G is a bipartite graph.

Part (2). Let f be an odd arithmetic labelling of G. For every $v_i \in V(G)$ $(i \in [1,p])$, let $deg(v_i)=d_i$ and $f(v_i)=x_i$. By adjacent relation, we have

$$\sum_{uv\in E(G)} f^*(uv) = \sum_{i=1}^{p} d_i x_i. \tag{i}$$

On the other hand

$$\sum_{uv\in E(G)} f^*(uv) = 1+3+\cdots+(2q-1) = q^2. \tag{ii}$$

It follows immediately from (i) and (ii) that the equation (I) holds. Since f is an injection, there exist solutions $(x_1, x_2, \ldots, x_p)$ satisfying $x_i \neq x_j$ if $i \neq j$ and $0 \leq x_i \leq 2q - \delta(G)$ for $i \in [1, p]$. $\square$

2. Preliminary results

Given n-dimensional vectors

$$A_1 = \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{n,1} \end{bmatrix}, \ A_2 = \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{n,2} \end{bmatrix}, \ \ldots, \ A_m = \begin{bmatrix} a_{1,m} \\ a_{2,m} \\ \vdots \\ a_{n,m} \end{bmatrix}. \qquad (*)$$

A vector A_k is called an EPD vector if the elements of A_k are pairwise distinct. A vectors group $A_1, A_2, \ldots, A_m$ (shortly $\{A_k\}$) is called an EPDVG if the elements of the $A_1, A_2, \ldots, A_m$ are pairwise distinct, i.e., $|\{a_{i,j} \mid j \in [1, m], i \in [1, n]\}|=mn$, and the matrix $[A_1, A_2, \ldots, A_m]$ is called EPD matrix. Denote by $\langle A_k \rangle$ the set of all elements of A_k. A_k is called an arithmetic vector if the elements $a_{1,k}, a_{2,k}, \ldots, a_{n,k}$ of $\langle A_k \rangle$ form an arithmetic progression, and the common difference of the arithmetic progression is called the common difference of the A_k. $\{A_1, A_2, \ldots, A_m\}$ is called a consecutive vector group if $a_{1,1}, a_{2,1}, \ldots, a_{n,1}, a_{1,2}, a_{2,2}, \ldots, a_{n,2}, \ldots, a_{1,m}, \ldots, a_{n,m}$ is an arithmetic progression, and A_{k+1} is called the successor of A_k.

Lemma 2.1 *If two vectors A_k with common difference d_1 and A_{k+1} with common difference d_2 in (*) are arithmetic vectors, and $d_1 + d_2 \neq 0$, then their sum $A_k + A_{k+1}$ is an EPD vector.*

Proof Let $a_{j,k}=a + d_1(j-1)$, $j \in [1, n]$, $a_{j,k+1}=b + d_2(j-1)$, $j \in [1, n]$. Then $a_{j,k} + a_{j,k+1}=a + b + (d_1 + d_2)(j-1)$, $j \in [1, n]$. Since $d_1 + d_2 \neq 0$, $A_k + A_{k+1}$ is an EPD vector with first term $a + b$ and last term $a + b + (d_1 + d_2)(n-1)$. $\square$

Lemma 2.2 *Let (*) be an arithmetic vectors group. If the common difference of A_k is d_1 if k is even, and the common difference of A_k is d_2 if k is odd, then $\{A_k + A_{k+1}\}$ is a consecutive vector group when $a_{1,j+2}=a_{1,j} + n(d_1 + d_2)$.*

Proof Suppose that $a_{j,k}=a_{1,k} + d_1(j-1)$, $j \in [1, n]$, and $a_{j,k+1}=a_{1,k+1} + d_2(j-1)$, $j \in [1, n]$ for any $k \in [1, m-1]$. If we want $A_{k+1} + A_{k+2}$ to become a successor of $A_k + A_{k+1}$, then $a_{1,k} + a_{1,k+1} + n(d_1 + d_2)=a_{1,k+2} + a_{1,k+1}$ must be satisfied. Therefore, $a_{1,k+2}=a_{1,k} + n(d_1 + d_2)$. $\square$

Lemma 2.3 *Let (*) be an arithmetic vectors group. If d_1 is the common difference of A_{2k}, and d_2 is the common difference of A_{2k-1} satisfying $d_1 + d_2 \neq 0$, then $[A_1 + A_2, A_2 + A_3, \ldots, A_{m-1} + A_m]$ is an EPD matrix when $a_{1,j+2}=a_{1,j} + n(d_1 + d_2)$.*

Proof To solve the recursive equation $a_{1,j+2}=a_{1,j} + n(d_1 + d_2)$, we obtain that

$a_{1,2t-1}=a_{1,1} + (t-1)n(d_1 + d_2)$, and $a_{1,2t}=a_{1,2} + (t-1)n(d_1 + d_2)$.

Since each $\{A_k + A_{k+1}\}$ is strictly monotone vectors group, the matrix $[A_1 + A_2, A_2 +$

$A_3, \ldots, A_{m-1} + A_m]$ is an EPD matrix from Lemma 2.2. □

Lemma 2.4 *Let n be odd, and A_{2t} and A_{2t-1} be an arithmetic vector with common difference -4 and 2, respectively. If $a_{1,2t}$=$2n(t+1)-2$ and $a_{1,2t-1}$=$2n(t-1)+1$, then both $[A_1, A_2, \ldots, A_m]$ and $[A_1, A_1 + A_2, A_2 + A_3, \ldots, A_{m-1} + A_m]$ are EPD matrices.*

Proof By the hypotheses, we have $a_{1,2t} \equiv 0 \pmod 4$ if t is even, and $a_{1,2t} \equiv 2 \pmod 4$ if t is odd. Since the common difference of the vector A_{2t} is -4, $\langle A_{2t}\rangle \bigcap \langle A_{2s}\rangle = \phi$ when s and t have opposite parity.

When t=$2k-1$, the greatest element of $\langle A_{2t}\rangle$ is $4nk-2$ and the least element is $4nk-4n+2$. When t=$2k+1$, the greatest element of $\langle A_{2t}\rangle$ is $4nk+4n-2$ and the least element is $4nk+2$. Therefore $\{A_{2t}\}$ is an EPDVG when t is odd. In the same way, we obtain that $\{A_{2t}\}$ is also an EPDVG when t is even. Thus $\{A_{2t}\}$ is EPDVG.

When $a_{1,2t-1} = 2n(t-1)+1$, $\{A_{2t-1}\}$ is a consecutive vector group because the vector A_{2t-1} is an arithmetic vector with common difference 2. Also, each $a_{i,2t} \in \langle A_{2t}\rangle$ is even and each $a_{i,2t-1} \in \langle A_{2t-1}\rangle$ is odd, then the matrix $[A_1, A_2, \ldots, A_m]$ is an EPD matrix. Furthermore, we can also get $[A_1 + A_2, A_2 + A_3, \ldots, A_{m-1} + A_m]$ is an EPD matrix. □

3. Main results

Let the one-point union of n copies of graph G be denoted by G^n. Then the common vertex is called the center of G^n, denoted by x_0. We denote by K_n the complete graph with n vertices, P_n the path with n vertices and C_n the cycle with n vertices.

Theorem 3.1 *If regardless of the order of pendant vertices, then the star graph $K_{1,n}$ exactly has two odd arithmetic labelings.*

Proof It is easy to see that the center of $K_{1,n}$ is only labelled 0 or 1. When the center of $K_{1,n}$ is labelled 0, then the pendant vertices are labelled $1, 3, \ldots, 2n-1$, successively; When the center of $K_{1,n}$ is labelled 1, then the pendant vertices are labelled $0, 2, \ldots, 2n-2$ successively. □

Theorem 3.2 *When n is odd, and $m \leq 3$ and n is a positive integer, the graph P_{m+1}^n is odd arithmetic.*

Proof Let x_0 be the center of P_{m+1}^n. The vertices of k'th path on the P_{m+1}^n from center to outside are $x_{k1}, x_{k2}, \ldots, x_{km}$ successively, and a_{kj}, $k \in [1, n], j \in [1, m]$ are the elements in $\{A_k\}$ satisfying the conditions of Lemma 2.4.

Put $f(x_{kj})$=$a_{kj}, k \in [1, n], j \in [1, m]$, i.e., for any $k \in [1, n]$,

$$f(x_{kj}) = \begin{cases} n(j-1) + 1 + 2(k-1), & j \in [1, m]_2 \\ n(j+2) - 2 - 4(k-1), & j \in [2, m]_2 \end{cases}$$

and $f(x_0)$=0. Then f is an odd arithmetic labeling of P_{m+1}^n from Lemma 2.4.

When $m \leq 3$ and n is a positive integer, applying Lemmas 2.2 and 2.3 immediately obtains that the graph P_{m+1}^n is odd arithmetic. $\square$

Theorem 3.3 *Let n and t be positive integers. Then*

(1) The windmill graph K_n^t is odd arithmetic if and only if $n = 2$;

(2) All complete bipartite graphs are odd arithmetic.

Proof From Lemma 1.2 and Theorem 3.1 it follows the conclusion (1). In the following we prove conclusion (2). Let (X, Y) be the bipartition of the complete bipartite graph $K_{m,n}$, where $X{=}\{x_j \mid j \in [1, m]\}$ and $Y{=}\{y_j \mid \ \in [1, n]\}$. Define $f(x_j){=}2j-1$, $j \in [1, m]$, $f(y_j){=}2m(j-1)$, $j \in [1, n]$. Then f is an odd arithmetic labelling of $K_{m,n}$. $\square$

Theorem 3.4 *If m is odd and t is a positive integer, where $m \equiv 2$ (mod4) and t is odd, then C_m^t is not odd arithmetic.*

Proof From Lemma 1.2, one can know that C_m^t is not odd arithmetic when m is odd and t is a positive integer.

We show the case $m \equiv 2$ (mod 4) and t is odd by contradiction. Suppose that f is odd arithmetic labeling of C_m^t and $m{=}4n+2$. Using Lemma 1.2, we have

$\sum_{v\in V} d(v)f(v) = 2\sum_{v\in V\setminus\{x_0\}} f(v) + 2tf(x_0){=}[t(4n+2)]^2$.

Thus $\sum_{v\in V\setminus\{x_0\}} f(v) + tf(x_0) = 2[t(2n+1)]^2$.

i) If $f(x_0)$ is even, then the sum of labels of all vertices except x_0 on each C_m of C_m^t is an odd number. Since t is odd, the first term on left side of the equality is odd and the second term is even. But the right side of the equality is even, this is a contradiction.

ii) If $f(x_0)$ is odd, then the sum of labels of all vertices except x_0 on each C_m of C_m^t is an even number. Thus the first term on left side of the equality is even. Since t is odd, the second term on left side of the equality is odd. But the right side of the equality is even, which also leads to a contradiction. $\square$

Theorem 3.5 *(1) The cycle C_m is odd arithmetic if and only if $m \equiv 0$ (mod4).*

(2) When $m = 2, 4$ and n is a positive integer, where $m = 3$ and n is even, C_{2m}^n is odd arithmetic.

(3) When $m = 4n$ and $t{=}2$, C_m^t is odd arithmetic.

Proof Part (1). If C_m is odd arithmetic, then C_m is a bipartite graph by Lemma 1.2. This implies that m is an even number. From Theorem 3.4 one can obtain C_m is not odd arithmetic if $m \equiv 2$ (mod 4). When $m \equiv 0$ (mod 4), define

$$f(x_{2i-1}) = 2(i-1), i \in [1, m/2];\ f(x_{2i}) = 2i-1, i \in [1, m/4];$$

$$f(x_{2i}) = 2i+1, i \in [m/4+1, m/2].$$

It is easy to verify that f is an odd arithmetic labeling of C_m.

Part (2). Let the vertices (except center x_0) of t'th C_{2m} on C_{2m}^n be $a_{2t-1,1}, a_{2t-1,2}, \ldots, a_{2t-1,m-1}$, $a_{t,m}, a_{2t,m-1}, a_{2t,m-2}, \ldots, a_{2t,1}$, successively. We define $f(x_0){=}0$, and write the labels of all vertices (except center x_0) of each C_{2m} on C_{2m}^n for vector.

When $m = 2$ and n is a positive integer, let arithmetic vector $A_1 = [a_{1,1}, a_{2,1}, \ldots, a_{2n,1}]^{\mathrm{T}}$ with common difference 2 and $a_{1,1} = 1$, and let vector $A_2 = [a_{1,2}, a_{1,2}, a_{2,2}, a_{2,2}, \ldots, a_{n,2}, a_{n,2}]^{\mathrm{T}}$, where $a_{k,2} = 8(n-k)+4$, $k \in [1, n]$. Then $[A_1, A_1 + A_2]$ is an EPD matrix.

When $m = 4$ and n is a positive integer, let the vectors A_1, A_2, A_3 be the same as the vectors in Lemma 2.4. Define $A_4 = [a_{1,4}, a_{1,4}, a_{2,4}, a_{2,4}, \ldots, a_{n,4}, a_{n,4}]^{\mathrm{T}}$, where $a_{k,4} = 12n + 4 - 8k$, $k \in [1, n]$. Then $A_4^*{=}[a_{1,4}, a_{2,4}, \ldots, a_{n,4}]^{\mathrm{T}}$ and A_1, A_2, A_3 are EPD vector group. Therefore, $[A_1, A_1 + A_2, A_2 + A_3, A_3 + A_4]$ is an EPD matrix.

When $m = 3$ and n is even, let the vectors A_1 and A_2 be the same as the vectors in Lemma 2.4. Define $A_3 = [a_{1,3}, a_{1,3}, a_{2,3}, a_{2,3}, \ldots, a_{n,3}, a_{n,3}]^{\mathrm{T}}$, where $a_{k,3} = 4(t{+}k){-}2{+}(-1)^k$, $k \in [1, n]$. Then $A_3^* = [a_{1,3}, a_{2,3}, \ldots, a_{n,3}]^{\mathrm{T}}$ and A_1, A_2 are EPD vector group. Therefore, $[A_1, A_1 + A_2, A_2 + A_3]$ is an EPD matrix. Combining the results, we obtain that the result (2) is true.

Part (3). Let the vertices of one C_{4n} be $x_1, x_2, \ldots, x_{4n}$, and let the vertices of another C_{4n} be $x_{4n+1}, x_{4n+2}, \ldots, x_{8n}$. The vertex x_{4n} is identified with x_{8n}, denoted by x_0. Define the function f as follows: $f(x_0) = 0$, $f(x_{2k-1}) = 2(k-1)$, $k \in [1, 4n]$,

$$f(x_{2k}) = \begin{cases} 2k-1, & k \in [1, n], \\ 2k+1, & k \in [n+1, 3n-1], \\ 2k+3, & k \in [3n, 4n-1]. \end{cases}$$

Then f is an odd arithmetic labeling of the C_{4n}^2. □

Theorem 3.6 *The graph $C_m^n \cdot P_t$ is obtained by identifying the center of C_m^n with the end vertex of P_t. Suppose n is a positive integer. Then $C_{2m}^n \cdot P_m$ and $C_{2m}^n \cdot P_{m+1}$ are odd arithmetic if $m \equiv 0 \pmod 2$, and $C_{2m}^n \cdot P_{m+1}$ is odd arithmetic if $m \equiv 1 \pmod 2$.*

Proof Let $f(x_0){=}0$, the vertices of t'th C_{2m} on the C_{2m}^n be

$$a_{2t-1,1}, a_{2t-1,2}, \ldots, a_{2t-1,m-1}, a_{t,m}, a_{2t,m-1}, a_{2t,m-2}, \ldots, a_{2t,1},$$

and the vertices of P_s be $x_0, a_{2n+1,1}, a_{2n+1,2}, \ldots, a_{2n+1,s-2}, a_{n+1,s-1}$.

(1) When $m \equiv 0 \pmod 2$, let the vectors $A_1, A_2, \ldots, A_{m-1}$ be the same as the vectors in Lemma 2.4. For the graph $C_{2m}^n \cdot P_m$, let $A_m{=}[a_{1,m}, a_{1,m}, a_{2,m}, a_{2,m}, \ldots, a_{n,m}, a_{n,m}, -]^{\mathrm{T}}$, where $a_{k,m}{=}(2n+1)m + 4n - 4 - 8(k-1)$, $k \in [1, n]$ and "-" denote no elements in the cell. Then $A_m^* {=}[a_{1,m}, a_{2,m}, \ldots, a_{n,m}]^{\mathrm{T}}$ and $A_1, A_2, \ldots, A_{m-1}$ are EPD vectors. Thus $[A_1, A_1 + A_2, A_2 + A_3, \ldots, A_{m-1} + A_m]$ is an EPD matrix.

Since

$$\begin{aligned} &\max\langle A_{m-1} + A_{m-2}\rangle = a_{1,m-1} + a_{1,m-2} = (2n+1)(2m-2) - 1, \\ &\min\langle A_{m-1} + A_m\rangle = a_{n,m} + a_{2n-1,m-1} = [(2n+1)m - 4n + 4] + [(2n+1)(m-2) + 4n - 3] \\ &\quad = (2n+1)(2m-2) + 1, \end{aligned}$$

and

$$\begin{aligned} \max\langle A_{m-1} + A_m\rangle &= a_{1,m} + a_{2,m-1} = [(2n+1)m + 4n - 4] + [(2n+1)(m-2) + 3] \\ &= (2n+1)(2m-2) + 4n - 1, \end{aligned}$$

$$\langle A_{m-1}+A_m\rangle=[(2n+1)(2m-2)+1,(2n+1)(2m-2)+4n-1]_2.$$

Let B_m denote the vector generated by arranging the elements of $A_{m-1}+A_m$ from small to large. Then $A_1, A_1+A_2, A_2+A_3, \ldots, A_{m-2}+A_{m-1}, B_m$ form a consecutive vector group.

For the graph $C_{2m}^n \cdot P_{m+1}$, let $A_m=[a_{1,m}, a_{1,m}, a_{2,m}, a_{2,m}, \ldots, a_{n,m}, a_{n,m}, a_{n+1,m}]^{\mathrm{T}}$, where $a_{k,m}=(2n+1)m+4n+4-8k$, $k\in[1,n]$, $a_{n+1,m}=(2n+1)m$.

Then $A_m^*=[a_{1,m}, a_{2,m}, \ldots, a_{n,m}, a_{n+1,m}]^{\mathrm{T}}$ and $A_1, A_2, \ldots, A_{m-1}$ are EPD vectors. Thus $[A_1, A_1+A_2, A_2+A_3, \ldots, A_{m-1}+A_m]$ is an EPD matrix.

$\langle A_{m-1}+A_m\rangle=[(2n+1)(2m-2)+1,(2n+1)(2m-2)+4n+1]_2$. Let B_m denote the vector generated by arranging the elements of $A_{m-1}+A_m$ from small to large. Then $A_1, A_1+A_2, A_2+A_3, \ldots, A_{m-2}+A_{m-1}, B_m$ form a consecutive vector group.

(2) When $m\equiv 1 \pmod 2$, let the vertices of P_{m+1} be $x_0, a_{2n+1,1}, a_{2n+1,2}, \ldots, a_{2n+1,m-1}, a_{n+1,m}$ successively, and the vectors $A_1, A_2, \ldots, A_{m-1}$ be the same as in Lemma 2.4. Define

$$A_m=[a_{1,m}, a_{1,m}, a_{2,m}, a_{2,m}, \ldots, a_{n,m}, a_{n,m}, a_{n+1,m}]^{\mathrm{T}},$$

where

$$a_{k,m}=(2n+1)(m-3)+4n+4k+(-1)^k, \quad k\in[1,n].$$

Then $A_m^*=[a_{1,m}, a_{2,m}, \ldots, a_{n,m}, a_{n+1,m}]^{\mathrm{T}}$ and $A_1, A_2, \ldots, A_{m-1}$ are EPD vectors. Thus $[A_1, A_1+A_2, A_2+A_3, \ldots, A_{m-1}+A_m]$ is an EPD matrix. Let B_m denote the vector yielded by arranging the elements of $A_{m-1}+A_m$ from small to large. Then $A_1, A_1+A_2, A_2+A_3, \ldots, A_{m-2}+A_{m-1}, B_m$ form a consecutive vector group. □

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