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Positive Solutions for Nonlinear Second-Order Boundary Value Problem of Delay Differential Equation

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Abstract In this paper, we study the nonlinear second-order boundary value problem of delay differential equation. Without the assumption of the nonnegativity of f, we still obtain the existence of the positive solution.

Keywords second-order boundary value problem; delay differential equation; positive solutions; fixed point index.

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1. Introduction and lemmas

Presently, many authors discussed the existence of the solutions for delay differential equations and functional differential equations[1,2]. However, the nonlinearity of the functions in the relational references is nonnegative. In this paper, we obtain the existence of positive solutions for a class of delay differential equations with the assumption that the nonlinearity is bounded below and not always nonnegative, which generalizes the corresponding results in [1], [2].

In this paper, we consider the following second-order boundary value problem of delay differential equation

$$\begin{cases} (p(t)u'(t))' + f(t, u(t-\tau)) = 0, \quad 0 < t < 1, \\ u(t) = 0, \quad -\tau \le t \le 0, \quad u(1) = 0, \end{cases} \tag{1}$$

where $\tau > 0$, $p \in C[0,1]$, $p(t) > 0$, $t \in [0,1]$. Throughout this paper, we assume that $f \in C([0,1] \times [0,\infty), (-\infty,\infty))$, and $f(t,u)$ is bounded below, i.e., there exists a real number $M > 0$ such that $f(t,u) + M \ge 0$, $\forall t \in [0,1], u \in [0,\infty)$. A function $u(t)$ is called a positive solution of (1) if $u(t)$ satisfies (1) and $u \in C([-\tau,1], [0,\infty))$, $u(t) > 0$, $t \in (0,1)$.

Let

$$G(t,s) = \begin{cases} \dfrac{1}{\rho}\omega(t)(\omega(1) - \omega(s)), \quad 0 \le t \le s \le 1, \\ \dfrac{1}{\rho}\omega(s)(\omega(1) - \omega(t)), \quad 0 \le s \le t \le 1, \end{cases} \tag{2}$$

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where $\rho = \omega(1), \omega(t) = \int_0^t \frac{1}{p(r)}\mathrm{d}r$.

Lemma 1[3,4] *The function given by (2) has the following property:*

$$G(s,s) \geq G(t,s) \geq \sigma(t)G(s,s), (t,s) \in [0,1]\times[0,1], \int_0^1 G(t,s)\mathrm{d}s \leq \gamma\sigma(t),$$

where $\sigma(t) = \min\{\frac{\omega(t)}{\omega(1)}, \frac{\omega(1)-\omega(t)}{\omega(1)}\}, \gamma = \omega(1)$.

Lemma 2 *(i) If $u_*(t)$ is a positive solution of (1), then $u_*(t)+w(t)$ is a positive solution of the following delay differential equation*

$$\begin{cases} (p(t)u'(t))' + F(t, u(t-\tau) - w(t-\tau)) = 0, \quad 0 < t < 1, \\ u(t) = 0, \quad -\tau \leq t \leq 0, \quad u(1) = 0, \end{cases} \tag{3}$$

where

$$F(t,u) = \begin{cases} \widetilde{f}(t,u), \quad t \in [0,1], u \geq 0, \\ \widetilde{f}(t,0), \quad t \in [0,1], u < 0, \end{cases}$$

the function $\widetilde{f}(t,u) = f(t,u) + M, \widetilde{f} : [0,1]\times[0,\infty) \to [0,\infty)$ *is continuous,*

$$w(t) = \begin{cases} 0, & -\tau \leq t \leq 0, \\ M\displaystyle\int_0^1 G(t,s)\mathrm{d}s, 0 \leq t \leq 1. \end{cases}$$

(ii) If $u(t)$ is a solution of (3) and $u(t) \geq w(t), t \in [-\tau, 1]$, then $u_*(t) = u(t) - w(t)$ is a positive solution of (1).

Proof Suppose that $u_*(t)$ is a positive solution of (1). Then we have

$$\begin{cases} (p(t)u_*'(t))' + f(t, u_*(t-\tau)) = 0, \quad 0 < t < 1, \\ u_*(t) = 0, \quad -\tau \leq t \leq 0, \quad u_*(1) = 0. \end{cases}$$

It follows from $w(t) = 0, -\tau \leq t \leq 0, w(1) = 0$ that $u_*(t)+w(t) = 0, -\tau \leq t \leq 0, u_*(1)+w(1) = 0$ and

$$\begin{aligned}(p(t)(u_*(t)+w(t))')' + F(t, u_*(t-\tau)) &= (p(t)u_*'(t))' + (p(t)w'(t))' + f(t, u_*(t-\tau)) + M \\ &= (p(t)w'(t))' + M = 0.\end{aligned}$$

i.e., $u_*(t)+w(t)$ satisfies (3). Hence (i) holds true. Similarly, it is easy to prove that (ii) holds true. The proof is completed. □

By (ii) of Lemma 2, in order to obtain the positive solution of (1), one only needs to find a solution $u(t)$ of (3) satisfying $u(t) \geq w(t), t \in [-\tau, 1]$. On the other hand, if $u(t)$ is a solution of (3), then $u(t)$ satisfies

$$u(t) = \begin{cases} 0, \quad -\tau \leq t \leq 0, \\ \displaystyle\int_0^1 G(t,s)F(s, u(s-\tau) - w(s-\tau))\mathrm{d}s, \quad 0 \leq t \leq 1. \end{cases} \tag{4}$$

We define an operator A as follows:

$$Au(t) = \begin{cases} 0, & -\tau \le t \le 0, \\ \displaystyle\int_0^1 G(t,s)F(s, u(s-\tau) - w(s-\tau))\mathrm{d}s, & 0 \le t \le 1. \end{cases} \tag{5}$$

It is easy to check that a solution $u(t)$ of (3) is equivalent to the fixed point $u(t)$ of the operator of A. Let $X = \{u \in C[-\tau, 1] | u(t) = 0, t \in [-\tau, 0], u(1) = 0\}$. Then X is a Banach space with the norm $\|u\| = \sup\{|u(t)| | t \in [-\tau, 1]\}$. For any $u \in X$, $\|u\| = \|u\|_{[0,1]} = \sup\{|u(t)| | t \in [0,1]\}$. Let $P = \{u \in X | u(t) \ge \sigma(t)\|u\|, t \in [0,1]\}$, where $\sigma(t)$ is given in Lemma 1. Then P is a cone of X. By standard argument, it is easy to prove that the operator $A : P \to P$ is completely continuous.

2. Main results

Let λ_1 be the first eigenvalue of the linear integral operator $Tu(t) = \int_0^1 G(t,s)u(s)\mathrm{d}s$ and $u_1(t)$ be the minimum positive eigenfunction corresponding to λ_1. Then $u_1 \in C^2[0,1]$. Denote $f_\infty = \liminf_{u\to\infty} \min_{t\in[0,1]} \frac{f(t,u)}{u}$, $f_\infty = \limsup_{u\to\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u}$ and

$$M_\tau = \frac{\|u_1\|_{[0,1]} + (1-\tau)\|u_1'\|_{[0,1]}}{\int_0^{1-\tau} \sigma(s)u_1(s+\tau)\mathrm{d}s}, \quad m_\tau = \frac{(1-\tau)\|u_1'\|_{[0,1]}}{\int_0^{1-\tau} \sigma(s)u_1(s+\tau)\mathrm{d}s}.$$

In this section, we assume that:

(H_1) $f^\infty < \lambda_1(1 - \tau m_\tau)$, there exists $Q : [0,1] \to (-\infty, \infty), \theta \in (0, \frac{1}{2})$ and some constant $t_0 \in [0,1]$ such that

$$f(t,u) + M \ge Q(t), t \in [\theta, 1-\theta], u \in [0, \gamma(M+1)], \tag{6}$$

$$\int_\theta^{1-\theta} G(t_0, s)Q(s)\mathrm{d}s \ge \gamma(M+1). \tag{7}$$

(H_2) $f_\infty > \lambda_1(1 + \tau M_\tau)$, there exists $Q : [0,1] \to (-\infty, \infty)$ such that

$$f(t,u) + M \le Q(t), t \in [0,1], u \in [0, \gamma(M+1)], \tag{8}$$

$$\int_0^1 G(s,s)Q(s)\mathrm{d}s < \gamma(M+1). \tag{9}$$

Theorem 1 *Suppose that (H_1) or (H_2) holds. Then the second-order boundary value problem of delay differential equation (1) has at least one positive solution.*

Proof (i) Suppose (H_1) is satisfied. By Lemma 2, it suffices to find a fixed point $u(t)$ of A satisfying $u(t) \ge w(t), t \in [-\tau, 1]$. By Lemma 1, for any $u \in P$ and $t \in [0,1]$, we have

$$u(t) - w(t) \ge u(t) - \gamma M\sigma(t) \ge u(t) - \gamma M \frac{u(t)}{\|u\|} = (1 - \frac{\gamma M}{\|u\|})u(t). \tag{10}$$

By $f^\infty < \lambda_1(1 - \tau m_\tau)$, we get

$$\lim_{u\to\infty} \sup \max_{t\in[0,1]} \frac{f(t,u) + M}{u} < \lambda_1(1 - \tau m_\tau).$$

Moreover, there exists $\varepsilon > 0$ and $R_1 > 0$ such that $f(t,u)+M \le (\lambda_1(1-\tau m_\tau)-\varepsilon)u, u \in [R_1,\infty)$. Let $b_1 = \max\{|f(t,u)||t \in [0,1], u \in [0,R_1]\}$. By the continuity of f, we know that

$$f(t,u) \le (\lambda_1(1-\tau m_\tau)-\varepsilon)u + b_1, t \in [0,1], u \ge 0.$$

Choose

$$R > \max\{R_1, \gamma(M+1), \frac{b_1\int_0^1 u_1(s)\mathrm{d}s}{\varepsilon\int_0^{1-\tau}\sigma(s)u_1(s+\tau)\mathrm{d}s}\}.$$

Let $B_R = \{u \in P|\|u\| < R\}$. Then for any $u \in \partial B_R$ and $t \in [0,1]$, by (10), we have $u(t)-w(t) \ge 0, t \in [0,1]$. Noting that $u(t) = w(t) = 0, t \in [-\tau, 0], w(t) \ge 0, t \in [-\tau, 1]$, one easily obtains that

$$\begin{aligned} F(t,u(t-\tau)-w(t-\tau)) \le & (\lambda_1(1-\tau m_\tau)-\varepsilon)(u(t-\tau)-w(t-\tau)) + b_1 \\ \le & (\lambda_1(1-\tau m_\tau)-\varepsilon)u(t-\tau) + b_1, t \in [0,1], u \in \partial B_R. \end{aligned} \tag{11}$$

Next, we will prove that

$$Au \ne \mu u, \forall\ \mu \ge 1, u \in \partial B_R. \tag{12}$$

We may assume that A has no fixed point on ∂B_R, otherwise the proof is completed. If (12) is not satisfied, then there exists $\mu_0 > 1$ and $u_0 \in \partial B_R$ such that $Au_0 = \mu_0 u_0$ and

$$(p(t)u_0'(t))' + \frac{1}{\mu_0}F(t,u_0(t-\tau)-w(t-\tau)) = 0, \quad 0 < t < 1, \tag{13}$$

$$u_0(t) = 0,\ -\tau \le t \le 0, u_0(1) = 0. \tag{14}$$

By (11),(13) and (14), we have

$$\begin{aligned} \lambda_1\int_0^1 u_0(s)u_1(s)\mathrm{d}s &= \frac{1}{\mu_0}\int_0^1 u_1(s)F(s,u_0(s-\tau)-w(s-\tau))\mathrm{d}s \\ &\le \int_0^1 u_1(s)(\lambda_1(1-\tau m_\tau)-\varepsilon)u_0(s-\tau)\mathrm{d}s + b_1\int_0^1 u_1(s)\mathrm{d}s \\ &= (\lambda_1(1-\tau m_\tau)-\varepsilon)\int_\tau^1 u_1(s)u_0(s-\tau)\mathrm{d}s + b_1\int_0^1 u_1(s)\mathrm{d}s \\ &= (\lambda_1(1-\tau m_\tau)-\varepsilon)\int_0^{1-\tau} u_1(s+\tau)u_0(s)\mathrm{d}s + b_1\int_0^1 u_1(s)\mathrm{d}s \end{aligned}$$

and so

$$(\lambda_1\tau m_\tau+\varepsilon)\int_0^{1-\tau} u_1(s+\tau)u_0(s)\mathrm{d}s \le \lambda_1\tau(1-\tau)\|u_1'\|_{[0,1]}\|u_0\| + b_1\int_0^1 u_1(s)\mathrm{d}s. \tag{15}$$

On the other hand, by Lemma 1, we have

$$\int_0^{1-\tau} u_1(s+\tau)u_0(s)\mathrm{d}s \ge \|u_0\|\int_0^{1-\tau}\sigma(s)u_1(s+\tau)\mathrm{d}s. \tag{16}$$

By virtue of (15) and (16), we get

$$(\lambda_1\tau m_\tau+\varepsilon)\|u_0\|\int_0^{1-\tau}\sigma(s)u_1(s+\tau)\mathrm{d}s \le \lambda_1\tau(1-\tau)\|u_1'\|_{[0,1]}\|u_0\| + b_1\int_0^1 u_1(s)\mathrm{d}s.$$

By the definition of m_τ, we get

$$R = \|u_0\| \le \frac{b_1\int_0^1 u_1(s)\mathrm{d}s}{\varepsilon\int_0^{1-\tau}\sigma(s)u_1(s+\tau)\mathrm{d}s}.$$

Evidently, it is a contradiction to the choice of R. Hence (12) holds true. It follows from the fixed point index in [5], [6] that

$$i(A, B_R, P) = 1. \tag{17}$$

Let $B_1 = \{u \in P | \|u\| < \gamma(M+1)\}$. Then for any $u \in \partial B_1$, by (10), we have $\gamma(M+1) \geq u(t) \geq u(t) - w(t) \geq 0, t \in [0,1]$. It follows from (6) and (7) that

$$\begin{aligned} Au(t_0) &= \int_0^1 G(t_0, s) F(s, u(s-\tau) - w(s-\tau)) \\ &> \int_\theta^{1-\theta} G(t_0, s)(f(s, u(s-\tau) - w(s-\tau)) + M)\mathrm{d}s \\ &\geq \int_\theta^{1-\theta} G(t_0, s) Q(s)\mathrm{d}s \geq \gamma(M+1). \end{aligned}$$

Hence $\|Au\| > \|u\|, u \in \partial B_1$, and so by the fixed point index [5], [6], we have

$$i(A, B_1, P) = 0. \tag{18}$$

It follows from (17), (18) and the additivity of the fixed point index that A has at least a fixed point u satisfying $\|u\| > \gamma(M+1)$. The proof is completed.

(ii) Suppose (H_2) holds. By $f^\infty > \lambda_1(1+\tau M_\tau)$, there exists $\varepsilon > 0$ and $R_1 > 0$ such that $f(t,u) + M \geq (\lambda_1(1+\tau M_\tau) + \varepsilon)u, u \in [R_1, \infty)$. By the continuity of f, there exists $b_1 > 0$ such that

$$f(t,u) \geq (\lambda_1(1+\tau M_\tau) + \varepsilon)u - b_1,\ t \in [0,1], \quad u \geq 0.$$

Choose

$$R > \max\big\{R_1, \gamma(M+1), \frac{b_1 \int_0^1 u_1(s)\mathrm{d}s + (\lambda_1(1+\tau M_\tau) + \varepsilon)\int_0^{1-\tau} u_1(s+\tau)w(s)\mathrm{d}s}{\varepsilon \int_0^{1-\tau} \sigma(s) u_1(s+\tau)\mathrm{d}s}\big\}$$

and set $B_R = \{u \in P | \|u\| < R\}$. Then by (10) we have

$$F(t, u(t-\tau) - w(t-\tau)) \geq (\lambda_1(1+\tau M_\tau) + \varepsilon)(u(t-\tau) - w(t-\tau) - b_1, \quad t \in [0,1], u \in \partial B_R. \tag{19}$$

Next, we will prove that

$$u - Au \neq \mu u_1, \forall\ \mu \geq 0, u \in \partial B_R. \tag{20}$$

We may assume that A has no fixed point on ∂B_R, otherwise the proof is completed. If (20) does not hold true, then there exist $\mu_0 > 0$ and $u_0 \in \partial B_R$ such that $u_0 = Au_0 + \mu_0 u_1$ and

$$(p(t)u_0'(t))' + F(t, u_0(t-\tau) - w(t-\tau)) + \lambda_1 \mu_0 u_1(t) = 0, \quad 0 < t < 1.$$

Similarly to the proof of (15), we have

$$\begin{aligned} &(\lambda_1 \tau M_\tau + \varepsilon)\int_0^{1-\tau} u_1(s+\tau)u_0(s)\mathrm{d}s \leq \lambda_1\tau(1-\tau)\|u_1'\|_{[0,1]}\|u_0\| + \\ &\lambda_1\tau\|u_1\|_{[0,1]}\|u_0\| + b_1\int_0^1 u_1(s)\mathrm{d}s + (\lambda_1(1+\tau M_\tau) + \varepsilon)\int_0^{1-\tau} u_1(s+\tau)w(s)\mathrm{d}s. \end{aligned}$$

By (16), we get

$$(\lambda_1\tau M_\tau+\varepsilon)\int_0^{1-\tau}\sigma(s)u_1(s+\tau)\mathrm{d}s\le \lambda_1\tau(\|u_1\|_{[0,1]}+(1-\tau)\|u_1'\|_{[0,1]})+$$
$$b_1\int_0^1 u_1(s)\mathrm{d}s+(\lambda_1(1+\tau M_\tau)+\varepsilon)\int_0^{1-\tau}u_1(s+\tau)w(s)\mathrm{d}s.$$

By the definition of M_τ, we have

$$R=\|u_0\|\le\frac{b_1\int_0^1 u_1(s)\mathrm{d}s+(\lambda_1(1+\tau M_\tau)+\varepsilon)\int_0^{1-\tau}u_1(s+\tau)w(s)\mathrm{d}s}{\varepsilon\int_0^{1-\tau}\sigma(s)u_1(s+\tau)\mathrm{d}s}.$$

It is a contradiction to the choice of R. Hence (20) holds true. It follows from the fixed point index in [5], [6] that

$$i(A,B_R,P)=0. \tag{21}$$

By (8) and (9), we have

$$\begin{aligned}Au(t)&=\int_0^1 G(t,s)F(s,u(s-\tau)-w(s-\tau))\mathrm{d}s\\&\le\int_0^1 G(s,s)F(s,u(s-\tau)-w(s-\tau))\mathrm{d}s\\&=\int_0^1 G(s,s)(f(s,u(s-\tau)-w(s-\tau))+M)\mathrm{d}s\\&\le\int_0^1 G(s,s)Q(s)\mathrm{d}s<\gamma(M+1),\ \forall\, u\in\partial B_1, t\in[0,1].\end{aligned}$$

i.e., $\|Au\|<\|u\|, u\in\partial B_1$. And so

$$i(A,B_1,P)=1. \tag{22}$$

It follows from (21), (22) and the additivity of the fixed point index that A has at least a fixed point u satisfying $\|u\|>\gamma(M+1)$. The proof is completed. □

3. Example

Consider the following second-order boundary value problem of delay differential equation

$$\begin{cases}u''+f(t,u)=0, & 0<t<1,\\ u(t)=0, \quad -\tau\le t\le 0, & u(1)=0.\end{cases} \tag{23}$$

Conclusions

(i) Suppose that $f(t,u)=M_1e^{10-u}-9t\cos u$ where $M_1>0$. Then for any $M_1>\frac{320}{3}$, (23) has at least one positive solution.

(ii) Suppose that $f(t,u)=M_1(\frac{u}{8})^\beta-7t\cos u$ where $M_1>0,\beta>1$. Then for any $M_1<34$, (23) has at least one positive solution.

Proof (i) Fix $M=9, t_0=\frac{1}{2}, \theta=\frac{1}{4}, p(t)\equiv 1$. Then we get

$$G(t,s)=\begin{cases}t(1-s), & 0\le t\le s\le 1,\\ s(1-t), & 0\le s\le t\le 1.\end{cases}$$

Direct calculation gives that $\int_0^1 G(t,s)\mathrm{d}s = \frac{t(1-t)}{2}$, $w(t) = M\frac{t(1-t)}{2}$, $\sigma(t) = \min\{t, 1-t\}, \gamma = 1, \gamma(M+1) = 10$. Evidently, $\lim_{u\to\infty} \frac{f(t,u)}{u} = 0$, $\forall\ t \in [0,1]$ and $f(t,u) + M \geq Q(t)e^{10-u} \geq M_1$, $\forall t \in [0,1]$, $\forall u \in [0,10]$, where $Q(t) \equiv M_1$, $t \in [0,1]$. Moreover, for any $M_1 > \frac{320}{3}$, we have $\int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)M_1\mathrm{d}s = \frac{3M_1}{32} > 10$. Hence ($\mathrm{H}_1$) is satisfied. It follows from Theorem 1 that (i) holds true.

(ii) Similarly to the proof of (i), we can prove that (ii) also holds true.

Remark The nonlinearity f in example 1 can get negative value, so the conclusions (i) and (ii) in example cannot be obtained by virtue of [1]. Hence Theorem 1 in this paper totally generalizes the corresponding results in [1].

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