Journal of Mathematical Research & Exposition
Aug., 2008, Vol. 28, No. 3, pp. 727–732
DOI:10.3770/j.issn:1000-341X.2008.03.038
Http://jmre.dlut.edu.cn

2-Harmonic Submanifolds in a Complex Space Form

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Abstract In the present paper, the authors study totally real 2-harmonic submanifolds in a complex space form and obtain a Simons' type integral inequality of compact submanifolds as well as some relevant conclusions.

Keywords 2-harmonic; minimal; totally geodesic; complex space form.

Document code A
MR(2000) Subject Classification 53C42
Chinese Library Classification O186.12

1. Introduction

Following the tentative ideas of Eell and Lemaire, Jiang[1,2] studied 2-harmonic map on Riemannian manifolds. After that, many new results concerning 2-harmonic submanifolds come out in succession. In 1999, Sun and Zhong[3] discussed real 2-harmonic hypersurface in a complex projective space. In this paper, we study totally real 2-harmonic submanifolds in a complex space form, generalize the conclusion in [3], and obtain a series of results.

2. Preliminaries

Let CN_c^n be a complex space form[4], of complex dimension n, with the Fubini-study metric of constant holomorphic sectional curvature c, and J be the complex structure of CN_c^n. Among all submanifolds of CN_c^n, there are two typical classes: one is the class of holomorphic submanifolds and the other is the class of totally real submanifolds. A submanifold M^n in CN_c^n is called holomorphic (resp. totally real) if each tangent space of M^n is mapped into itself (resp. the normal space) by the complex structure J. In this paper we study totally real submanifolds in CN_c^n and use the following convention on the ranges of indices unless otherwise stated:

$$A, B, C \ldots = 1, \ldots, n, 1^*, \ldots, n^*; i, j, \ldots = 1, \ldots, n.$$

We choose local field of orthonormal frames $e_1, \ldots, e_n, e_{1^*} = Je_1, \ldots, e_{n^*} = Je_n$ in CN_c^n, in such a way that, restricted to M^n, $e_1, \ldots e_n$ are tangent to M^n. Let $\{\omega_n\}$ be the field of dual frames.

Received date: 2006-05-07; **Accepted date**: 2007-03-23
Foundation item: Natural Science Foundation of Education Department of Anhui Province (No. 2004kj166zd).

Then the structure equations of CN_c^n are given by

$$\mathrm{d}\omega_A = -\sum_B \omega_{AB}\wedge\omega_B, \quad \omega_{AB}+\omega_{BA}=0; \tag{2.1}$$

$$\mathrm{d}\omega_{AB} = -\sum_C \omega_{AC}\wedge\omega_{CB} + \frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C\wedge\omega_D; \tag{2.2}$$

$$K_{ABCD} = \frac{c}{4}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD}), \tag{2.3}$$

where J_{AB} is the component of complex structure J of CN_c^n with the following form

$$J_{AB} = \begin{array}{cc} \begin{pmatrix} 0 & \vdots & -I_n \\ \cdots & \cdots & \cdots \\ I_n & \vdots & 0 \end{pmatrix} & \begin{array}{l} \}i \\ \}i^* \end{array} \\ \underbrace{\quad}_{j} \quad \underbrace{\quad}_{j^*} & \end{array} \tag{2.4}$$

Restricting these formulas to M^n, we have

$$\omega_{k^*} = 0; \quad \omega_{ij} = \omega_{i^*j^*}; \quad \omega_{i^*j} = \omega_{j^*i}; \tag{2.5}$$

$$\omega_{k^*i} = \sum_j h_{ij}^{k^*}\omega_j; \quad h_{ij}^{k^*} = h_{ji}^{k^*} = h_{jk}^{i^*} = h_{ik}^{j^*}; \tag{2.6}$$

$$R_{ijkl} = K_{ijkl} + \sum_{m^*}(h_{ik}^{m^*}h_{jl}^{m^*} - h_{il}^{m^*}h_{jk}^{m^*}); \tag{2.7}$$

$$R_{i^*j^*kl} = K_{i^*j^*kl} + \sum_m (h_{km}^{i^*}h_{lm}^{j^*} - h_{km}^{j^*}h_{lm}^{i^*}). \tag{2.8}$$

Let $B(=\sum_{m^*,i,j} h_{ij}^{m^*}\omega_i\otimes\omega_j\otimes e_{m^*})$ be the second fundamental form of M^n and $\tau(=\sum_{m^*,i} h_{ii}^{m^*}e_{m^*} = n\eta)$ be tensile field of M^n, where η is the mean curvature vector of M^n.

Let $h_{ijk}^{m^*}$ and $h_{ijkl}^{m^*}$ be the first and second covariant derivatives of $h_{ij}^{m^*}$. Then we get

$$h_{ijk}^{m^*} = h_{ikj}^{m^*}, \tag{2.9}$$

$$h_{ijkl}^{m^*} - h_{ijlk}^{m^*} = \sum_p h_{ip}^{m^*}R_{pjkl} + \sum_p h_{jp}^{m^*}R_{pikl} - \sum_{p^*} h_{ij}^{p^*}R_{m^*p^*kl}. \tag{2.10}$$

Let the Laplacian of $h_{ij}^{m^*}$ be as follows:

$$\triangle h_{ij}^{m^*} = \sum_k h_{ijkk}^{m^*}.$$

Then we have

$$\triangle h_{ij}^{m^*} = \sum_k h_{kkij}^{m^*} + \sum_{k,l}(h_{kl}^{m^*}R_{lijk} + h_{li}^{m^*}R_{lkjk}) - \sum_{l^*,k} h_{ik}^{l^*}R_{m^*l^*jk}. \tag{2.11}$$

From (2.11) and other relevant formulas, we can get

$$\frac{1}{2}\triangle\|B\|^2 = \sum_{m^*,i,j,k}(h_{ijk}^{m^*})^2 + \sum_{m^*,i,j,k} h_{ij}^{m^*}h_{kkij}^{m^*} + \frac{c}{4}(n+1)\|B\|^2 -$$

$$\frac{c}{2}\|\tau\|^2 + 2\sum_{i^*,j^*}[\mathrm{tr}(H_{i^*}H_{j^*})^2 - \mathrm{tr}(H_{i^*}^2H_{j^*}^2)] +$$
$$\sum_{i^*,j^*}\mathrm{tr}(H_{i^*}^2H_{j^*})\mathrm{tr}H_{j^*} - \sum_{i^*,j^*}[\mathrm{tr}(H_{i^*}H_{j^*})]^2. \tag{2.12}$$

From (2.3), (2.4) and [2], we have

Lemma 1 *Let M^n be a totally real submanifold in a complex space form CN_c^n. Then $M^n$is a 2-harmonic submanifold if and only if M^n satisfies the following conditions.*

$$\sum_{m^*,i,k}(2h_{iik}^{m^*}h_{jk}^{m^*} + h_{ii}^{m^*}h_{kkj}^{m^*}) = 0, \quad \forall j; \tag{2.13}$$

$$\sum_{i,k}h_{iikk}^{m^*} - \sum_{p^*,i,j,k}h_{ii}^{p^*}h_{jk}^{p^*}h_{jk}^{m^*} + \frac{c}{4}(n+3)\sum_i h_{ii}^{m^*} = 0, \quad \forall m^*. \tag{2.14}$$

Lemma 2[5] *Let $A_1, A_2, \ldots, A_m$ be $(n\times n)$-symmetric matrices $(m \geq 2)$. Then*

$$\sum_{\alpha,\beta=1}^{m}\mathrm{tr}(A_\alpha A_\beta - A_\beta A_\alpha)^2 - \sum_{\alpha,\beta=1}^{m}(\mathrm{tr}(A_\alpha A_\beta))^2 \geqslant -\frac{3}{2}(\sum_{\alpha=1}^{m}\mathrm{tr}(A_\alpha^2))^2. \tag{2.15}$$

Lemma 3 *Let M^n be a totally real submanifold in a complex space form CN_c^n, then we have*

(1) $2\sum_{i^*,j^*}[\mathrm{tr}(H_{i^*}H_{j^*})^2 - \mathrm{tr}(H_{i^*}^2H_{j^*}^2)] - \sum_{i^*,j^*}[\mathrm{tr}(H_{i^*}H_{j^*})]^2 \geqslant -\frac{3}{2}\|B\|^4$; (2.16)

(2) $\sum_{i^*,j^*}\mathrm{tr}(H_{i^*}H_{j^*})\mathrm{tr}H_{i^*}\mathrm{tr}H_{j^*} \geqslant 0$; (2.17)

(3) $\sum_{i^*,j^*}\mathrm{tr}(H_{i^*}^2H_{j^*})\mathrm{tr}H_{j^*} \geqslant -\|\tau\|\cdot\|B\|^3$. (2.18)

Proof From Lemma 2, (1) is clear.

(2) Obviously, $\sum_{i^*,j^*}\mathrm{tr}(H_{i^*}H_{j^*})\mathrm{tr}H_{i^*}\mathrm{tr}H_{j^*} = \sum_{l,m}(\sum_{j^*}(\sum_k h_{kk}^{j^*})h_{lm}^{j^*})^2 \geqslant 0$.

(3) For fixed i^*, let $h_{kl}^{i^*} = \lambda_k^{i^*}\delta_{kl}$. By Schwarz inequality, we have

$$\begin{aligned}\mathrm{tr}(H_{i^*}^2H_{j^*}) &= \sum_k(\lambda_k^{i^*})^2h_{kk}^{j^*} \leqslant \sqrt{\sum_k(\lambda_k^{i^*})^2\sum_l(\lambda_l^{i^*})^2(h_{ll}^{j^*})^2}\\ &\leqslant \sqrt{\sum_k(\lambda_k^{i^*})^2\sum_l(\lambda_l^{i^*})^2\sum_{k,l}(h_{kl}^{j^*})^2}\\ &= \mathrm{tr}(H_{i^*}^2)\cdot\sqrt{\mathrm{tr}(H_{j^*}^2)}.\end{aligned}$$

Then

$$\begin{aligned}|\sum_{i^*,j^*}\mathrm{tr}(H_{i^*}^2H_{j^*})\mathrm{tr}H_{j^*}| &\leqslant \sqrt{\sum_{j^*}(\sum_{i^*}\mathrm{tr}(H_{i^*}^2H_{j^*}))^2\cdot\sum_{k^*}(\mathrm{tr}H_{k^*})^2}\\ &\leqslant \sqrt{\sum_{j^*}\Big[\sum_{i^*}\mathrm{tr}(H_{i^*}^2)\sqrt{\mathrm{tr}(H_{j^*}^2)}\ \Big]^2\cdot\sum_{k^*}(\mathrm{tr}H_{k^*})^2}\\ &= \|\tau\|\cdot\|B\|^3.\end{aligned}$$

3. Main results

Firstly, we study the relations between the totally real 2-harmonic submanifold and the minimal submanifold in CN_c^n.

Theorem 1 *Let M^n be a totally real 2-harmonic submanifold in a complex space form CN_c^n $(c<0)$. If the mean curvature vector of M^n is parallel, then M^n is minimal.*

Proof Since the mean curvature vector of M^n is parallel, we have

$$\sum_i h_{iik}^{m^*}=0,\quad \sum_i h_{iikj}^{m^*}=0,\quad m^*=1^*,\dots,n^*.$$

Multiplying $\sum_l h_{ll}^{m^*}$ on the both sides of (2.14) and summing up with respect to m^*, we get

$$\begin{aligned}0&=\sum_{m^*,p^*,i,j,k,l} h_{ll}^{m^*}h_{ii}^{p^*}h_{jk}^{p^*}h_{jk}^{m^*}-\frac{c}{4}(n+3)\sum_{m^*,i,l}h_{ii}^{m^*}h_{ll}^{m^*}\\&=\sum_{j,k}(\sum_{p^*}(\sum_i h_{ii}^{p^*})h_{jk}^{p^*})^2-\frac{c}{4}(n+3)\sum_{m^*}(\sum_i h_{ii}^{m^*})^2\\&\geqslant -\frac{c}{4}(n+3)\|\tau\|^2.\end{aligned}$$

From $c<0$, we have $\|\tau\|^2=0$. Therefore M is minimal submanifold.

Theorem 2 *Let M^n be a totally real 2-harmonic submanifold in a complex space form CN_c^n $(c>0)$. If the mean curvature vector of M^n is parallel and $\|B\|^2<\frac{c}{4}(n+3)$, then M^n is minimal.*

Proof Similarly to the proof of Theorem 1, we have

$$\begin{aligned}0&=\sum_{j,k}(\sum_{p^*}(\sum_i h_{ii}^{p^*})h_{jk}^{p^*})^2-\frac{c}{4}(n+3)\sum_{m^*}(\sum_i h_{ii}^{p^*})^2\\&\leqslant\sum_{j,k}(\sum_{p^*}(\sum_i h_{ii}^{p^*})^2\sum_{m^*}(h_{jk}^{m^*})^2)-\frac{c}{4}(n+3)\sum_{m^*}(\sum_i h_{ii}^{m^*})^2\\&=\|\tau\|^2(\|B\|^2-\frac{c}{4}(n+3)).\end{aligned}$$

From $\|B\|^2<\frac{c}{4}(n+3)$, it follows that $\|\tau\|^2=0$.

Theorem 3 *Let M^n be a totally real 2-harmonic submanifold in a complex space form CN_c^n $(c\geqslant 0)$. If the mean curvature vector η of M^n is parallel and $S_\eta\neq\frac{c}{4}(n+3)$, then M^n is minimal. Where S_η is the square of the second fundamental form of M^n with respect to η.*

Proof Suppose M^n is not a minimal submanifold. Then $\eta\neq 0$. Choosing e_{1^*} which has the same direction as η, we have

$$\eta=\frac{1}{n}\sum_i h_{ii}^{1^*}e_{1^*};\quad \sum_i h_{ii}^{1^*}=n\|\eta\|,\quad \sum_i h_{ii}^{m^*}=0,\quad m^*\neq 1^*.$$

Since the mean curvature vector of M^n is parallel, from (2.14) we get

$$\sum_{i,j,k}h_{ii}^{1^*}h_{jk}^{1^*}h_{jk}^{m^*}=0,\quad m^*\neq 1^*;\tag{3.1}$$

$$\sum_{i,j,k} h_{ii}^{1^*} h_{jk}^{1^*} h_{jk}^{1^*} - \frac{c}{4}(n+3) \sum_{i} h_{ii}^{1^*} = 0. \tag{3.2}$$

Since $\eta \neq 0$, from (3.2), we have

$$\sum_{j,k} (h_{jk}^{1^*})^2 - \frac{c}{4}(n+3) = 0.$$

Hence $S_\eta = \frac{c}{4}(n+3)$, which results in a contradiction.

Secondly, we discuss the relation between $\|B\|$ and $\|\tau\|$ in the totally real 2-harmonic submanifold and obtain a J.Simons's type integral inequality.

Theorem 4 *Let M^n be a compact totally real 2-harmonic submanifold in a complex space form CN_c^n. Then we have*

$$\int_{M^n} (\frac{c}{4}(n+5)\|\tau\|^2 + \|\tau\|\|B\|^3 - \frac{c}{4}(n+1)\|B\|^2 - \frac{3}{2}\|B\|^4) \mathrm{d}v_M \geqslant 0.$$

Proof Taking the covariant derivative with respect to j on the both sides of (2.13), and summing up with respect to j, we have

$$\sum_{m^*,i,j,k} (2h_{iikj}^{m^*} h_{kj}^{m^*} + 2h_{iik}^{m^*} h_{kjj}^{m^*} + h_{kkj}^{m^*} h_{iij}^{m^*} + h_{kkjj}^{m^*} h_{ii}^{m^*}) = 0.$$

Adjusting the indices of upper formula properly gives

$$\begin{aligned}\sum_{m^*,i,j,k} h_{ij}^{m^*} h_{kkjj}^{m^*} &= -\frac{1}{2} \sum_{m^*,i,j,k} (3h_{iik}^{m^*} h_{jjk}^{m^*} + h_{ii}^{m^*} h_{jjkk}^{m^*}) \\ &= -\frac{3}{2} \sum_{m^*,i,j,k} (h_{iik}^{m^*} h_{jjk}^{m^*} + h_{ii}^{m^*} h_{jjkk}^{m^*}) + \sum_{m^*,i,j,k} h_{ii}^{m^*} h_{jjkk}^{m^*}.\end{aligned} \tag{3.3}$$

Since

$$\frac{1}{2} \triangle \|\tau\|^2 = \sum_{m^*,i,j,k} (h_{iik}^{m^*} h_{jjk}^{m^*} + h_{ii}^{m^*} h_{jjkk}^{m^*}). \tag{3.4}$$

From (2.14), we have

$$\sum_{m^*,i,j,k} h_{ii}^{m^*} h_{jjkk}^{m^*} = \sum_{m^*,p^*} \mathrm{tr} H_{m^*} \mathrm{tr} H_{p^*} \mathrm{tr}(H_{m^*} H_{p^*}) - \frac{c}{4}(n+3)\|\tau\|^2. \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), we obtain

$$\sum_{m^*,i,j,k} h_{ij}^{m^*} h_{kkij}^{m^*} = -\frac{3}{4} \triangle \|\tau\|^2 + \sum_{m^*,p^*} \mathrm{tr} H_{m^*} \mathrm{tr} H_{p^*} \mathrm{tr}(H_{m^*} H_{p^*}) - \frac{c}{4}(n+3)\|\tau\|^2. \tag{3.6}$$

From (2.12) and (3.6), we have

$$\begin{aligned}\frac{1}{2} \triangle \|B\|^2 + \frac{3}{4} \triangle \|\tau\|^2 = &\sum_{m^*,i,j,k} (h_{ijk}^{m^*})^2 + \sum_{m^*,p^*} \mathrm{tr} H_{m^*} \mathrm{tr} H_{p^*} \mathrm{tr}(H_{m^*} H_{p^*}) - \frac{c}{4}(n+5)\|\tau\|^2 + \\ &\frac{c}{4}(n+1)\|B\|^2 + 2\sum_{i^*,j^*} [\mathrm{tr}(H_{i^*} H_{j^*})^2 - \mathrm{tr}(H_{i^*}^2 H_{j^*}^2)] + \\ &\sum_{i^*,j^*} \mathrm{tr}(H_{i^*}^2 H_{j^*}) \mathrm{tr} H_{j^*} - \sum_{i^*,j^*} [\mathrm{tr}(H_{i^*} H_{j^*})]^2.\end{aligned} \tag{3.7}$$

Noting Lemma 3, we get

$$\frac{1}{2}\triangle\|B\|^2+\frac{3}{4}\triangle\|\tau\|^2 \geqslant -\frac{c}{4}(n+5)\|\tau\|^2+\frac{c}{4}(n+1)\|B\|^2-\frac{3}{2}\|B\|^4-\|\tau\|\|B\|^3. \tag{3.8}$$

Since M^n is compact, we have

$$\int_{M^n}(\frac{c}{4}(n+5)\|\tau\|^2+\|\tau\|\|B\|^3+\frac{3}{2}\|B\|^4-\frac{c}{4}(n+1)\|B\|^2)\mathrm{d}v_M \geqslant 0.$$

At last, we study the relations between the totally real 2-harmonic submanifold and the totally geodesic submanifold in CN_c^n.

Theorem 5 *Let M^n be a compact totally real 2-harmonic submanifold in a complex space form CN_c^n $(c<0)$. If the mean curvature vector of M^n is parallel, then M^n is totally geodesic.*

Proof From Theorem 1, we know M^n is minimal. Therefore $\|\tau\|=0$. Since M^n is compact, from Theorem 4, we have

$$\int_{M^n}\frac{3}{2}\|B\|^2(\|B\|^2-\frac{c}{6}(n+1))\mathrm{d}V_M \geqslant 0. \tag{3.9}$$

From $c<0$ and (3.9), we have

$$\|B\|^2=0.$$

Hence M^n is totally geodesic.

Theorem 6 *Let M^n be a compact totally real 2-harmonic submanifold in a complex space form CN_c^n $(c>0)$. If the mean curvature vector of M^n is parallel and $\|B\|^2\leqslant\frac{c}{6}(n+1)$, then M^n is totally geodesic, or $\|B\|^2=\frac{c}{6}(n+1)$.*

Proof Since $\|B\|^2\leqslant\frac{c}{6}(n+1)<\frac{c}{4}(n+3)$, from Theorem 2, we know M^n is minimal. Then $\|\tau\|=0$. Since M^n is compact, from Theorem 4, we have

$$\int_{M^n}\frac{3}{2}\|B\|^2(\|B\|^2-\frac{c}{6}(n+1))\mathrm{d}v_M \geqslant 0.$$

Noting $\|B\|^2\leqslant\frac{c}{6}(n+1)$, we know M^n is totally geodesic, or $\|B\|^2=\frac{c}{6}(n+1)$.

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