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Classification about Non-Solvable Groups with Exactly 40 Maximal Order Elements

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Abstract Let φ be a homomorphism from a group H to a group $\mathrm{Aut}(N)$. Denote by $H_\varphi \times N$ the semidirect product of N by H with homomorphism φ. This paper proves that: Let G be a finite nonsolvable group. If G has exactly 40 maximal order elements, then G is isomorphic to one of the following groups: (1) $Z_{4\varphi} \times A_5, \ker\varphi = Z_2$; (2) $D_{8\varphi} \times A_5$, $\ker\varphi = Z_2 \times Z_2$; (3) $G/N = S_5$, $N = Z(G) = Z_2$; (4) $G/N = S_5$, $N = Z_2 \times Z_2$, $N \cap Z(G) = Z_2$.

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1. Introduction and main result

The number of maximal order elements of finite groups has strong influence on the structure of finite groups. It was shown in [1–5] that if the number of maximal order elements of finite groups is odd, or $2p, 2p^2, 2p^3, 4p$, then the groups are solvable groups. Papers [1–5] concern about that when the number of maximal order elements is given in special, the finite groups are solvable. But we find that if the number of maximal order elements is $8p$, there exists non-solvable groups. The main target of this paper is to classify the non-solvable groups if the number of maximal order elements is 40 (8p, p=5). We have the following result:

Theorem 3.2 *Suppose that G is a finite nonsolvable group. Then G has 40 maximal order elements if and only if G is isomorphic to one of the following groups:*

(1) $Z_{4\varphi} \times A_5$, $\ker\varphi = Z_2$;
(2) $D_{8\varphi} \times A_5$, $\ker\varphi = Z_2 \times Z_2$;
(3) $G/N = S_5$, $N = Z(G) = Z_2$;
(4) $G/N = S_5$, $N = Z_2 \times Z_2$, $N \cap Z(G) = Z_2$.

For the sake of convenience, we introduce some notations. Let G be a finite group, $\pi_e(G)$ be the set of orders of elements of G, $\pi(n) = \{p \mid p \text{ is a prime}, p \mid n\}$, $\pi(G)$=$\pi(|G|)$, $M_i(G) = \{x \in G, \mid o(x) = i, i \in \pi_e(G)\}$, $\phi(x)$ be Eular function, $M(G)$ be the set of maximal order elements

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of G, $k = \max \pi_e(G)$, and $o(x)$ be the order of an element of x. We use $H_\varphi \times N$ to denote the semidirect product of N by H with homomorphism φ, where φ is a homomorphism from a group H to a group $\mathrm{Aut}(N)$[6]. The other symbols are standard. We begin with some lemmas.

2. Preliminaries

Lemma 2.1 *Let G be a finite group and G have n cyclicsubgroups of order k, say, $A_1, A_2, \ldots, A_n$. We classify A_i, $i = 1, \ldots, n$, by conjugation. Assume it has s orbits, and the length of orbit containing A_i is n_i, $i = 1, 2, \ldots, s$. Without loss of generality, let A_i be the representative of each orbit, $i = 1, 2, \ldots, s$. Then:*

(1) $n_i = |G : N_G(A_i)|$, $n = n_1 + n_2 + \cdots + n_s$;

(2) $\pi(C_G(A_i)) = \pi(A_i) = \pi(k)$, $\pi(G) = \pi(n_i) \cup \pi(N_G(A_i))$, $i = 1, 2, \ldots, s$;

(3) $|N_G(A_i) : C_G(A_i)| \mid \phi(k)$, $\pi(N_G(A_i)) \subseteq \pi(k) \cup \pi(\phi(k))$, $i = 1, 2, \ldots, s$;

(4) $|G| = n_i |N_G(A_i) : C_G(A_i)||C_G(A_i)|$, $i = 1, 2, \ldots, s$.

Proof See Lemma 2.1 in the [5].

Lemma 2.2 *Let $N \lhd G$ and N be a non-Abelin simple group. Then $G/C_G(N)N \cong$ a subgroup of* $\mathrm{Out}(N)$.

Proof Since N is a simple group, $Z(N) = 1$. Thus $N \cong N/Z(N) \cong \mathrm{Inn}(N)$. Since $N \lhd G$ and N is simple, $C_G(N)N = C_G(N) \times N$. Therefore $\mathrm{Inn}(N) \cong N \cong C_G(N)N/C_G(N)$. By *n-c* theorem, $G/C_G(N) \cong$ a subgroup of $\mathrm{Aut}(N)$.

Hence $G/C_G(N)/C_G(N)N/C_G(N) \cong$ a subgroup of $\mathrm{Aut}(N)/\mathrm{Inn}(N)$, i.e., a subgroup of $G/C_G(N)N \cong \mathrm{Out}(N)$.

Lemma 2.3 *Let G be a finite group. If $|M(G)| = \phi(k)$, i.e., $n = 1$, then G is a supersolvable group.*

Proof See [1].

3. Main theorem

In the following proof, we always assume that G has n cyclicsubgroups of order k and there are s orbits among those cyclicsubgroups under the conjugacy action of G. Suppose $A_1, A_2, \ldots, A_s$ are the representatives of each orbit and the length of the orbit A_i is n_i, $i = 1, 2, \ldots, s$. Furthermore, we assume that $n_1 \leq n_1 \leq n_2 \leq \cdots \leq n_s$ and $A = A_1$. Clearly, $|M(G)|$ (the number of maximal order elements), n (the number of cyclicsubgroups of order k), and Euler function $\phi(k)$ have the relation of $|M(G)| = n\phi(k)$. We list the relation in following Table 1:

n	1	2	4	5	10	20	40
$\phi(k)$	40	20	10	8	4	2	1
k	k	$5^2, 3.11, 4.11, 6.11, 2.3.11$	$11, 2.11$	$16, 20, 24, 15, 30$	$5, 8, 10, 12$	$3, 4, 6$	2

Table 1

Lemma 3.1 *In above cases, G is solvable except for $k = 12$.*

Proof

Case 1. If $n = 1$ and $\phi(k) = 40$, then $|M(G)| = \phi(k)$. By Lemma 2.3, G is solvable.

Case 2. If $n = 2$ and $\phi(k) = 20$. Since $n = 2$, $|G : N_G(A)| \leq 2$ and $N_G(A) \lhd G$ follows. Also since $|N_G(A) : C_G(A)| \mid \phi(k) = 20$, $N_G(A)/C_G(A)$ is solvable. We know $|A| = k$a and $\phi(k) = 2^2.5$. So let $k = p^a q^b \cdots r^c$. Then $\phi(k) = p^{a-1}(p-1)q^{b-1}(q-1)\cdots r^{c-1}(r-1)$. Firstly, we prove that k has no more than two distinct prime factors, except $k = 2.3.11$. Let p be the minimal prime factor of k, and $p = 2$. (i) If $a = 1$, that is, $\phi(k) = q^{b-1}(q-1)\cdots r^{c-1}(r-1)$. Since $2^2 \mid\mid \phi(k) = 2^2.5$, if k has more than two prime factors, then $q = 3, r = 11$, i.e., $k = 2.3.11$. (ii) If $a \geq 2$, then k has no more than two prime factors since $2^2 \mid\mid \phi(k) = 2^2.5$. If $p > 2$, since $p-1$, $q-1$, $r-1$ all have factor 2, k has no more than two prime factors. Secondly, we prove G is solvable. By Lemma 2.1(2), $\pi(C_G(A)) = \pi(A) = \pi(k)$. If $k = 2.3.11$, then $C_G(A)$ is $\{2, 3, 11\}$ group. G is nonsovable implies that $C_G(A)$ is nonsolvable and $C_G(A)$ has a simple section to be a K_3-simple group. By [9], the order of the K_3-simple group does not have prime factor 11, which is a contraction. Therefore k has at most two prime factors, and $C_G(A)$ is a $\{p, q\}$ group. It follows that G is solvable.

Case 3. If $n = 4$ and $\phi(k) = 10 = 2.5$, then $k = 11, 22$ and $\pi(C_G(A)) = \pi(k) \subseteq \{2, 11\}$. Let $|N_G(A) : C_G(A)| = 2^a.5^b$ and $|C_G(A)| = 2^u.11^v$. By Lemma 2.1(4), $|G| = n_1|N_G(A) : C_G(A)||C_G(A)| = n_1.2^a.5^b.2^u.11^v$. Clearly, $v = 1$ and $n_1 \leq 4$. Let $P_{11} \in Syl_{11}(G)$. Then $P_{11} \lhd G$. Hence G/P_{11} is a $\{2, 5\}$ group and a solvable group. So G is solvable.

Case 4. If $n = 5$, $\phi(k) = 8$, $k = 16, 24, 15, 20, 30$. Let $P_3 \in Syl_3(G)$ and $P_5 \in Syl_5(G)$. In the following, we discuss various cases.

(1) If $k = 16$ and $|G| = n_1.2^a.2^u$, then G is solvable.

(2) If $k = 24$, by Lemma 2.1, $|G| = n_1|N_G(A) : C_G(A)||C_G(A)| = n_1.2^a.2^u.3^v$. Since $n_1 \leq n_2 \leq \cdots \leq n_s$, we have $n_1 = 1, 2, 5$. G is nonsolvable implies that $n_1 = 5$, $v = 1$. Since P_3 char $C_G(A) \lhd N_G(A)$, $P_3 \lhd N_G(A)$ and $N_G(A) \leq N_G(P_3)$. By $|G : N_G(A)| = n_1 = 5$, we have $|G : N_G(P_3)| = 3l + 1 \mid 5$, and $|G : N_G(P_3)| = 1$ follows. That is, $P_3 \lhd G$ and G/P_3 is solvable. Therefore G is solvable, a contradiction.

(3) If $k = 15$, by Lemma 2.1, $|G| = n_1.2^a.2^u.5^v$. Since $|C_G(A)| = 3^u.5^v$, and G has exactly 40 maximal order elements, we have $u = 1$, $v = 1$. Since $n_1 \leq 5$, if $n_1 < 5$, then $P_5 \lhd G$ and G is solvable since $n_1 \leq 5$. If $n_1 = 5$, as the same as (2), we have $P_3 \lhd G$. So G/P_3 is solvable, which means that G is solvable, a contradiction.

(4) If $k = 20$, by Lemma 2.1, $|G| = n_1.2^a.2^u.5^v$. Since $|M(G)| = 40$, we have $v = 1$. Considering the minimum of n_1 and $n_1 + n_2 + \cdots + n_s = 5$, we have $n_1 = 1$, 2, or 5. So G is a $\{2, 5\}$ group, which means that G is solvable.

(5) If $k = 30$, by Lemma 2.1, $|G| = n_1.2^a.2^b.3^u.5^v$. Since the number of maximal order elements of G is 40, we have $u = v = 1$. In the same way as (3), if $n_1 < 5$, then $P_5 \lhd G$; if $n_1 = 5$,

then $P_3 \lhd G$. Both lead to that G is solvable.

Case 5. $n = 10$, $\phi(k) = 4$, $k = 5, 8, 10, 12$.

(1) If $k = 5$, by Lemma 2.1, $|G| = n_1.2^a.5^u$. Since $n_1 + n_2 + \cdots + n_s = 10$ and $n_1 \leq n_2 \leq \cdots \leq n_s$, $n_1 = 1, 2, 3, 4, 5, 10$. If $n_1 = 3$, considering $|G| = n_i.2^a.5^u$, $i = 1, 2, \ldots, s$, we have $s = 3$ and $n_2 = 3$, $n_3 = 4$. So $|G| = 3.2^a.5^u = |G| = 4.2^a.5^u$, a contradiction. Hence $n_1 = 1, 2, 4, 5, 10$. Clearly, G is a $\{2, 5\}$ group, so G is solvable.

(2) If $k = 8$, 10, we can conclude that G is a $\{2, 5\}$ group and G is solvable. We discuss the case $k = 12$ later.

Case 6. $n = 20$, $\phi(k) = 2$, $k = 3, 4, 6$. It is easy to see that if $k = 3, 4$, then G is solvable. When $k = 6$, by Lemma 2.1, $|G| = n_1.2^a.2^u.3^v$, $n_1 + n_2 + \cdots + n_s = 20$. If G is nonsolvable, then $5 \mid n_1$, $n_1 = 5$, 10. So $\pi_e(G) = \{1, 2, 3, 4, 5, 6\}$, or $\pi_e(G) = \{1, 2, 3, 5, 6\}$. By Refs. [7], [8], G is solvable.

Case 7. $n = 40$, $\phi(k) = 1$, $k = 2$. It is easy to see that G is solvable.

Now we begin to prove Theorem 3.2.

Proof of Theorem 3.2 Firstly, we verify the groups in Theorem 3.2 are nonsolvable groups with exactly 40 maximal order elements. Considering the length of this paper, we only verify $G/N \cong S_5$. The other groups can be verified similarly.

Let $G/N = S_5$ and $N = Z_2 \times Z_2$. By $n - c$ theorem, $G/C_G(N) \leq Aut(N) = S_3$. G is nonsolvable implies that $C_G(N)$ is nonsolvable. So we have $3 \mid |C_G(N)|$. By $C_G(N) \lhd G$, we know that $C_G(N)$ contains all Sylow 3-subgroups of G. Evidently, the maximal order of elements of G is not greater than 12. We assert that G has exactly 40 elements of order 12.

Firstly, we prove that G has elements of order 12.

Otherwise, let a, b be any elements of order 3 of G. Then $\bar{a}$, $\bar{b}$ are also elements of order 3 of G/N. Since all elements of order 3 of S_5 lie in one conjugacy class, $\bar{a}$, $\bar{b}$ are conjugate in G/N, that is, there exists $g \in G$, such that $a^g = bc$, $c \in N$. So bc is an element of order 3. But $b \in C_G(N)$, so $c = 1$. Therefore a, b are conjugate in G, i.e., all elements of order 3 of G lie in one conjugacy class. Now we know that a and b are conjugate in G if and only if $\bar{a}$ and $\bar{b}$ are conjugate in G/N. Since $|(\bar{a})^{\bar{G}}| = 20$, $|a^G| = 20$. Therefore $|C_G(a)| = |G|/20 = 2^5.3.5/2^2.5 = 2^3.3$. Let $P_2 \in \mathrm{Syl}_2(C_G(a))$ and $P_3 \in Syl_3(C_G(a))$. Then P_3 is a Sylow-3 subgroup of G too, and $C_G(a) = P_2 \times \langle a \rangle$. G has no element of order 12 implies that P_2 is an elementary Abelian 2-group. So $C_G(a)$ has $7 \times 2 = 14$ elements of order 6. Since $G/N = S_5$ has 10 Sylow 3-subgroups, G has 10 Sylow 3-subgroups too. Assume they are P_3, $P_3^{x_2}, \ldots, P_3^{x_{10}}$. Because $P_3 \neq P_3^{x_i}$, $C_G(a) \cap (C_G(a))^{x_i}$ has no elements of order 6 and G has at least $10 \times 14 = 140$ elements of order 6. But we know that $G/N = S_5$ has 20 elements of order 6, the inverse images of them under the natural homomorphism of $G \to G/N$ also are the elements of order of 6. Therefore G has at most $20|N| = 80$ elements of order of 6. This contradicts the former result.

Secondly, we prove that G has exactly 40 elements of order 12.

Let $ab = ba$ be an element of order 12, where $o(a) = 3, o(b) = 4$. We have proven that the elements of order 3 lie in one conjugacy class of G and the length of the conjugacy class is 20, so $12 \le |C_G(ab)| \le |C_G(a)| = 24$. Furthermore, $|(ab)^G| = 40$, or $|(ab)^G| = 20$.

Let $1 \ne z \in N \cap Z(G)$. Then zb is element of order 4 of $C_G(a)$. Let $C_G(a) = \langle a\rangle \times P_2$. Since a Sylow 2-subgroup of $G/N = S_5$ is D_8, $P_2 = D_8$ and two elements of order 4 of $P_2 = D_8$ are conjugate. So there exists $h \in P_2 \le C_G(a)$, such that $zb = b^h$. If $|(ab)^G| = 20$, then $|C_G(ab)| = |C_G(a)| = 24$. Thus $C_G(ab) = C_G(a)$. Since $h \in C_G(a) = C_G(ab)$, we have $zab = (ab)^h = ab$, a contradiction. Hence $|(ab)^G| = 40$. Now we prove that all elements of order 12 of G are in one conjugacy class. Let $x = a_1b_1$ be an element of order 12 of G, where $o(a_1) = 3$, $o(b_1) = 4$. Since all elements of order 3 of G are in one conjugacy class, there exists $g \in G$, such that $a = a_1^g$. Therefore, $x^g = a_1^g b_1^g = b_1^g a_1^g = ab_1^g = b_1^g a$, and $b_1^g \in C_G(a) = \langle a\rangle \times D_8$. So $b_1^g \in D_8$. Evidently, b is an element of order 4 of $C_G(a)$, so there exists $h \in D_8 \le C_G(a)$, such that $b_1^g = b^h$. Hence $x^g = ab_1^g = ab^h = (ab)^h$, and G has exactly 40 elements of order 12.

In the same way, we can prove that in other cases, the maximal order of elements of G is 12, and G has exactly 40 maximal order elements.

Conversely, if G is a nonsolvable group with 40 maximal order elements, by Lemma 3.1 and table 1, we know that the maximal order of elements of G is $k = 12$, $\phi(k) = 4$, $n = 10$. By Lemma 2.1, $n = n_1 + n_2 + \cdots + n_s = 10$. G is nonsolvable implies that $5 \mid n_1$. Considering n_1 is minimal, we have $n_1 = 5, 10$ and $|G| = n_1.2^a.2^u.3^v$. Let $A = A_1 = \langle a\rangle$, $o(a) = 12$, $H = C_G(A)$ and $P_3 \in Syl_3(G)$. If $v \ge 3$, then $H = C_G(A)$ has at least $2 \times (3^3 - 1) = 52$ elements of order 12 since $|C_G(A)| = 2^u.3^v$ and $k = 12$. It is contrary to that G has 40 maximal order elements. So $v \le 2$.

(I) Let $v = 2$. Then G has a simple section to be K_3- simple group. By Refs. [9] and [10], we know that the simple section is A_5 or A_6. If G has a simple section $M/K = A_5$ (M, K are normal subgroup of G), by Lemma 2.2, $\bar{G}/C_{\bar{G}}(\bar{M})\bar{M} \le \mathrm{Out}(\bar{M}) = Z_2$. If $3 \mid |K|$, then G has a principal factor L/N to be a group of order 3 (where $N \lhd L \lhd N \lhd K$). Since $5 \mid |G/N|$, and a Sylow 5-subgroup of G/N acts on L/N by conjugacy, the action is trivial. So we can conclude that G has elements of order 15, a contradiction. Therefore, $3 \dagger |K|$, and $3 \mid |C_{\bar{G}}(\bar{M})|$. But $\bar{M} = M/K = A_5$, we also get a contradiction of G having elements of order 15.

Hence G does not have a simple section A_5, and it is only possible that G has a simple section A_6. Let $M/K = A_6$. We know that A_6 has 80 elements of order 3, and $3^2 \mid |A_6|$, $3^2 \| |G|$, so $3 \dagger |K|$. Therefore, we can conclude that under natural homomorphism ($G \to \bar{G}$), the 3-part of inverse image of $M/K = A_6$ are different, i.e., M, moreover, G has at least 80 elements of order 3. On the other hand, $|H| = |C_G(a)| = 2^u.3^2$, and any Sylow 3-subgroup of G is contained in $H = C_G(a)$ conjugately. So any element of order 3 corresponds to at least two elements of order 12 since G has 40 elements of order 12, and distinct element of order 3 corresponds to distinct elements of order 12. Hence G has at most 20 elements of order 3, a contradiction.

(II) If $v = 1$, since $H = C_G(a)$, by Lemma 2.1(2), $\pi(H) = \pi(A)$. So $|H| = 2^u.3$ and P_3 char $H = C_G(A) \lhd N_G(A)$, furthermore, $N_G(A) \le N_G(P_3)$. If $n_1 = |G : N_G(A)| = 5$, since $n_1 = |G : N_G(A)| = 5$, 10, $|G : N_G(P_3)| = 1, 5$. The former implies $P_3 \lhd G$, and G is solvable, a contradiction, and the later by Sylow theorem implies $5 = 3l + 1$, a contradiction. So $n_1 = |G : N_G(A)| = 10$. Since G is nonsolvable, P_3 is not normal subgroup of G, and $|G : N_G(P_3)| = 10$

follows. That is, G has 10 Sylow 3-subgroups. Suppose they are P_3, $P_3^{x_2}$, $P_3^{x_3}, \ldots, P_3^{x_{10}}$, $x_i \in G$, $i = 2, 3, \ldots, 10$. Therefore, $H \cap H^{x_i}$ doses not contain any Sylow 3-subgroup of G (otherwise, we have a contradiction of $P_3 = P_3^{x_i}$) and $H \cap H^{x_i}$ has no elements of order 12 of G, which means that $H \cup H^{x_2} \cup H^{x_3} \cup \cdots \cup H^{x_{10}}$ has ten times as many elements of order 12 as H does. Since $|M(G)| = 40$, $u = 2$ by simple calculating. Thus $|H| = 2^2.3$, and $H = A$, $|G| = 2.5.2^{\alpha}.2^2.3$.

(A) If $\alpha = 1$, that is, $|G| = 2.5.2.2^2.3$. Let $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_m = G$ be a principal series of G. Since G is nonsolvable, G has a simple section, say, A_5. Furthermore, four cases may occur in the principal series of G: (i) $m = 3$, $1 = G_0 \lhd G_1 \lhd G_2 \lhd G_3 = G$, $G_1 = Z_2$, $G_2/G_1 = A_5$, $G/G_2 = Z_2$, (ii) $m = 3$, $G_1 = A_5$, $G_2/G_1 = Z_2$, $G/G_2 = Z_2$, (iii) $m = 2$, $G_1 = Z_2 \times Z_2$, $G/G_1 = A_5$, (iv) $m = 2$, $G_1 = A_5$, $G/G_1 = Z_2 \times Z_2$. If $G/A_5 = Z_2 \times Z_2$, we can conclude that G has no elements of order 12, a contradiction, which means that case (iv) will not occur. If (iii) is true, since $G/G_1 = A_5$, $C_G(G_1) = G_1$, moreover, we have a contradiction of $G/C_G(G_1) = A_5 \leq Aut(G_1) = S_3$. If (ii) is true, since $A_5 \lhd G$, by Lemma 2.2, $G/C_G(A_5) \times A_5 \leq \mathrm{Out}(A_5) = Z_2$. Thus $G = A_5 \times Z_2 \times Z_2$, or $G = Z_{4\varphi} \times A_5$, but the former implies that G has no elements of order 12. Therefore, $G = Z_{4\varphi} \times A_5$, where $\ker\varphi = Z_2$. If (i) is true, then G/G_1 is a nonsolvable group of order 120. So $G/G_1 \cong Z_2 \times A_5$, or $G/G_1 \cong SL(2,5)$, or $G/G_1 \cong S_5$. It is easy to see that there are at most 20 elements of order 6 in $Z_2 \times A_5, SL(2,5)$. Therefore, G has at most 20 elements of order 12, which contradicts the fact G has 40 elements of order 12. So it is only possible that $G/G_1 \cong S_5$.

Now we have proven that if $\alpha = 1$, $G \cong Z_{4\varphi} \times A_5$, where $\ker\varphi = Z_2$, or $G/Z_2 \cong S_5$, where $Z_2 \leq Z(G)$.

(B) If $\alpha = 2$, $|G| = 2.5.2^2.2^2.3 = 2^5.3.5$. We can know from the order of G that the only simple section of G is A_5. Let N, H be normal subgroups of G satisfying $N = A_5$ or $H/N = A_5$.

Case 1. If $N = A_5$, by Lemma 2.2, $G/C_G(A_5) \times A_5 \leq \mathrm{Out}(A_5) = Z_2$. If $G = C_G(A_5) \times A_5$, then $C_G(A_5) = Z_2 \times Z_2 \times Z_2$ since the maximal order of elements of G is 12. But there is no elements of order 12 in G, a contradiction. So $G/C_G(A_5) \times A_5 = Z_2$, and $C_G(A_5) = Z_2 \times Z_2$. Also, since the maximal order of elements of G is 12, we have $G \cong D_{8\varphi} \times A_5$, where $\ker\varphi = Z_2 \times Z_2$.

Case 2. If $H/N = A_5$, then N is a 2- subgroup. Let $|N| = 2^s$. Then $s \leq 3$.

(1) If $|N| = 2^3$, then $H = G$ and $G/N = A_5$ is a simple group. So $C_G(N) = N$, or $C_G(N) = G$. If $C_G(N) = N$, we have $A_5 = G/N \leq \mathrm{Aut}(N)$, furthermore, $|A_5| = 2^2.3.5$ divides $|\mathrm{Aut}(N)|$. But $|\mathrm{Aut}(N)|$ divides $2^{3(3-1)/2}.(2^3-1).(2^2-1).(2-1) = 2^3.3.7$, this is a contradiction. If $C_G(N) = G$, then $N \leq Z(G)$. Furthermore if $\exp(N) = 2$, then G has no elements of order 12; if $\exp(N) = 4$, then G has elements of order 20, which contradicts that the maximal order of elements of G is 12.

(2) If $|N| = 2^2$ (Clearly, N is an elementary Abelian subgroup, otherwise, we have a contradiction of G having elements of order 20), then $H/N = A_5$. Since $|G/N| = 120$ and $H/N = A_5$, G/N is isomorphic to $A_5 \times Z_2$ or S_5. If $G/N \cong A_5 \times Z_2 = A_5 \times \langle \bar{a} \rangle$, then $N\langle a \rangle$ is a normal subgroup with order 8 of G, we have a contradiction as same as (1). Thus $G/N \cong S_5$. Now we prove $N \cap Z(G) = Z_2$. If $N \leq Z(G)$, let a be an element of order 12 of G. Then $\bar{a}$ is an element

of order 6 of $\bar{G} = S_5$. Since all elements of order 6 of S_5 belong to one conjugacy class, for any element $\bar{b}$ of order 6 of S_5, there exists $g \in G$ such that $b = a^g z$, $z \in N \leq Z(G)$. Therefore b is an element of order 12 of G, that is, the inverse images (under the natural homomorphism) of all elements of order 6 of $\bar{G} = S_5$ are the elements of order 12 of G. So we can conclude that G has 80 elements of order 12, a contradiction. Moreover, $C_H(N) = N$ or $C_H(N) = H$ since $H/N = A_5$. If $C_H(N) = N$, then $A_5 = H/N = H/C_H(N) \leq \mathrm{Aut}(N) = S_3$, a contradiction. Hence $C_H(N) = H$, from here we can know that all Sylow 3-subgroups and Sylow-5 subgroups of G are contained in $C_H(N) = H$. Let P_2 be a Sylow 2-subgroup of G, then $N \lhd P_2$. So let $a \in N \cap Z(P_2) > 1$. Then $P_2 \leq C_G(a)$, and $a \in Z(G)$ follows. Therefore, $N \cap Z(G) = Z_2$.

(3) If $|N| = 2$, let $\bar{G} = G/N$, $\bar{H} = H/N = A_5$. Then $\bar{G}/C_{\bar{G}}(\bar{H}) \leq Aut(\bar{H}) = S_5$. Since $\bar{H}$ is a simple group and the maximal order of elements of G is 12, $C_{\bar{G}}(\bar{H})$ does not contain elements of order 3, 5. So $C_{\bar{G}}(\bar{H}) = 1$, or $C_{\bar{G}}(\bar{H})$ is a 2-group. If the former is true, then $\bar{G} \leq S_5$, but $|\bar{G}| = 2^4.3.5 > |S_5|$, a contradiction. Therefore, $C_{\bar{G}}(\bar{H}) = \bar{K}$ is a 2-group. Since $\bar{K} \lhd \bar{G}$, we have $K \lhd G$. But $|K| = 2^t$, $t > 2$, we have solved this case in (1), or (2).

Therefore we have proven that: if $\alpha = 2$, then $G \cong D_{8\varphi} \times A_5$, where $\ker\varphi = Z_2 \times Z_2$, or $G/N \cong S_5$, where $N = Z_2 \times Z_2$, $N \cap Z(G) = Z_2$.

This proves the sufficiency of Theorem 3.2. The proof is completed.

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