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The Dimensions of Spline Spaces on Quasi-Rectangular Meshes

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Abstract A quasi-rectangular mesh (denoted by Δ_{QR}) is basically a rectangular mesh (Δ_R) that allows local modifications, including T-mesh (Δ_T) and L-mesh (Δ_L). In this paper, the dimensions of the bivariate spline spaces $S_k^\mu(\Delta_{QR})$ are discussed by using the Smoothing Cofactor-Conformality method. The dimension formulae are obtained with some constraints depending on the order of the smoothness, the degree of the spline functions and the structure of the mesh as well.

Keywords bivariate spline; smoothing cofactor-conformality method; dimension formula; quasi-rectangular mesh; T-mesh; L-mesh.

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1. Introduction

As we know, tensor-product B-splines which are a basic tool in computer aided geometric design, are defined on rectangular meshes. B-spline surfaces have a weakness that the control points must lie topologically on a rectangular grid. To overcome this limitation, Sederberg et al.[1,2] invented T-spline, which is a pointbased spline defined on T-mesh. Then Deng et al.[3] proposed a method based on B-nets to calculate the dimension of a spline space $\mathcal{S}(m,n,\alpha,\beta,\Delta_T)$ on a T-mesh with constraints $m \geq 2\alpha+1$ and $n \geq 2\beta+1$. Recently, Li et al.[4] improved the dimension formulae of the same spline spaces on the T-mesh by using the Smoothing Cofactor-Conformality method[5,6].

In this paper, the dimension of the bivariate spline space $S_k^\mu(\Delta_{QR})$ on quasi-rectangular mesh is further discussed by using the Smoothing Cofactor-Conformality method. The dimension formulae are obtained with some constraints depending on the order of the smoothness, the degree of the spline functions and the structure of the mesh as well.

We use the same definitions and notations as in [4]. Given a rectangular mesh (denoted by Δ_R, as shown in Fig.1(a)), we modify it locally and get two new meshes (as shown in Fig.1(b) and Fig.1(c)). There are three kinds of interior mesh segments.

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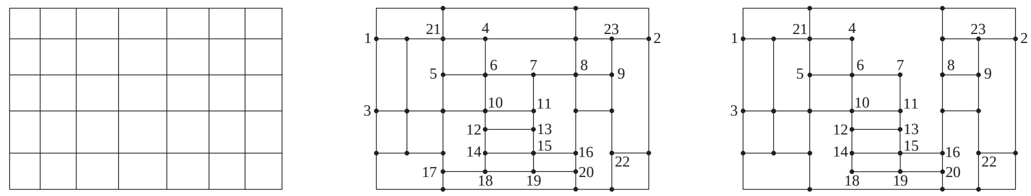


(a) Δ_R (b) Δ_T (c) Δ_L

Figure 1 A rectangular mesh (a) and the corresponding locally modified T-mesh (b) and L-mesh (c).

a) cross-cut: both of its endpoints lie on the boundary of the mesh, e.g., v_1v_2 in Fig.1(b).

b) ray: only one of its endpoints lies on the boundary of the mesh, e.g., v_3v_{11} in Fig.1(b) and v_1v_4 in Fig.1(c).

c) Truncated-segment (or T-segment) : both of its endpoints do not lie on the boundary of the mesh, e.g., v_5v_9 in Fig.1(b) and v_5v_7 in Fig.1(c).

According to the three kinds of mesh segments, there are also three kinds of interior vertices.

a) free-vertex: the vertex is an intersection point of two cross-cuts, or two rays, or one cross-cut and one ray, e.g., v_{21}, v_{23} in Fig.1(b) and Fig.1(c).

b) mono-vertex: the vertex is an intersection point of one T-segment and one cross-cut or one ray, e.g., v_4, v_{10} in Fig.1(b) and Fig.1(c).

c) multi-vertex: the vertex is an intersection point of two T-segments, e.g., v_6, v_{15} in Fig.1(b) and Fig.1(c).

We define some relations between two mesh segments.

a) T-connected: two segments are T-connected if an endpoint of one segment is an interior point of the other segment. The intersection point is called a T-junction, e.g., v_5v_9 and v_7v_{19} are T-connected at v_7 in Fig.1(b), and v_7 is a T-junction vertex.

b) L-connected: two segments are L-connected if they have one common endpoint. The intersection point is called an L-junction, e.g., v_5v_7 and v_7v_{19} are L-connected at v_7 in Fig.1(c), and v_7 is an L-junction vertex.

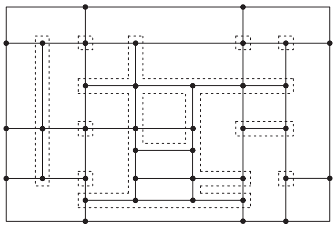
c) connected component: the union of all connected T-segments and their vertices.

A connected component is called a T-connected component if it contains T-junction vertices, as every independent part closed by dotted line shown in Fig.2(a), then the mesh is called a T-mesh. Similarly, a connected component is called an L-connected component if it contains L-junction vertices, as the central part closed by dotted line shown in Fig.2(b), then the mesh is called an L-mesh. By the definition, the whole T-mesh (or L-mesh) can be decomposed into many different T-connected components (or L-connected components) without intersections.

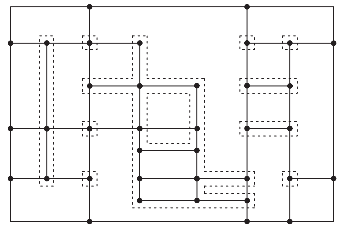
Given a quasi-rectangular mesh Δ_{QR}. Let $\mathcal{F}$ denote all the cells in Δ_{QR}, and Ω denote the region occupied by all the cells in Δ_{QR}. A bivariate spline space defined on Δ_{QR} is

$$S_k^{\mu}(\Delta_{QR}) := \{s(x,y) \in C^{\mu}(\Omega)|\ s(x,y)|_{\phi} \in \mathbb{P}_k, \forall \phi \in \mathcal{F}\},$$

where $\mathbb{P}_k$ denotes the space of all bivariate polynomials of total degree at most k.



(a) Δ_T



(b) Δ_L

Figure 2 (a) A T-mesh and its T-connected components, (b) an L-mesh and its L-connected components.

2. The dimensions of spline spaces $S_k^\mu(\Delta_{QR})$

2.1 The conformality conditions along T-segments

At first, we introduce the conformality conditions on $S_k^\mu(\Delta_{QR})$. By using the Smoothing Cofactor-Conformality method[5,6], there exist $p_i(x,y) \in \mathbb{P}_{k-\mu-1}$, the corresponding smoothing cofactors of each horizontal grid edge, and $q_j(x,y) \in \mathbb{P}_{k-\mu-1}$, the corresponding smoothing cofactors of each vertical grid edge. For each interior vertex (x_i, y_j) denoted by $v_{i,j}$, the conformality condition is

$$(p_i(x,y) - p_{i-1}(x,y))(y-y_j)^{\mu+1} + (q_{j-1}(x,y) - q_j(x,y))(x-x_i)^{\mu+1} = 0. \tag{1}$$

When $k \geq 2\mu+2$, there exists a conformality cofactor $d_{i,j}(x,y) \in \mathbb{P}_{k-2\mu-2}$ at the interior vertex $v_{i,j}$, satisfying

$$\begin{cases} p_i(x,y) - p_{i-1}(x,y) = d_{i,j}(x,y)(x-x_i)^{\mu+1}, \\ q_j(x,y) - q_{j-1}(x,y) = d_{i,j}(x,y)(y-y_j)^{\mu+1}. \end{cases} \tag{2}$$

In order not to get the zero conformality cofactor, we suppose $k \geq 2\mu+2$. Otherwise, the T-segment dose not exist actually.

By Eq. (2), for a horizontal T-segment including N interior vertices, the corresponding conformality factor $d_{i,j}$ $(i = 1, 2, \ldots, N)$ at each vertex must satisfy the conformality condition along the horizontal T-segment as follows

$$\sum_{i=1}^{N} d_{i,j}(x,y)(x-x_i)^{\mu+1} = 0. \tag{3}$$

Similarly, the conformality condition along the vertical T-segment is

$$\sum_{j=1}^{N} d_{i,j}(x,y)(y-y_j)^{\mu+1} = 0, \tag{4}$$

where N denotes the number of the interior vertices on the vertical T-segment.

Therefore, the global conformality conditions of a connected component are composed of the conformality equations along its all T-segments, and the global conformality conditions of different connected components are indpendent of each other. Similar to the spline space on the cross-cut patition[6], the source cell has $\frac{(k+2)(k+1)}{2}$ degree of freedom, and each cross-cut has a free smoothing cofactor. Thus, the dimensions of the spline spaces on the whole partition are given by the following lemma.

Lemma 1 *Given a quasi-rectangular mesh Δ_{QR} with L_C cross-cuts and T different connected components $\mathcal{T}_i,\ i=1,2,\ldots,T$. Then the dimension of the spline space defined on Δ_{QR} is*

$$\dim S_k^{\mu}(\Delta_{QR}) = \binom{k+2}{2} + L_C\binom{k-\mu+1}{2} + \sum_{i=1}^{T}\dim\mathcal{T}_i,$$

where $\binom{k}{\mu} = \dfrac{k!}{\mu!(k-\mu)!}$, and $\dim\mathcal{T}_i$ denotes the dimension of the global conformality conditions corresponding to the i-th connected component.

2.2 The dimensions of the conformality conditions along the T-segments

By Lemma 1, if we obtain the dimension of each connected component, then we obtain the dimension of the whole spline space. Now we consider the dimension of the conformality condition along any T-segment including N vertices. We have the following results.

Lemma 2 *Let $k \geq 2\mu+2$, $d_i(x,y) \in \mathbb{P}_{k-2\mu-2}$. The dimension of the solution space of the system of equations $\sum_{i=1}^{N} d_i(x,y)(x-x_i)^{\mu+1} = 0$ is*

$$\dim_T(N) = \frac{1}{2}(k-2\mu-1-[\frac{\mu+1}{N-1}])_+((N-1)k-2N\mu-2+(N-1)[\frac{\mu+1}{N-1}]), \tag{5}$$

where $[x]$ denotes the largest integer not greater than x, $\mu_+ = \max\{0,\mu\}$.

If $N > \mu+2$,

$$\dim_T(N) = (N-K_0)\binom{k-2\mu}{2}, \tag{6}$$

where $K_0 = \dfrac{k+2}{k-2\mu}$.

Proof Since $d_i(x,y) \in \mathbb{P}_{k-2\mu-2}$, there exists $a_j^i(x) \in \mathbb{P}_{k-2\mu-2-j},\ j=0,1,\ldots,k-2\mu-2$, such that

$$d_i(x,y) = a_0^i(x) + a_1^i(x)y + \cdots + a_{k-2\mu-2}^i(x)y^{k-2\mu-2} = \sum_{j=0}^{k-2\mu-2} a_j^i(x)y^j.$$

Then

$$\sum_{i=1}^{N} d_i(x,y)(x-x_i)^{\mu+1} = \sum_{i=1}^{N}\sum_{j=0}^{k-2\mu-2} a_j^i(x)(x-x_i)^{\mu+1}y^j = 0.$$

Since all coefficients of y^j equal to 0, we have $k-2\mu-1$ independent systems of equations

$$\sum_{i=1}^{N} a_j^i(x)(x-x_i)^{\mu+1} = 0,\ j=0,1,\ldots,k-2\mu-2.$$

Now we focus on the system of equations

$$\sum_{i=1}^{N} a^i(x)(x+x_i)^{\mu+1} = 0. \tag{7}$$

We can expand each $a^i(x) \in \mathbb{P}_{k-2\mu-2-j}$ as

$$a^i(x) = c_0^i + c_1^i(x+x_i) + \cdots + c_{k-2\mu-2-j}^i (x+x_i)^{k-2\mu-2-j},$$

where $c_m^i \in \mathbb{R},\ m = 0, 1, \ldots, k-2\mu-2-j$. Then

$$a^i(x)(x+x_i)^{\mu+1} = c_0^i(x+x_i)^{\mu+1} + c_1^i(x+x_i)^{\mu+2} + \cdots + c_{k-2\mu-2-j}^i (x+x_i)^{k-\mu-1-j}.$$

Denote

$$C = (c_1, c_2, \ldots, c_N)^{\mathrm{T}},$$

where

$$c_i = (c_{k-2\mu-2-j}^i, \ldots, c_0^i)^{\mathrm{T}}$$

and

$$A = (A_1, A_2, \ldots, A_N),$$

where $A_i\ (i = 1, 2, \ldots, N)$ is a $(k-\mu-j) \times (k-2\mu-1-j)$ matrix of the following form

$$A_i = \begin{pmatrix} 1 & & & \\ \binom{k-\mu-1-j}{1} x_i & 1 & & \\ \binom{k-\mu-1-j}{2} x_i^2 & \binom{k-\mu-2-j}{1} x_i & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ \binom{k-\mu-1-j}{k-\mu-1-j} x_i^{k-\mu-1-j} & \binom{k-\mu-2-j}{k-\mu-2-j} x_i^{k-\mu-2-j} & \cdots & \binom{\mu+1}{\mu+1} x_i^{\mu+1} \end{pmatrix}.$$

Then the systems of equations (7) is equivalent to the linear system $AC = 0$. Using the method similar to that in [7], we have

$$\mathrm{rank}(A) = \min\{k-\mu-j, N(k-2\mu-1-j)\}.$$

Then the dimension of the solution space is

$$(N(k-2\mu-1-j) - (k-\mu-j))_+,\ j = 0, 1, \ldots, k-2\mu-2.$$

Thus, the dimension of the solution space $\sum_{i=1}^N d_i(x,y)(x-x_i)^{\mu+1} = 0$ is

$$\begin{aligned} \dim_T(N) &= \sum_{j=0}^{k-2\mu-2} (N(k-2\mu-1-j) - (k-\mu-j))_+ \\ &= \frac{1}{2}(k-2\mu-1-[\frac{\mu+1}{N-1}])_+((N-1)k - 2N\mu - 2 + (N-1)[\frac{\mu+1}{N-1}]). \end{aligned}$$

This result agrees with the formula of the conformality condition at one point given in [6].

If $N > \mu+2$, then

$$\dim_T(N) = N\binom{k-2\mu}{2} - \binom{k-\mu+1}{2} + \binom{\mu+2}{2} = (N-K_0)\binom{k-2\mu}{2}.$$

Similarly, the dimension of the solution space $\sum_{i=1}^N d_j(x,y)(y-y_j)^{\mu+1} = 0$ is the same. $\square$

Let $K_1 = \dfrac{k-\mu}{k-2\mu-1}$ $(\le K_0 \le \mu+2,\ \text{when}\ k \ge 2\mu+2)$. By the above result of dimension, $N > K_1$ is a necessary condition for one T-segment with N interior vertices not to degenerate.

The corresponding matrix A shows that, when $N > \mu+2$, there are $N-K_0$ conformality cofactors belonging to the basis of the solution space. Table 1 shows how K_0 and K_1 vary with k.

k	$2\mu+2$	$2\mu+3$	$2\mu+4$	$2\mu+5$	$\cdots$
K_0	$\mu+2$	$\frac{2\mu+5}{3}$	$\frac{2\mu+6}{4}$	$\frac{2\mu+7}{5}$	$\cdots$
K_1	$\mu+2$	$\frac{\mu+3}{2}$	$\frac{\mu+4}{3}$	$\frac{\mu+5}{4}$	$\cdots$

Table 1 K_0 and K_1 vary with k

2.3 The dimensions of spline spaces $S_k^\mu(\Delta_T)$

Given a T-mesh, by using the regular order of T-segments defined in [4]. we can obtain the dimensions of the global conformality conditions on T-connected component with some constraints as in the following lemma.

Lemma 3 *Let $\mathcal{T}_i$ be the i-th T-connected component. There are N_i T-segments and V_i interior vertices in $\mathcal{T}_i$. Each T-segment L_s contains h_s interior vertices including $h_s^{(1)}$ mono-vertices and $h_s^{(2)}$ multi-vertices except the two end vertices (i.e. $h_s = h_s^{(1)} + h_s^{(2)} + 2$). Suppose $h_s > \mu + 2$ and $h_s^{(1)} + 2 \geq K_0$. Then the dimension of $\mathcal{T}_i$ is*

$$\dim \mathcal{T}_i = V_i\binom{k-2\mu}{2} - N_i(\binom{k-\mu+1}{2} - \binom{\mu+2}{2}). \tag{8}$$

Proof Case 1 When $\mathcal{T}_i$ has no T-cycle, the N_i T-segments can be arranged in a regular order as

$$L_{N_i} \to L_{N_i-1} \to \cdots \to L_2 \to L_1.$$

Denote by t_s the number of interior multi-vertices on L_s, which are common vertices with the latter $s-1$ T-segments $L_{s-1}, \ldots, L_1$. It is clear that $t_1 = 0$, $t_s \leq h_s^{(2)}$, $s = 2, \ldots, N_i$. The global conformality conditions of $\mathcal{T}_i$ are composed of N_i conformality equations along the N_i T-segments. Then the dimension of the global conformality conditions can be determined by the inverse of the regular order.

1) Let L_1 have its all degree of freedom $\dim L_1 = \dim_T(h_1)$.

2) By $h_s^{(1)} + 2 \geq K_0$, $s = 2, \ldots, N_i$, for L_s, even if its $h_s^{(2)}$ conformality cofactors are determined by $L_1, \ldots, L_{s-1}$, the rest $h_s^{(1)} + 2$ conformality cofactors have $(h_s^{(1)} + 2 - K_0)\binom{k-2\mu}{2}$ degree of freedom. Then L_s has its real degree of freedom $\dim L_s - t_s\binom{k-2\mu}{2}$. Notice that $\sum_{s=1}^{N_i}(h_s - t_s) = V_i$, we have

$$\begin{aligned}
\dim \mathcal{T}_i &= \sum_{s=1}^{N_i}(\dim L_s - t_s\binom{k-2\mu}{2}) \\
&= \sum_{s=1}^{N_i}(h_s - K_0 - t_s)\binom{k-2\mu}{2} \\
&= \sum_{s=1}^{N_i}(h_s - t_s)\binom{k-2\mu}{2} - N_i K_0\binom{k-2\mu}{2}
\end{aligned}$$

$$= V_i\binom{k-2\mu}{2} - N_i(\binom{k-\mu+1}{2} - \binom{\mu+2}{2}).$$

Csse 2 When $\mathcal{T}_i$ has T-cycles, by using the same method as in [4], each T-cycle can be dismissed by adding a virtual mono-vertex. Then the dimension formula (8) is right. □

Remark 1 (a) When the T-connected component contains only one free-vertex, $\dim \mathcal{T}_i = \binom{k-2\mu}{2}$. That is the degree of freedom of one conformality cofactor.

(b) When the T-connected component contains only one T-segment, by Lemma 2 and Eq. (5), the constraint $h_s^{(1)}+2 \geq K_0$ can be relaxed as $h_s > K_1$, then $\dim \mathcal{T}_i = \dim_T(h_s)$. If $h_s \leq K_1$, then $\dim \mathcal{T}_i = 0$, the T-segment will vanish from the T-mesh.

By Lemmas 1 and 3, we can obtain the dimension of spline space $S_k^\mu(\Delta_T)$ with constraints $h_s > \mu+2$ and $h_s^{(1)}+2 \geq K_0$.

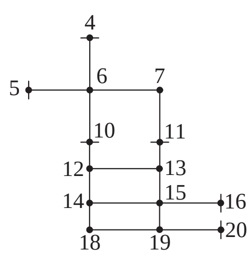
Theorem 1 *Given a T-mesh Δ_T, which includes L_C cross-cuts, L_T T-segments and V interior vertices, and the s-th T-segment L_s contains h_s interior vertices, including $h_s^{(1)}$ mono-vertices except the two end vertices. Suppose $h_s > \mu+2$, $h_s^{(1)}+2 \geq K_0$. Then the dimension of the spline space $S_k^\mu(\Delta_T)$ $(k \geq 2\mu+2)$ is*

$$\dim S_k^\mu(\Delta_T) = \binom{k+2}{2} + V\binom{k-2\mu}{2} + (L_C - L_T)\binom{k-\mu+1}{2} + L_T\binom{\mu+2}{2}. \tag{9}$$

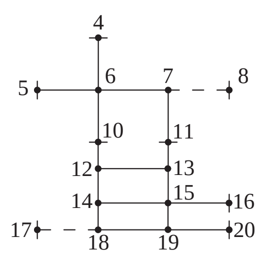
Remark 2 If each T-connected component contains one T-segment at most, and the constraints can be relaxed as $h_s > K_1$ (such that the T-segment does not degenerate), then the dimension $\dim S_k^\mu(\Delta_T)$ can be calculated by Lemma 1 and Eq. (5) in Lemma 2.

2.4 The dimensions of spline spaces $S_k^\mu(\Delta_L)$

The difference between T-mesh and L-mesh is that L-mesh contains L-junctions (as shown in Fig.1 (c)), that is, two T-segments have one common end point. Hence, each L-junction can be changed into a T-junction by adding a virtual mono-vertex to one of the T-segments. As shown in Fig. 3, an L-connected component is changed into a T-connected component by adding two virtual mono-vertices v_8 and v_{17}. Similarly, an L-cycle is changed into a T-cycle by adding six virtual mono-vertices $v_4, v_5, v_8, v_{14}, v_{18}, v_{21}$, as shown in Fig.4.



(a) Δ_L



(b) Δ_T

Figure 3 An L-connected component is changed into a T-connected component by adding two virtual mono-vertices.

When an L-mesh (Δ_L) is changed into a T-mesh (Δ_T) by adding N virtual mono-vertices,

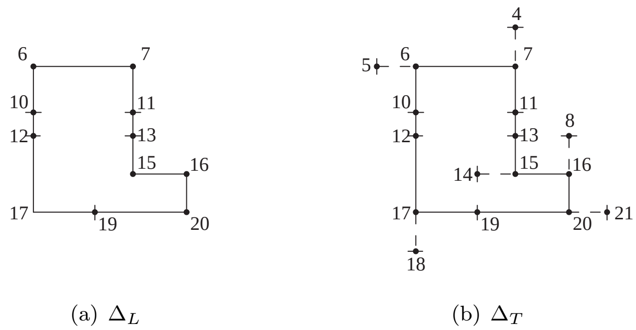


Figure 4 An L-cycle is changed into a T-cycle by adding six virtual mono-vertices.

the dimension of the spline space on Δ_T can be obtained by Eq. (9) ($\dim S_k^\mu(\Delta_T)$). Then let all added virtual mono-vertices vanish again. We can get the dimension of spine space on L-mesh $\dim S_k^\mu(\Delta_L)$. In fact, if the N virtual mono-vertices have all degree of freedom in $\dim S_k^\mu(\Delta_T)$, then

$$\dim S_k^\mu(\Delta_L) = \dim S_k^\mu(\Delta_T) - N\binom{k-2\mu}{2}.$$

The correctness of the above procedure can be guaranteed by hypothesis $h_s > \mu+2$ and $h_s^{(1)}+1 \geq K_0$ according to the proof of Lemma 3. Thus, the dimension formula can be generalized to the spline space on L-mesh $S_k^\mu(\Delta_L)$.

Theorem 2 *Given an L-mesh Δ_L, which includes L_C cross-cuts, L_T T-segments and V interior vertices, and the s-th T-segment L_s contains h_s interior vertices, including $h_s^{(1)}$ mono-vertices except the two end vertices. Suppose $h_s > \mu+2$ and $h_s^{(1)}+1 \geq K_0$. Then the dimension of the spline space $S_k^\mu(\Delta_L)$ $(k \geq 2\mu+2)$ is*

$$\dim S_k^\mu(\Delta_L) = \binom{k+2}{2} + V\binom{k-2\mu}{2} + (L_C - L_T)\binom{k-\mu+1}{2} + L_T\binom{\mu+2}{2}. \tag{10}$$

Remark 3 Similarly to the dimension formula in [4], if the number of mono-vertices does not satisfy the constraints in Theorem 1 or Theorem 2, then the corresponding dimension formula is a lower bound of the real dimension.

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