Journal of Mathematical Research & Exposition
Nov., 2008, Vol. 28, No. 4, pp. 753–758
DOI:10.3770/j.issn:1000-341X.2008.04.002
Http://jmre.dlut.edu.cn

Mean Bounded Variation Condition and Applications in Fourier Analysis

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Abstract This announcement is to raise an ultimate generalization to monotonicity condition on the Fourier (trigonometric) coefficient sequences. We prove this condition cannot be weakened any further to guarantee the uniform convergence of the sine series. Some interesting and important classical results in Fourier analysis are re-established under this ultimate condition. Over ninty year research history is surveyed in this announcement.The first original paper of this series of papers is posted in arXiv:math.CA/0611805 v1, November 27, 2006.

Keywords MVBVS; trigonometric series; convergence; integrability; best approximation.

Document code A
MR(2000) Subject Classification 42A20; 42A32
Chinese Library Classification O174.2

1. Introduction

It is well known that there are a great number of interesting results in Fourier analysis established by assuming monotonicity of coefficients. For example, Chaundy and Jolliffe[1] proved that if $\{a_n\}$ is a nonnegative and non-increasing (monotonic) real sequences (in symbol, $\{a_n\} \in$ MS, i.e, Monotone Sequence) with $\lim_{n\to\infty} a_n = 0$, then a necessary and sufficient condition for the uniform convergence of sine series

$$\sum_{n=1}^{\infty} a_n \sin nx \tag{1.1}$$

is $\lim_{n\to\infty} na_n = 0$. This classical result, together with many other convergence results of series (1.1), such as L^1-convergence, L^p-convergence, and best approximation, have been generalized by loosing the condition $\{a_n\} \in$ MS to $\{a_n\} \in$ CQMS (Classical Quasi-monotone Sequences), $\{a_n\} \in$ RVQMS (O-regularly Varying Quasi-monotone Sequences), $\{a_n\} \in$ RBVS (Rest Bounded Variation Sequences), $\{a_n\} \in$ GBVS (Group Bounded Variation Sequences), and $\{a_n\} \in$ NBVS (Non-onesided Bounded Variation Sequences). Readers can find details of the definitions of all above sequences and generalizations of Chaundy and Jolliff's results in [9]. We introduce the

Received date: 2006-12-22; **Accepted date**: 2007-03-26
Foundation item: Open Funds of State Key Laboratory of Oil and Gas Reservoir and Exploitation of Southwest Petroleum University (No. PCN0613); NSERC of Canada; the NSERC RCD grant and AARMS of Cananda.

following new kind of sequences, which contains all classes of sequences mentioned earlier:

Definition 1 *A nonnegative sequence $\mathbf{A}=\{a_n\}$ is said to be a mean value bounded variation sequence ($\{a_n\}\in$ MVBVS) if there is a $\lambda\ge 2$ such that*

$$\sum_{k=n}^{2n}|a_k-a_{k+1}|\le\frac{C(\mathbf{A})}{n}\sum_{k=[\lambda^{-1}n]}^{[\lambda n]}a_k$$

holds for all $n=1,2,\ldots$ and some constant $C(\mathbf{A})$ depending only upon the sequence A.

We can generalize the definition of MVBVS to complex case as follows:

Definition 2 *Let $\mathbf{C}:=\{c_n\}_{n=0}^{\infty}$ be a sequence satisfying $c_n\in K(\theta_1):=\{z:|\arg z|\le\theta_1\}$ for some $\theta_1\in[0,\pi/2)$ and $n=1,2,\ldots$. If there is a number $\lambda\ge 2$ such that*

$$\sum_{k=m}^{2m}|\Delta c_k|:=\sum_{k=m}^{2m}|c_k-c_{k+1}|\le C(\mathbf{C})\frac{1}{m}\sum_{k=[\lambda^{-1}m]}^{\lambda m}|c_k|$$

holds for all $m=1,2,\ldots$, then we say that the sequence $\mathbf{C}$ belongs to the class MVBVS in complex sense.

A nonnegative sequence $\mathbf{A}:=\{a_n\}$ is said to be an almost monotonic sequence ($\mathbf{A}\in$ AMS) if there is a positive constant $C(\mathbf{A})$ such that $a_k\le C(\mathbf{A})b_n$ for all $k\ge n$. Evidently, AMS contains RVQMS $\cup$ RBVS, but it is not comparable with GBVS, NBVS or MVBVS (we refer to [9] for more discussion on this). We summarize the generalization of the monotone conditions in the following two figures. Figure 1 shows the development of the generalization successively, while Figure 2 shows the relations of the different generalized classes of monotonic sequences.

Figure 1 History of Development

Figure 2 Relationship of Coefficient Sequences

2. Main results

An AMS looks easy to manage, but the following result shows that it cannot guarantee the uniform convergence of a trigonometric series.

Theorem 1 *There exists a sequence* $\{a_n\} \in \mathrm{AMS}$ *with* $\lim_{n\to\infty} na_n = 0$ *such that the series (1.1) is not uniformly convergent.*

The following theorem extends the monotonic condition on the sequence $\{a_n\}$ in the classical Chaundy-Jollif Theorem to MVBV condition:

Theorem 2 *If* $\mathbf{A} := \{a_n\} \in \mathrm{MVBVS}$*, then a necessary and sufficient condition either for the uniform convergence of series (1.1), or for the continuity of its sum function* f*, is that* $\lim_{n\to\infty} na_n = 0$.

We conclude that MVBV condition is the ultimate condition to generalize the Chaundy-Jolliffe's result, that is, MVBV condition cannot be weakened any further to guarantee the uniform convergence of the series (1.1). In fact, we have

Theorem 3 *Let* M_n *be a given nonnegative increasing sequence tending to infinity. Then there exists a sine series of the form (1.1) with* $\lim_{n\to\infty} na_n = 0$ *such that for any given* $\lambda \geq 2$,

$$\lim_{n\to\infty} \frac{\sum_{k=n}^{2n} |\Delta a_k|}{\frac{M_n}{n}\sum_{k=[\lambda^{-1}n]}^{[\lambda n]} |a_k|} = 0,$$

however, the series is not uniformly convergent.

Given a trigonometric series $\sum_{k=-\infty}^{\infty} c_k e^{ikx} := \lim_{n\to\infty} \sum_{k=-n}^{n} c_k e^{ikx}$, write

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

for those points x where the series converges. Denote its nth partial sum $\sum_{k=-n}^{n} c_k e^{ikx}$ again by $S_n(f,x)$. We generalize the Chaundy-Jolliffe's result to the complex spaces by establishing the following:

Theorem 4 *Let* $\mathbf{C} = \{c_n\}$ *be a complex sequence satisfying*

$$c_n \in K(\theta_0) \text{ and } c_n + c_{-n} \in K(\theta_0),\ n = 1, 2, \dots$$

for some $\theta_0 \in [0, \pi/2)$ *and* $\mathbf{C} \in \mathrm{MVBVS}$. *Then the necessary and sufficient conditions for* $f \in C_{2\pi}$ *and* $\lim_{n\to\infty} \|f - S_n(f)\| = 0$ *are that*

$$\lim_{n\to\infty} nc_n = 0$$

and

$$\sum_{n=1}^{\infty} |c_n + c_{-n}| < \infty.$$

Next we show some important classical results (previously appearing in [2–11] etc.) still keep true by applying MVBV condition.

Denote by $E_n(f)$ the best approximation of f by trigonometric polynomials of degree n. Then

Theorem 5 *Let* $\left\{\hat{f}(n)\right\}_{n=0}^{\infty} \in \mathrm{MVBVS}$, $\left\{\hat{f}(n)+\hat{f}(-n)\right\}_{n=0}^{\infty} \in \mathrm{MVBVS}$, *and*

$$f(x)=\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}.$$

Then $f \in C_{2\pi}$ *if and only if*

$$\lim_{n\to\infty} n\hat{f}(n)=0,$$

and

$$\sum_{n=1}^{\infty} |\hat{f}(n)+\hat{f}(-n)|<\infty.$$

Furthermore, if $f \in C_{2\pi}$, *then*

$$E_n(f) \sim \max_{1\le k\le n} k\left(|\hat{f}(n+k)|+|\hat{f}(-n-k)|\right)+\max_{k\ge 2n+1} k|\hat{f}(k)-\hat{f}(-k)|+\sum_{k=2n+1}^{\infty} |\hat{f}(k)+\hat{f}(-k)|.$$

Let $f(x)=\sum_{n=0}^{\infty} c_n \cos nx$. The following corollary is an interesting application of Theorem 5 to a hard problem in classical Fourier analysis (see [12], for background).

Let $\{\hat{f}(n)\} \in \mathrm{MVBVS}$ be a real sequence. If $f \in C_{2\pi}$ and

$$\sum_{k=n+1}^{2n} \hat{f}(k)=O\left(\max_{1\le k\le n} k\hat{f}(n+k)\right),$$

then

$$\|f-S_n(f)\|=O(E_n(f)).$$

Let $L_{2\pi}$ be the space of all complex valued integrable functions $f(x)$ of period 2π with the norm $\|f\|_L=\int_{-\pi}^{\pi} |f(x)|dx$. Denote the Fourier series of $f \in L_{2\pi}$ by $\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}$.

Theorem 6 *Let* $f(x) \in L_{2\pi}$ *be a complex valued function. If the Fourier coefficients* $\hat{f}(n)$ *of* f *satisfy that* $\{\hat{f}(n)\}_{n=0}^{+\infty} \in \mathrm{MVBVS}$ *and*

$$\lim_{\mu\to 1+0} \limsup_{n\to\infty} \sum_{k=n}^{[\mu n]} |\Delta\hat{f}(k)-\Delta\hat{f}(-k)|\log k=0.$$

Then

$$\lim_{n\to\infty} \|f-S_n(f)\|_L=0$$

if and only if

$$\lim_{n\to\infty} \hat{f}(n)\log|n|=0.$$

Let $E_n(f)_L$ be the best approximation of a complex valued function $f \in L_{2\pi}$ by trigonometric polynomials of degree n in the integral metric. We establish the following L^1-approximation theorem:

Theorem 7 *Let $f(x) \in L_{2\pi}$ be a complex valued function and $\{\psi_n\}$ a decreasing sequence tending to zero with*

$$\psi_n \sim \psi_{2n}.$$

If both $\{\hat{f}(n)\}_{n=0}^{+\infty} \in \mathrm{MVBVS}$ and $\{\hat{f}(-n)\}_{n=0}^{+\infty} \in \mathrm{MVBVS}$, then

$$\|f - S_n(f)\|_L = O(\psi_n)$$

if and only if

$$E_n(f)_L = O(\psi_n) \quad \text{and} \quad \hat{f}(n)\log|n| = O(\psi_{|n|}).$$

Let $L^p,\ 1 < p < \infty$, be the space of all p-power integrable functions of 2π equipped with the norm

$$\|f\|_p = \left(\int_{-\pi}^{\pi} |f(x)|^p \mathrm{d}x\right)^{1/p}.$$

Write

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx, \quad g(x) = \sum_{k=1}^{\infty} b_k \sin kx$$

for those x where the series converge. Denote by $\phi(x)$ either $f(x)$ or $g(x)$ and let λ_n be its associated Fourier coefficients, i.e., λ_n is either a_n or b_n.

Theorem 8 *Let $1 < p < \infty$. If $\{\lambda_n\} \in \mathrm{MVBVS}$, then $x^{-\gamma}\phi(x) \in L^p,\ 1/p - 1 < \gamma < 1/p$, if and only if*

$$\sum_{n=1}^{\infty} n^{p+p\gamma-2}\lambda_n^p < \infty. \tag{2.1}$$

Let $f(x) \in L^p,\ 1 < p < \infty$ and $1/p - 1 < \gamma < 1/p$. Define the weighted modulus of continuity in L^p norm as follows:

$$\omega(f,h)_{p,x^{-\gamma}} := \omega(f,h)_{p,\gamma} := \sup_{|t|\le h} \left\|x^{-\gamma}\left(f(x+t) - f(x)\right)\right\|_p.$$

Theorem 9 *Let $1 < p < \infty$. If $\{\lambda_n\} \in \mathrm{MVBVS}$ satisfies (2.1), then for $1/p - 1 < \gamma < 1/p$, we have*

$$\omega(\phi, \frac{1}{n})_{p,\gamma} \le Cn^{-1}\left(\sum_{k=1}^{n-1} k^{2p+p\gamma-2}\lambda_k^p\right)^{1/p} + C\left(\sum_{k=n}^{\infty} k^{p+p\gamma-2}\lambda_k^p\right)^{1/p}.$$

Theorem 9 above is the first result on the relations among Fourier coefficients and the weighted modulus of continuity in L^p-norm.

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