Operator-Valued Semicircular Distribution and Its Asymptotically Free Matrix Models

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Abstract The moments of operator-valued semicircular distribution are calculated and a new relation between random variables which is called semi-independence is introduced. The asymptotically free matrix models of operator-valued semicircular distribution are given and a method is found to determine the freeness of some semicircular variables.

Keywords   operator-valued semicircular distribution; operator-valued random matrix; semi-independence.

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1. Introduction

Free probability theory, due to Voiculescu D., is a new subject of operator algebras. Originally, Voiculescu D. developed this new theory in order to understand the elusive II_1 free group factor. At present, the research on free probability has attracted many mathematicians in the past ten years, and it has been developed into many new directions when it intersected with harmonic analysis, random matrices, combination theory, etc.

Let A be a unital C^*-algebra or von-Neumann algebra and let φ be a state on A. Then (A, φ) will be called a noncommutative probability space and an element in A is called a random variable. It is easy to see that free probability is a kind of noncommutative probability. A key notion in free probability is "freeness" instead of the classical independence in probability theory and then some new notions have appeared such as free convolution, free entropy, and free Fish information, etc[1]. Many essential problems in operator algebras have been solved by using this theory. For instance, in 1996, Voiculescu proved free group factor $\mathcal{L}(F_n)\,(2 \leq n < \infty)$ has no Cartan subalgebras; In 1998, GE Liming showed $\mathcal{L}(F_n)\,(2 \leq n < \infty)$ is prime, that is, $\mathcal{L}(F_n) \neq \mathcal{L}(F_{n_1}) \otimes \mathcal{L}(F_{n_2})\,(2 \leq n, n_1, n_2 < \infty)$.

Now, we fix a subalgebra B of $A, (1 \in B)$ and a conditional expectation φ from A onto B, and then (A, φ) will be called an operator-valued noncommutative probability space. Obvi-

ously, noncommutative probability space is a special case of the operator-valued noncommutative probability space, and so we hope operator-valued free probability can be more useful in wider fields.

The most important distribution is semicircular distribution in noncommutative probability space. We will give its definition in Section 2 after introducing the canonical forms of operator-valued random variables. These results belong to Voiculescu D.[2]. In Section 3, we calculate the moments of operator-valued semicircular distribution. In Section 4, 5 we discuss the asymptotically free random matrix models of semicircular distribution, and this is our main result in this paper. Usually, determining the freeness of some random variables is not easy, and constructing their matrix models is an effective way. In scalar valued probability space, we can find a class of random matrices whose elements are independent and mixed moments satisfy a restrictive condition, such that they can serve as the matrix models of semicircular distribution[1,3]. While in operator-valued case, the elements in matrix are required to be noncommutative, and so we should find a new relation instead of independence (since independence can only be defined for commutative random variables) and it is easy to be verified. This is so called semi-independence which we first introduce in this paper. By using this notion, we obtain our main result:

Let (A, φ) be C^*-operator-valued noncommutative probability space, $\eta_s : B \longrightarrow B, s \in \mathbb{N}$ be a family of linear maps and $\forall n \in \mathbb{N}, \{Y(s, n)\}_{s \in \mathbb{N}}$ be a family of symmetric random matrices. $Y(s, n) := \left[\frac{a(i, j; s, n)}{\sqrt{n}} \right]_{1 \leq i, j \leq n}, \forall s \in \mathbb{N}$, satisfying:

(1) $\varphi(a(i, j; s, n)) = 0$; $\varphi(a(i, j; s, n)ba(i, j; s, n)) = \eta_s(b), \forall s \in \mathbb{N}, 1 \leq i, j \leq n.$;

(2) $c_k := \sup_{\substack{1 \leq m_1, \ldots, m_k \leq n \\ s_1, \ldots, s_k \in \mathbb{N}}} \|\varphi(a(m_1, m_2; s_1, n)a(m_2, m_3; s_2, n) \cdots a(m_k, m_1; s_k, n))\| = O(1);$

(3) $\{a(i, j; s, n) | 1 \leq i, j \leq n, s \in \mathbb{N}\}$ is semi-independent.

Then $(Y(s, n))_{s \in \mathbb{N}}$ is asymptotically free and each $Y(s, n)$ tends to $\lambda^*(1) + \lambda(\eta_s)$ in the sense of distribution.

2. Operator-valued noncommutative probability space and algebra $A(B)$

In this section we introduce some notions and give the canonical forms of random variables.

Definition 2.1[2] *Let A be a unital algebra over $\mathbb{C}$, and let B be a subalgebra of A, $1 \in B$. $\varphi : A \longrightarrow B$ is a conditional expectation, that is, φ is linear over $\mathbb{C}$, and $\varphi(b_1 a b_2) = b_1 \varphi(a) b_2$, $\varphi(b) = b, \forall b, b_1, b_2 \in B, a \in A$. Then we call (A, φ) an operator-valued (or B-valued) noncommutative probability space; elements in A will be called B-valued random variables.*

Definition 2.2[2] *The algebra freely generated by B and an indeterminate X will be denoted by $B\langle X \rangle$. Let (A, φ) be as in Definition 2.1 and let $a \in A$ be a B-valued random variable. The distribution of a is the conditional expectation $\mu_a : B\langle X \rangle \longrightarrow B, \mu_a = \varphi \circ \tau_a$, where $\tau_a : B\langle X \rangle \longrightarrow A$ is the unique homomorphism such that $\tau_a(b) = b, \forall b \in B, \tau_a(X) = a$. $\mu_a(b_0 X b_1 X \cdots b_{n-1} X b_n)$ is called the moments of a and $\mu_a(bXbX \cdots Xb)$ is called the symmetric moments of a. For convenience, we only consider symmetric moments in this paper.*

In the following, we will review the notion of canonical forms of random variables which was

introduced by Voiculescu in [2].

Let $\chi_n(B) = \mathcal{L}(B^{\otimes n}, B)$ be the set of all linear maps from $B^{\otimes n}$ to B (the $\otimes$ and linearity are over $\mathbb{C}$) and $\chi_0(B) = B, \chi(B) := \bigoplus_{n \geq 0} \chi_n(B)$ which is a right B-module.

If $\xi \in \chi_n(B)$, we define the modular endomorphism $\lambda(\xi) : \chi(B) \longrightarrow \chi(B)$ by:

$\forall \eta \in \chi_k(B)$, $k > 0$ (namely, $\deg\eta := k > 0$), $\lambda(\xi)\eta \in \chi_{n+k}(B)$, satisfying:

$$(\lambda(\xi)\eta)(m_1 \otimes m_2 \cdots \otimes m_{n+k})$$
$$= \eta(m_{n+1}\xi(m_1 \otimes \cdots \otimes m_n) \otimes m_{n+2} \otimes \cdots \otimes m_{n+k}), \quad \forall m_i \in B, \, i = 1, 2, \cdots n + k$$

and if $k = 0$, namely, $\eta \in B$, put $\lambda(\xi)\eta := \xi\eta$.

We also need to define $\lambda^*(m), m \in B$ by:

$\lambda^*(m)\eta := 0$, if $\deg\eta = 0$; and $\deg(\lambda^*(m)\eta) = \deg\eta - 1$,

$$(\lambda^*(m)\eta)(m_1 \otimes \cdots \otimes m_{k-1}) := \eta(m \otimes m_1 \otimes \cdots \otimes m_{k-1}),$$

if $\deg\eta = k > 0$.

Denote by $A(B)$ the algebra generated by $\{\lambda(\xi) | \xi \in \chi_n(B) : n \geq 0\} \cup \{\lambda^*(m) : m \in B\}$.

By a direct calculation, we have the following proposition.

Proposition 2.3[2]

(1) $\lambda(\xi_1)\lambda(\xi_2) = \lambda(\lambda(\xi_1)\xi_2)$;

(2) $\lambda^*(m)\lambda(\xi) = \lambda(\lambda^*(m)\xi)$, if $\deg\xi > 0$;

(3) $\lambda^*(m)\lambda(\xi) = \lambda^*(m\xi)$, if $\deg\xi = 0$.

From this proposition, we know every monomial in $A(B)$ can be converted into the form: $\lambda(\xi_n)\lambda^*(m_1) \cdots \lambda^*(m_k)$.

Define a conditional expectation ε_B from $A(B)$ onto B by

$$\varepsilon_B(\lambda(\xi_n)\lambda^*(m_1) \cdots \lambda^*(m_k)) = 0, \text{ if } n + k > 0;$$

$$\varepsilon_B(\lambda(\xi_0)) = \xi_0, \text{ if } \xi_0 \in \chi_0(B) = B.$$

It is not difficult to verify that ε_B is indeed a conditional expectation, and so $(A(B), \varepsilon_B)$ is a B-valued probability space.

Remark $\chi(B)$ and $\lambda(\xi)$ are similar to full Fock space and creating operator, respectively.

Definition 2.4[2] *The elements in $A(B)$ of the form $\lambda^*(1) + \sum_{n \geq 0} \lambda(\xi_n)$ $(\xi_n \in \chi_n(B))$, are called canonical.*

Proposition 2.5[2] *Given a distribution μ, there is a unique canonical element $a = \lambda^*(1) + \sum_{n \geq 0} \lambda(\xi_n) \in A(B)$, such that $\mu_a = \mu$, where μ_a denotes the distribution of a.*

3. Semicircular distribution

Definition 3.1 *A random variable a will be called a B-semicircular variable, if its canonical form is $\lambda^*(1) + \lambda(\xi_0) + \lambda(\xi_1)$. Generally, we consider central B-semicircular distribution, namely, $\varphi(a) = 0$ i.e. $\xi_0 = 0$.*

Next, we calculate the moments of semicircular distribution.

Proposition 3.2 *The m-th moment of semicircular distribution is*

$$\varepsilon_B(\lambda^*(1) + \lambda(\xi_1))^m = \begin{cases} 0, & m \text{ odd} \\ \displaystyle\sum_{i \in I} M_i, & m = 2k \text{ even} \end{cases}$$

where $|I| = \frac{1}{k+1}\binom{2k}{k}$; *and* $\{M_i\}_{i \in I}$ *are all the combinatorial possibilities of the following factors:* $\xi_1(1), \xi_1(\xi_1(1)), \ldots, \xi_1(\xi_1(\xi_1 \cdots \xi_1(\xi_1(1))))$, *and* ξ_1 *appears just k times.*

Proof We calculate $\varepsilon_B(\lambda^*(1) + \lambda(\xi_1))^m = \sum \varepsilon_B(a_{i(m)} a_{i(m-1)} \cdots a_{i(1)})$, where $a_{i(k)} \in \{\lambda^*(1), \lambda(\xi_1)\}$, and the sum is over all the possibilities. We can associate to each monomial a polygonal line on the X-Y-plane. We draw line $y = -1$ on X-Y-plane. If $a_{i(1)} = \lambda(\xi_1)$, then we draw a segment from $(0,0)$ to $(1,1)$; if $a_{i(1)} = \lambda^*(1)$, we draw a segment from $(0,0)$ to $(1,-1)$. Next, we draw a segment from the end point of the former line, and if $a_{i(2)} = \lambda(\xi_1)$, we draw a $\sqrt{2}$-length segment with slop 1; if $a_{i(2)} = \lambda^*(1)$, we draw a $\sqrt{2}$-length segment with slop -1; .... Following this way, we can obtain a polygonal line corresponding to $a_{i(m)} \cdots a_{i(1)}$.

From the definition of ε_B, we can learn that if the polygonal line touches the line $y = -1$, or its endpoint is not on the X-axis, then $\varepsilon_B(a_{i(m)} \cdots a_{i(1)}) = 0$. Obviously, $\varepsilon(a_{i(m)} \cdots a_{i(1)}) = 0$ if m is odd.

When $m = 2k$, there are $\frac{1}{k+1}\binom{2k}{k}$ polygonal lines that end on X-axis and do not touch $y = -1$. We denote these lines by $L_1, L_2, \ldots, L_t, t = \frac{1}{k+1}\binom{2k}{k}$, and $\forall L_i$, it must correspond to unique factor $a_{i(m)} a_{i(n-1)} \cdots a_{i(1)}$. Writing $M_i := \varepsilon_B(a_{i(m)} a_{i(m-1)} \cdots a_{i(1)})$ and simplifying it, we obtain the result.

4. Matrix models of semicircular distribution

Definition 4.1 *Suppose (A, ϕ) is a B-valued C^*-noncommutative probability space. $A_n = [a_{ij}]_{n \times n}, a_{ij} \in A$ is called a B-valued random matrix. The C^*-algebra consisting of such matrices is denoted by $\mathcal{A}_n$. Define a conditional expectation on $\mathcal{A}_n$ by: $\tau_n(A_n) = \frac{1}{n}\sum_{i=1}^{n} \phi(a_{ii})$. Then $(\mathcal{A}_n, \tau_n)$ is a B-valued noncommutative probability space (in fact it is $B \otimes I_n$-valued).*

Definition 4.2 *Suppose (A, ϕ) is B-valued probability space, $A_\iota \subseteq A$ $(\iota \in I)$ is a family of subalgebras of A, and B is a subalgebra of A_ι, $\forall \iota \in I$. Then $\{A_\iota\}_{\iota \in I}$ will be called semi-independent if it satisfies:*

$$\phi(a_1 a_2 \cdots a_{k-1} a_k \ldots a_{k+j} a_{k+j+1} \ldots a_n)$$
$$= \phi(a_1 a_2 \ldots a_{k-1} \phi(a_k \ldots a_{k+j}) a_{k+j+1} \ldots a_n),$$

where $a_i \in A_{\iota_i}$, and $\{\iota_k, \ldots, \iota_{k+j}\} \bigcap \{\iota_1, \ldots, \iota_{k-1}, \iota_{k+j+1}, \iota_n\} = \emptyset$.

Given a set of elements $\{a_s\}$ in A. Denote by A_s the subalgebra generated by a_s and B, if $\{A_s\}_{s \in S}$ is semi-independent, then we call $\{a_s\}_{s \in S}$ semi-independent.

Remark The above definition is well defined. In fact, if

$$\{\iota_k, \iota_{k+1}, \ldots, \iota_{k+q}, \ldots, \iota_{k+j}\} \bigcap \{\iota_1, \ldots, \iota_{k-1}, \iota_{k+j+1}, \ldots, \iota_n\} = \emptyset;$$

$$\{\iota_{k+p}, \iota_{k+p+1}, \ldots, \iota_{k+q}\} \bigcap \{\iota_1, \ldots, \iota_k, \ldots, \iota_{k+p-1}, \iota_{k+q+1}, \ldots, \iota_n\} = \emptyset,$$

it is easy to see

$$\phi(a_1 a_2 \cdots a_k \cdots a_{k+p} \cdots a_{k+q} \cdots a_{k+j} \cdots a_n)$$

$$= \phi(a_1 a_2 \cdots a_{k-1} \phi(a_k \cdots a_{k+p} \cdots a_{k+q} \cdots a_{k+j}) a_{k+j+1} \cdots a_n)$$

$$= \phi(a_1 a_2 \cdots a_k \cdots a_{k+p-1} \phi(a_{k+p} \cdots a_{k+q}) a_{k+q+1} \cdots a_{k+j} \cdots a_n)$$

$$= \phi(a_1 \cdots a_{k-1} \phi(a_k \cdots a_{k+p-1} \phi(a_{k+p} \cdots a_{k+q}) \cdots a_{k+j}) \cdots a_n).$$

If

$$\{\iota_i, \ldots, \iota_{i+m}\} \bigcap \{\iota_1, \ldots, \iota_{i-1}, \iota_{i+m+1}, \ldots, \iota_n\} = \emptyset;$$

$$\{\iota_j, \ldots, \iota_{j+p}\} \bigcap \{\iota_1, \iota_2 \ldots, \iota_{j-1}, \iota_{j+p+1} \ldots, \iota_n\} = \emptyset,$$

then

$$\phi(a_1 a_2 \cdots a_{i-1} a_i \cdots a_{i+m} a_{i+m+1} \cdots a_{j-1} a_j \cdots a_{j+p} a_{j+p+1} \cdots a_n)$$

$$= \phi(a_1 a_2 \cdots a_{i-1} \phi(a_i \cdots a_{i+m}) a_{i+m+1} \cdots a_{j-1} a_j \cdots a_{j+p} a_{j+p+1} \cdots a_n)$$

$$= \phi(a_1 a_2 \cdots a_{i-1} a_i \cdots a_{i+m} a_{i+m+1} \cdots a_{j-1} \phi(a_j \cdots a_{j+p}) a_{j+p+1} \cdots a_n)$$

$$= \phi(a_1 \cdots a_{i-1} \phi(a_i \cdots a_{i+m}) a_{i+m+1} \cdots a_{j-1} \phi(a_j \cdots a_{j+p}) a_{j+p+1} \cdots a_n).$$

We will give an example of semi-independence later.

Theorem 4.3 *Let (A, ϕ) be a B-valued C^*-noncommutative probability space, $\eta : B \longrightarrow B$ be a linear map and $\forall n \in \mathbb{N}$, $A_n = \left[\frac{a_{ij}}{\sqrt{n}}\right]_{1 \leq i,j \leq n}$ be a symmetric random matrix, where $a_{ij} \in (A, \phi)$ are self-adjoint. If A_n satisfies:*

(1) $\phi(a_{ij}) = 0$, $\phi(a_{ij} b a_{ij}) = \eta(b)$ $(1 \leq i < j \leq n)$, $\forall b \in B$;

(2) $c_k = \sup_{1 \leq m_1, \ldots, m_k \leq n} \|\phi(a_{m_1 m_2} a_{m_2 m_3} \cdots a_{m_k m_1})\| = O(1)$;

(3) $\{a_{ij} : 1 \leq i \leq j \leq n\}$ *is semi-independent,*

then $\{A_n\}_{n=1}^{\infty}$ tends in the distribution sense to the semicircular element $\lambda^*(1) + \lambda(\eta)$, where the 'limit' is in the sense of norm.

Proof Calculate the A_n's kth moment:

$$\tau_n(A_n^k) = \frac{1}{n^{\frac{k}{2}+1}} \sum_{1 \leq m_1, \ldots, m_k \leq n} \phi(a_{m_1 m_2} a_{m_2 m_3} \cdots a_{m_k m_1}). \tag{4.3.1}$$

We group the term $\phi(a_{m_1 m_2} a_{m_2 m_3} \cdots a_{m_k m_1})$ according to $\sharp\{m_1, m_2, \ldots, m_k\}$, where $\sharp$ denotes the number of the elements which are not equal to each other in a set.

Obviously, the number of terms such that $\sharp\{m_1, m_2, \ldots, m_k\} = l$ is less than $\leq \binom{n}{l} l^k$, so that

$$\frac{1}{n^{\frac{k}{2}+1}} \| \sum_{\sharp\{m_1, \ldots, m_k\}=l} \phi(a_{m_1 m_2} a_{m_2 m_3} \cdots a_{m_k m_1}) \| \leq \binom{n}{l} \frac{l^k c_k}{n^{\frac{k}{2}+1}},$$

and it goes to 0 whenever $l < \frac{k}{2} + 1$.

When $l > \frac{k}{2} + 1$, there is a factor $a_{m_i m_{i+1}}(m_{k+1} := m_1)$ which appears only once in $a_{m_1 m_2} a_{m_2 m_3} \cdots a_{m_k m_1}$ such that $m_i \neq m_{i+1}$. In fact, inducting on k, we assert there must be an m_p appearing only once in $m_1, \ldots, m_k$. If $m_{p-1} \neq m_{p+1}$, then either $a_{m_{p-1} m_p}$ or $a_{m_p m_{p+1}}$ is a desired factor. If $m_{p-1} = m_{p+1}$, we may remove m_{p-1}, m_p from $m_1, \ldots, m_k$ and apply the induction assumption.

Thus from semi-independence:

$$\phi(a_{m_1 m_2} \cdots a_{m_{i-1} m_i} a_{m_i m_{i+1}} a_{m_{i+1} m_{i+2}} \cdots a_{m_k m_1})$$
$$= \phi[a_{m_1 m_2} \cdots a_{m_{i-1} m_i} \phi(a_{m_i m_{i+1}}) a_{m_{i+1} m_{i+2}} \cdots a_{m_k m_1}] = 0.$$

From the above, we know (4.3.1) goes to 0 when k is odd.

When k is even, we replace k by $2k$. From the above argument, we have only to consider the following case:

$$\frac{1}{n^{k+1}} \sum_{(a)(b)} \phi(a_{m_1 m_2} a_{m_2 m_3} \cdots a_{m_{2k} m_1}),$$

where the sum is over $1 \leq m_1, m_2, \ldots, m_{2k} \leq n$, satisfying:

(a) $\sharp\{m_1, m_2, \ldots, m_{2k}\} = k + 1$;

(b)   every element of the sequence $\{m_1, m_2\}, \{m_2, m_3\}, \ldots, \{m_{2k}, m_1\}$ appears at least twice.

Next, we will prove: If $m_1, m_2, \ldots, m_{2k}$ satisfy (a),(b), then every pair of elements in $\{m_1, m_2\}, \{m_2, m_3\}, \ldots, \{m_{2k}, m_1\}, m_i \neq m_{i+1}, 1 \leq i \leq 2k$, appears just twice, and partition $\mathcal{V}$ of $[2k]$ is non-crossing (the readers can find the notion of non-crossing in [4]), where $\mathcal{V}$ is defined by

$$\{i, j\} \in \mathcal{V} \Longleftrightarrow \{m_i, m_{i+1}\} = \{m_j, m_{j+1}\}.$$

We carry out the proof by induction on k. The case of $k = 1$ is trivial. We assume the conclusion is true for $k - 1$. There exists p, such that m_p has no repetitions in $\{m_1, m_2, \ldots, m_{2k}\}$. Then we have $m_{p-1} = m_{p+1} \neq m_p$. We remove m_{p-1}, m_p from $m_1, \ldots, m_{2k}$ and obtain a shorter sequence $n_1, \ldots, n_{2k-2}$. From the reduction assumption, the partition of this shorter sequence as the above is non-crossing, and it is still non-crossing when adding $\{p - 1, p\}$.

On the other hand, for a pair partition $\mathcal{V}$ of $[2k]$, we have $\{m_1, m_2, \ldots, m_{2k}\}$, which can be done by induction on $\mathcal{V}$ as the above and the number of such sequences is $n(n - 1) \cdots (n - k)$. We also know the number of pair partitions of $[2k]$ is Catalan number, i.e., $\frac{1}{k+1}\binom{k}{2k}$

Suppose $\{m_1, \ldots, m_{2k}\}$ satisfy (a),(b). From condition (1) and semi-independence, we know there is an $i \in I$, such that $\phi(a_{m_1 m_2} \cdots a_{m_{2k} m_1}) = M_i$. Therefore, there is a 1–1 correspondence among $\{M_i\}$, $\{\phi(a_{m_1 m_2} \cdots a_{m_{2k} m_1})\}$ and $\{\mathcal{V}\}$, where $\{M_i\}_{i \in I}$ just as in Proposition 3.2 (just replacing ξ_1 by η). So we have

$$\frac{1}{n^{k+1}} \sum_{(a),(b)} \phi(a_{m_1 m_2} a_{m_2 m_3} \cdots a_{m_{2k} m_1}) = \frac{n(n - 1) \cdots (n - k)}{n^{k+1}} \Big[\sum_{i \in I} M_i\Big],$$

and obviously, it tends to $\sum_{i \in I} M_i$.

Example 4.4   Let $(\Omega, \mathcal{F}, P)$ be a classical probability space, and $L := \bigcap_{p=1}^{\infty} L^p(\Omega, \mathcal{F}, P)$.

Consider $M_2(\mathbb{C})$-valued probability space: $(L \bigotimes M_2(\mathbb{C}), E \bigotimes id_2)$, where $E(f) = \int f(\omega) \, dP(\omega)$, $\forall f \in L$, that is, $\forall A_2 = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \in L \bigotimes M_2(\mathbb{C})$, its expectation is defined by

$$(E \bigotimes id_2)(A) = \begin{pmatrix} E(f_{11}) & E(f_{12}) \\ E(f_{21}) & E(f_{22}) \end{pmatrix} \in M_2(\mathbb{C}).$$

Let $X_{ij} := [f_{kl}^{(ij)}]_{1 \le k,l \le 2}$ $(1 \le i,j \le n)$ be a family of 2×2 symmetric random matrices and $X = [\frac{X_{ij}}{\sqrt{n}}]_{1 \le i,j \le n}$ a $M_2(\mathbb{C})$-valued symmetric matrix in $(L \bigotimes M_2(\mathbb{C}) \bigotimes M_n(\mathbb{C}))$.

If $\{f_{kl}^{(ij)} : 1 \le k \le l \le 2, 1 \le i \le j \le n\}$ is a set of independent Gauss elements, then X satisfies the conditions of Theorem 3.3, namely, in the sense of norm (for instance, choosing $\|A\|_2^2 = \mathrm{tr}_2(A^*A) \, \forall A \in M_2(\mathbb{C}))$, X tends to $\lambda^*(1) + \lambda(1)$ according to distribution.

We only need to show $\{X_{ij} : 1 \le i \le j \in n\}$ is semi-independent. If

$$\{i_k j_k, \ldots, i_{k+p} j_{k+p}\} \bigcap \{i_1 j_1, \ldots, i_{k-1} j_{k-1}, i_{k+p+1} j_{k+p+1}, \ldots, i_n j_n\} = \emptyset,$$

then from the independence, $\forall p_i \in M_2(\mathbb{C})\langle X \rangle, i = 1, 2, \ldots, n$, we have

$$(E \otimes id_2)(p_1(X_{i_1 j_1}) p_2(X_{i_2 j_2}) \cdots p_k(X_{i_k j_k}) \cdots p_{k+p}(X_{i_{k+p} j_{k+p}}) \cdots p_n(X_{i_n j_n}))$$

$$= (E \otimes id_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

$$= (E \otimes id_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} E(c_{11}) & E(c_{12}) \\ E(c_{21}) & E(c_{22}) \end{pmatrix} \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix},$$

where

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = p_1(X_{i_1 j_1}) p_2(X_{i_2 j_2}) \cdots p_{k-1}(X_{i_{k-1} j_{k-1}});$$

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = p_k(X_{i_k j_k}) \cdots p_{k+p}(X_{i_{k+p} j_{k+p}});$$

$$\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = p_{k+p+1}(X_{i_{k+p+1} j_{k+p+1}}) \cdots p_n(X_{i_n j_n}).$$

5. Asymptotic freeness

Definition 5.1[2] *Let (A, φ) be a B-valued noncommutative probability space. $(A_\iota)_{\iota \in I}$ will be called free if*

$$\varphi(a_1 a_2 \cdots a_n) = 0, \quad \text{whenever } a_i \in A_{\iota_i}, \varphi(a_i) = 0, \text{ and } \iota_1 \ne \iota_2 \ne \cdots \ne \iota_n.$$

Definition 5.2 *Let $(\mathcal{A}_n, \tau_n)$ be $n \times n$ B-valued C^* random matrix probability space and $(X(s, n))_{s \in S} \subseteq \mathcal{A}_n$ be a family of random matrices. We say they have the limit distribution μ when n tends to infinity, if μ is a distribution on $B\langle X_s : s \in S \rangle$ and $\forall s_1, s_2, \ldots, s_m \in S$,*

$$\mu(X_{s_1} \cdots X_{s_m}) = \lim_{n \to \infty} \tau_n(X(s_1, n) X(s_2, n) \cdots X(s_m, n)).$$

The limit is always in the sense of norm. In addition, if $\{X_s : s \in S\}$ is free in $(B\langle X_s : s \in S \rangle, \mu)$, then we say $\{X(s, n) : s \in S\}$ is asymptotically free.

Theorem 5.3 *Let (A, φ) be a B-valued C^*-noncommutative probability space, and let $\eta_s :$ $B \longrightarrow B, s \in \mathbb{N}$ be a set of linear maps. $\forall n \in \mathbb{N}, \{Y(s, n)\}_{s \in \mathbb{N}}$ is a family of $n \times n$ symmetric random matrices on A, where $Y(s, n) := [\frac{a(i,j;s,n)}{\sqrt{n}}]_{1 \leq i,j \leq n}, \forall s \in \mathbb{N}$. If $Y(s, n)$ satisfies:*

(1) $\varphi(a(i, j; s, n)) = 0$; $\varphi(a(i, j; s, n)ba(i, j; s, n)) = \eta_s(b), \forall s \in \mathbb{N}, 1 \leq i, j \leq n$;

(2) $c_k := \sup_{\substack{1 \leq m_1, \ldots, m_k \leq n \\ s_1, \ldots, s_k \in \mathbb{N}}} \|\varphi(a(m_1, m_2; s_1, n)a(m_2, m_3; s_2, n) \cdots a(m_k, m_1; s_k, n))\| = O(1)$;

(3) $\{a(i, j; s, n)|1 \leq i, j \leq n, s \in \mathbb{N}\}$ is semi-independent,

then $(Y(s, n))_{s \in \mathbb{N}}$ is asymptotically free and each $Y(s, n)$ tends in the distribution sense to $\lambda^*(1) + \lambda(\eta_s)$.

Proof From Theorem 4.3 we know $\forall s \in \mathbb{N}, Y(s, n)$ tends to $\lambda^*(1) + \lambda(\eta_s)$ in the sense of distribution.

In order to prove $(Y(s, n))_{s \in \mathbb{N}}$ is asymptotically free, we first prove it has limit distribution. Suppose μ_n is the joint distribution of $(Y(s, n))_{s \in \mathbb{N}}$. $\forall X_{s_1} X_{s_2} \cdots X_{s_k} \in B\langle X_\iota : \iota \in \mathbb{N} \rangle$, letting $\tau_n = \varphi \otimes tr_n$, we have

$$\mu_n(X_{s_1} X_{s_2} \cdots X_{s_k}) = \tau_n(Y(s_1, n)Y(s_2, n) \cdots Y(s_k, n))$$
$$= \sum_{1 \leq m_1, \ldots, m_k \leq n} n^{-\frac{k}{2}} \varphi[a(m_1, m_2; s_1, n)a(m_2, m_3; s_2, n) \cdots a(m_k, m_1; s_k, n)].$$

Similar to the proof of Theorem 4.3, we can show the above equation tends to $\varepsilon_B((\lambda^*(1) + \lambda(\eta_{s_1})) \cdots (\lambda^*(1) + \lambda(\eta_{s_k})))$.

Assuming

$$\lim_{n \to \infty} \tau_n(P_i(Y(s_i, n))) = 0, \quad i = 1, 2, \ldots, k; k > 1,$$

we will show

$$\lim_{n \to \infty} \tau_n(P_1(Y(s_1, n))P_2(Y(s_2, n)) \cdots P_k(Y(s_k, n))) = 0,$$

where $P_i \in B\langle X \rangle, i = 1, 2, \ldots, k; s_1 \neq s_2 \neq \cdots \neq s_k$, and for convenience, we may suppose $P_i(X) = (b_i X)^m + (b_i X)^{m-1} + \cdots + b_i$ without loss of generality. Then we have

$$\tau_n[P_1(Y(s_1, n))P_2(Y(s_2, n)) \cdots P_k(Y(s_k, n))]$$
$$= \tau_n\{[(b_1 Y(s_1, n))^m + (b_1 Y(s_1, n))^{m-1} + \cdots + b_1]$$
$$[(b_2 Y(s_2, n))^m + (b_2 Y(s_2, n))^{m-1} + \cdots + b_2]$$
$$\cdots$$
$$[(b_k Y(s_k, n))^m + (b_k Y(s_k, n))^{m-1} + \cdots + b_k]\}$$
$$= \sum_{0 \leq m_1, m_2, \ldots, m_k \leq m} \tau_n[(b_1 Y(s_1, n))^{m_1}(b_2 Y(s_2, n))^{m_2} \cdots (b_k Y(s_k, n))^{m_k}]$$
$$= \sum_{0 \leq m_1, \ldots, m_k \leq m} n^{-\frac{m_0}{2}+1} \varphi\{ \sum_{1 \leq i_1, i_2, \ldots, i_{m_0} \leq n} [b_1 a(i_1, i_2; s_1, n)b_1 a(i_2, i_3; s_i, n) \cdots$$
$$b_1 a(i_{m_1}, i_{m_1+1}; s_1, n)]$$
$$[b_2 a(i_{m_1+1}, i_{m_1+2}; s_2, n)b_2 a(i_{m_1+2}, i_{m_1+3}; s_2, n) \cdots b_2 a(i_{m_1+m_2}, i_{m_1+m_2+1}; s_2, n)]$$
$$\cdots$$

$$[b_k a(i_{m_1+\cdots+m_{k-1}+1}, i_{m_1+\cdots+m_{k-1}+2}, s_k, n) \cdots b_k a(i_{m_0}, i_1, ; s_k, n)]\} \tag{5.3.1}$$

where $m_0 := m_1 + m_2 + \cdots + m_k$. From the proof of Theorem 4.3, we know the terms which are not equal to 0 must satisfy:

(1) $\forall (i_p, i_{p+1}, s_{i_p}, n)$ there is a unique $(i_q, i_{q+1}, s_{i_q}, n), (p \neq q)$, such that $\{i_p, i_{p+1}\} = \{i_q, i_{q+1}\}, s_{i_p} = s_{i_q}$;

(2) Define the pair partition $\mathcal{V}$ of $[m_0]$:

$$(p, q) \in \mathcal{V} \Longleftrightarrow \{i_p, i_{p+1}\} = \{i_q, i_{q+1}\}, s_{i_p} = s_{i_q}.$$

Then $\mathcal{V}$ is non-crossing.

Since $s_1 \neq s_2 \neq \cdots \neq s_k$, $\forall \mathcal{V}$ defined as the above, there must be a nonzero monomial $b_j a(i_{m_1+\cdots+m_{j-1}+1}, i_{m_1+\cdots+m_{j-1}+2}, s_j, n) \cdots b_j a(i_{m_1+\cdots+m_j}, i_{m_1+\cdots+m_j+1}, s_j, n)$ such that $\forall l \in \{m_1 + \cdots + m_{j-1} + 1, \ldots, m_1 + \cdots m_j\}$, there exists $l' \in \{m_1 + \cdots + m_{j-1} + 1, \ldots, m_1 + \cdots m_j\}$ satisfying $(l, l') \in \mathcal{V}$.

For a fixed $b_j a(i_{m_1+\cdots+m_{j-1}+1}, i_{m_1+\cdots+m_{j-1}+2}, s_j, n) \cdots b_j a(i_{m_1+\cdots+m_j}, i_{m_1+\cdots+m_j+1}, s_j, n)$, $\mathcal{V}_{j_s}$ is the pair partition as the above and satisfies the following condition: $(l, l') \in \mathcal{V}_{j_s}$ and $l \in \{m_1 + \cdots + m_{j-1} + 1, \ldots, m_1 + \cdots m_j\}$ implies $l' \in \{m_1 + \cdots + m_{j-1} + 1, \ldots, m_1 + \cdots m_j\}$. We denote all of such $\mathcal{V}_{j_s}$ by $\{\mathcal{V}_{j_s}\}_{s \in S_j}$. Then from (5.3.1) we have:

$$\tau_n[P_1(Y(s_1, n)) P_2(Y(s_2, n)) \cdots P_k(Y(s_k, n))]$$

$$= \sum_{j=1}^{k} \sum_{0 \leq m_1, \ldots, m_{j-1}, m_{j+1}, \ldots, m_k \leq m} n^{-\frac{m_0 - m_j}{2} + 1} \varphi \{ \sum_{s \in S_j} \sum_{\substack{1 \leq i_1, i_2, \ldots, i_{m_0} \leq n \\ \mathcal{V}_{|_f}}}$$

$$[b_1 a(i_1, i_2; s_1, n) b_1 a(i_2, i_3; s_i, n) \cdots b_1 a(i_{m_1}, i_{m_1+1}; s_1, n)]$$

$$[b_2 a(i_{m_1+1}, i_{m_1+2}; s_2, n) b_2 a(i_{m_1+2}, i_{m_1+3}; s_2, n) \cdots b_2 a(i_{m_1+m_2}, i_{m_1+m_2+1}; s_2, n)]$$

$$\cdots$$

$$[\sum_{0 \leq m_j \leq m} n^{-\frac{m_j}{2}} b_j a(i_{m_1+\cdots+m_{j-1}+1}, i_{m_1+\cdots+m_{j-1}+2}, s_j, n) \cdots$$

$$b_j a(i_{m_1+\cdots+m_j}, i_{m_1+\cdots+m_j+1}, s_j, n)]$$

$$\cdots$$

$$[b_k a(i_{m_1+\cdots+m_{k-1}+1}, i_{m_1+\cdots+m_{k-1}+2}, s_k, n) \cdots b_k a(i_{m_0}, i_1, ; s_k, n)]\}.$$

Since

$$\frac{1}{n} \varphi [\sum_{0 \leq m_j \leq m} \sum_{\mathcal{V}_{j_s}} n^{-\frac{m_j}{2}} b_j a(i_{m_1+\cdots+m_{j-1}+1}, i_{m_1+\cdots+m_{j-1}+2}, s_j, n) \cdots$$

$$b_j a(i_{m_1+\cdots+m_j}, i_{m_1+\cdots+m_j+1}, s_j, n)]$$

$$= \tau_n(P_j Y(s_j, n)),$$

we can infer that (5.3.1) tends to 0.

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