Journal of Mathematical Research & Exposition
Nov., 2008, Vol. 28, No. 4, pp. 769–778
DOI:10.3770/j.issn:1000-341X.2008.04.004
Http://jmre.dlut.edu.cn

A Conjecture on the Relation between Three Types of Oriented Triple Systems

TIAN Zi Hong, KANG Qing De
(Institute of Mathematics, Hebei Normal University, Hebei 050016, China)
(E-mail: tianzh68@163.com)

Abstract A Mendelsohn (directed, or hybrid) triple system of order v, denoted by MTS(v,λ) (DTS(v,λ), or HTS(v,λ)), is a pair $(X,\mathcal{B})$ where X is a v-set and $\mathcal{B}$ is a collection of some cyclic (transitive, or cyclic and transitive) triples on X such that every ordered pair of X belongs to λ triples of $\mathcal{B}$. In this paper, a relation between three types of oriented triple systems was discussed. We conjecture: the block-incident graph of MTS(v,λ) is 3-edge colorable. Then we obtain three disjoint DTS(v,λ)s and four disjoint HTS(v,λ)s from a given MTS(v,λ).

Keywords cyclic triple; transitive triple; oriented triple system.

Document code A
MR(2000) Subject Classification 05B07
Chinese Library Classification O157.2

1. Introduction

Let X be a finite set. In what follows, an *ordered pair* of X will always be an ordered pair (x,y) where $x \neq y \in X$. A cyclic triple on X is a set of three ordered pairs $(x,y),(y,z)$ and (z,x) of X, which is denoted by $\langle x,y,z\rangle$ (or $\langle y,z,x\rangle$, or $\langle z,x,y\rangle$). A *transitive triple* on X is a set of three ordered pairs $(x,y),(y,z)$ and (x,z) of X, which is denoted by (x,y,z).

An oriented triple system of order v is a pair $(X,\mathcal{B})$, where X is a v-set and $\mathcal{B}$ is a collection of cyclic or transitive triples on X, called blocks, such that each ordered pair of X is contained in exactly λ triples of $\mathcal{B}$. If $\mathcal{B}$ consists of only cyclic triples, the system is called Mendelsohn triple system and denoted by MTS$(v,\ \lambda)$. If $\mathcal{B}$ consists of only transitive triples, the system is called directed triple system and denoted by DTS$(v,\ \lambda)$. But, if there are both cyclic and transitive triples in $\mathcal{B}$, then the system is called hybrid triple system and denoted by HTS$(v,\ \lambda)$. It is easy to see that if $(X,\mathcal{B})$ is an MTS(v,λ) (resp. DTS(v,λ) or HTS(v,λ)), then $|\ \mathcal{B}\ |= \lambda v(v-1)/3$. Thus, a necessary condition for the existence of an MTS(v,λ) (resp. DTS(v,λ) or HTS(v,λ)) is $\lambda v(v-1)\equiv 0 \pmod 3$. Usually, MTS$(v,1)$ (resp. DTS$(v,1)$ or HTS$(v,1)$) is briefly written as MTS(v) (resp. DTS(v) or HTS(v)).

For a v-set X, some cyclic (or transitive, or cyclic and transitive) triples on X are said to be a parallel class if their elements form a partition of X. Some cyclic (or transitive, or

Received date: 2006-12-07; **Accepted date**: 2007-03-23
Foundation item: the National Natural Science Foundation of China (No. 10671055).

cyclic and transitive) triples on X are said to be an almost parallel class if they form a partition of $X\backslash\{x\}$ for some $x \in X$. An MTS(v,λ) (or DTS(v,λ), or HTS(v,λ)) is resolvable, denoted by RMTS(v,λ) (or RDTS(v,λ), or RHTS(v,λ)), if its block set can be partitioned into parallel classes. An MTS(v,λ) (or DTS(v,λ), or HTS(v,λ)) is almost resolvable, denoted by ARMTS(v,λ) (or ARDTS(v,λ), or ARHTS(v,λ)), if its block set can be partitioned into almost parallel classes.

An oriented triple system is called simple if there are no repeat blocks in its block set. A simple MTS(v,λ) $(X,\mathcal{B})$ is called pure and denoted by PMTS(v,λ), if $\langle x,y,z\rangle \in \mathcal{B}$ implies $\langle z,y,x\rangle \notin \mathcal{B}$. Similarly, a PDTS$(v,\lambda)$ $(X,\mathcal{B})$ is a simple DTS(v,λ) in which $(x,y,z) \in \mathcal{B}$ implies $(z,y,x) \notin \mathcal{B}$. A PHTS$(v,\lambda)$ $(X,\mathcal{B})$ is a simple HTS(v,λ) in which $\langle x,y,z\rangle$ (or (x,y,z)) $\in \mathcal{B}$ implies $\langle z,y,x\rangle$ (or (z,y,x)) $\notin \mathcal{B}$.

Two oriented triple system $(X,\mathcal{A})$ and $(X,\mathcal{B})$ are called disjoint if $\mathcal{A} \cap \mathcal{B} = \phi$. A large set of pairwise disjoint Mendelsohn triple systems of order v and denoted by LMTS$(v,\ \lambda)$, is a collection $\{(X,\mathcal{B}_i) : 1 \le i \le \frac{v-2}{\lambda}\}$, where each $(X,\mathcal{B}_i)$ is an MTS$(v,\ \lambda)$ and all $\mathcal{B}_i$'s form a partition of all cyclic triples on X. A large set of pairwise disjoint directed triple systems of order v and denoted by LDTS(v,λ), is a collection $\{(X,\mathcal{B}_i^r) : 1 \le i \le \frac{v-2}{\lambda},\ r = 1,2,3\}$, where each $(X,\mathcal{B}_i^r)$ is a DTS(v,λ) and all $\mathcal{B}_i^r$'s form a partition of all transitive triples on X. A large set of pairwise disjoint hybrid triple systems of order v and denoted by LHTS(v,λ), is a collection $\{(X,\mathcal{A}_i^r) : 1 \le i \le \frac{v-2}{\lambda},\ r = 0,1,2,3\}$, where each $(X,\ \mathcal{A}_i^r)$ is an HTS$(v,\ \lambda)$ and all $\mathcal{A}_i^r$'s form a partition of all cyclic and transitive triples on X. Corresponding to the above definitions, we can define LRMTS(v,λ), LARMTS(v,λ), LPMTS(v,λ), and LRDTS(v,λ), LARDTS(v,λ), LPDTS(v,λ), LRHTS(v,λ), LARHTS(v,λ), LPHTS(v,λ), respectively.

Let Y be a $(v+1)$-set. An overlarge set of pairwise disjoint Mendelsohn triple systems of order v and denoted by OLMTS(v), is a collection $\{(Y\backslash\{y\},\mathcal{B}_y) : y \in Y\}$, where each $(Y\backslash\{y\},\mathcal{B}_y)$ is an MTS(v), and all $\mathcal{B}_y$'s form a partition of all cyclic triples on Y. An overlarge set of pairwise disjoint directed triple systems of order v and denoted by OLDTS(v), is a collection $\{(Y\backslash\{y\},\mathcal{B}_y^r) : y \in Y,\ r = 1,2,3\}$, where each $(Y\backslash\{y\},\mathcal{B}_y^r)$ is a DTS(v) and all $\mathcal{B}_y$'s form a partition of all transitive triples on Y. Similarly, an overlarge set of pairwise disjoint hybrid triple systems of order v and denoted by OLHTS(v), is a collection $\{(Y\backslash\{y\},\mathcal{A}_y^r) : y \in Y,\ r = 0,1,2,3\}$, where each $(Y,\ \mathcal{A}_y^r)$ is an HTS(v) and all $\mathcal{A}_y^r$'s form a partition of all cyclic and transitive triples on Y. Corresponding to the above definitions, we can define OLRMTS(v), OLARMTS(v), OLPMTS(v), and OLRDTS(v), OLARDTS(v), OLPDTS(v), OLRHTS(v), OLARHTS(v), OLPHTS(v), respectively.

Example 1 LMTS$(10) = \{(\{a,b\} \cup Z_8, \mathcal{B}_x) : x \in Z_8\}$, where

$$\begin{array}{lllllllll}
\mathcal{B}_0: & \langle 0\ 1\ 2\rangle & \langle 7\ 0\ 2\rangle & \langle 3\ 4\ 7\rangle & \langle 5\ 6\ 2\rangle & \langle 6\ 7\ 4\rangle & \langle 7\ 1\ 3\rangle & \langle 6\ 0\ 3\rangle & \\
& \langle 2\ 4\ 3\rangle & \langle 1\ 4\ 2\rangle & \langle 5\ 0\ 7\rangle & \langle 5\ 1\ 6\rangle & \langle 4\ 0\ 6\rangle & \langle 1\ 5\ 4\rangle & \langle 3\ 0\ 5\rangle & \\
& \langle a\ 1\ 0\rangle & \langle a\ 2\ 6\rangle & \langle a\ 3\ 1\rangle & \langle a\ 4\ 5\rangle & \langle a\ 5\ 7\rangle & \langle a\ 6\ 3\rangle & \langle a\ 7\ 2\rangle & \langle a\ b\ 4\rangle \\
& \langle b\ 0\ 4\rangle & \langle b\ 1\ 7\rangle & \langle b\ 2\ 3\rangle & \langle b\ 3\ 5\rangle & \langle b\ 5\ 2\rangle & \langle b\ 6\ 1\rangle & \langle b\ 7\ 6\rangle & \langle b\ a\ 0\rangle
\end{array}$$

and $\mathcal{B}_x = \mathcal{B}_0 + x, x \in Z_8$.

Example 2 LDTS(10) $= \{(\{a,b\} \cup Z_8, \mathcal{B}_x^r) : x \in Z_8, r = 1,2,3\}$, where

$$\mathcal{B}_0^1: \begin{array}{llllllll} (0\ 1\ 2) & (2\ 7\ 0) & (3\ 4\ 7) & (6\ 2\ 5) & (7\ 4\ 6) & (7\ 1\ 3) & (6\ 0\ 3) & \\ (4\ 3\ 2) & (2\ 1\ 4) & (5\ 0\ 7) & (5\ 1\ 6) & (0\ 6\ 4) & (4\ 1\ 5) & (3\ 0\ 5) & \\ (1\ 0\ a) & (2\ 6\ a) & (a\ 3\ 1) & (5\ a\ 4) & (7\ a\ 5) & (3\ a\ 6) & (a\ 7\ 2) & (b\ 4\ a) \\ (4\ b\ 0) & (1\ 7\ b) & (2\ 3\ b) & (5\ b\ 3) & (b\ 5\ 2) & (b\ 6\ 1) & (6\ b\ 7) & (a\ 0\ b) \end{array}$$

$$\mathcal{B}_0^2: \begin{array}{llllllll} (2\ 0\ 1) & (7\ 0\ 2) & (7\ 3\ 4) & (2\ 5\ 6) & (4\ 6\ 7) & (1\ 3\ 7) & (0\ 3\ 6) & \\ (2\ 4\ 3) & (1\ 4\ 2) & (0\ 7\ 5) & (1\ 6\ 5) & (6\ 4\ 0) & (5\ 4\ 1) & (5\ 3\ 0) & \\ (a\ 1\ 0) & (6\ a\ 2) & (3\ 1\ a) & (a\ 4\ 5) & (5\ 7\ a) & (a\ 6\ 3) & (2\ a\ 7) & (4\ a\ b) \\ (b\ 0\ 4) & (7\ b\ 1) & (3\ b\ 2) & (b\ 3\ 5) & (5\ 2\ b) & (6\ 1\ b) & (b\ 7\ 6) & (0\ b\ a) \end{array}$$

$$\mathcal{B}_0^3: \begin{array}{llllllll} (1\ 2\ 0) & (0\ 2\ 7) & (4\ 7\ 3) & (5\ 6\ 2) & (6\ 7\ 4) & (3\ 7\ 1) & (3\ 6\ 0) & \\ (3\ 2\ 4) & (4\ 2\ 1) & (7\ 5\ 0) & (6\ 5\ 1) & (4\ 0\ 6) & (1\ 5\ 4) & (0\ 5\ 3) & \\ (0\ a\ 1) & (a\ 2\ 6) & (1\ a\ 3) & (4\ 5\ a) & (a\ 5\ 7) & (6\ 3\ a) & (7\ 2\ a) & (a\ b\ 4) \\ (0\ 4\ b) & (b\ 1\ 7) & (b\ 2\ 3) & (3\ 5\ b) & (2\ b\ 5) & (1\ b\ 6) & (7\ 6\ b) & (b\ a\ 0) \end{array}$$

and $\mathcal{B}_x^r = \mathcal{B}_0^r + x, x \in Z_8, r = 1,2,3$.

Example 3 LHTS(10) $= \{(\{a,b\} \cup Z_8, \mathcal{A}_x^r) : x \in Z_8, r = 0,1,2,3\}$, where

$$\mathcal{A}_0^0: \begin{array}{llllllll} \langle 0\ 1\ 2\rangle & \langle 2\ 7\ 0\rangle & (3\ 4\ 7) & (6\ 2\ 5) & (7\ 4\ 6) & (7\ 1\ 3) & (6\ 0\ 3) & \\ (4\ 3\ 2) & (2\ 1\ 4) & (5\ 0\ 7) & (5\ 1\ 6) & (0\ 6\ 4) & (4\ 1\ 5) & (3\ 0\ 5) & \\ (1\ 0\ a) & (2\ 6\ a) & (a\ 3\ 1) & (5\ a\ 4) & (7\ a\ 5) & (3\ a\ 6) & (a\ 7\ 2) & (b\ 4\ a) \\ (4\ b\ 0) & (1\ 7\ b) & (2\ 3\ b) & (5\ b\ 3) & (b\ 5\ 2) & (b\ 6\ 1) & (6\ b\ 7) & (a\ 0\ b) \end{array}$$

$$\mathcal{A}_0^1: \begin{array}{llllllll} (2\ 0\ 1) & (7\ 0\ 2) & \langle 7\ 3\ 4\rangle & (2\ 5\ 6) & \langle 4\ 6\ 7\rangle & (1\ 3\ 7) & (0\ 3\ 6) & \\ (2\ 4\ 3) & (1\ 4\ 2) & (0\ 7\ 5) & (1\ 6\ 5) & (6\ 4\ 0) & (5\ 4\ 1) & (5\ 3\ 0) & \\ (a\ 1\ 0) & (6\ a\ 2) & (3\ 1\ a) & (a\ 4\ 5) & (5\ 7\ a) & (a\ 6\ 3) & (2\ a\ 7) & (4\ a\ b) \\ (b\ 0\ 4) & (7\ b\ 1) & (3\ b\ 2) & (b\ 3\ 5) & (5\ 2\ b) & (6\ 1\ b) & (b\ 7\ 6) & (0\ b\ a) \end{array}$$

$$\mathcal{A}_0^2: \begin{array}{llllllll} (1\ 2\ 0) & (0\ 2\ 7) & (4\ 7\ 3) & (5\ 6\ 2) & (6\ 7\ 4) & (3\ 7\ 1) & \langle 3\ 6\ 0\rangle & \\ (3\ 2\ 4) & (4\ 2\ 1) & (7\ 5\ 0) & (6\ 5\ 1) & (4\ 0\ 6) & (1\ 5\ 4) & \langle 0\ 5\ 3\rangle & \\ (0\ a\ 1) & (a\ 2\ 6) & (1\ a\ 3) & (4\ 5\ a) & (a\ 5\ 7) & (6\ 3\ a) & (7\ 2\ a) & (a\ b\ 4) \\ (0\ 4\ b) & (b\ 1\ 7) & (b\ 2\ 3) & (3\ 5\ b) & (2\ b\ 5) & (1\ b\ 6) & (7\ 6\ b) & (b\ a\ 0) \end{array}$$

$$\mathcal{A}_0^3: \begin{array}{llllllll} (0\ 1\ 2) & (2\ 7\ 0) & (7\ 3\ 4) & \langle 5\ 6\ 2\rangle & (4\ 6\ 7) & \langle 7\ 1\ 3\rangle & (3\ 6\ 0) & \\ \langle 2\ 4\ 3\rangle & \langle 1\ 4\ 2\rangle & \langle 5\ 0\ 7\rangle & \langle 5\ 1\ 6\rangle & \langle 4\ 0\ 6\rangle & \langle 1\ 5\ 4\rangle & (0\ 5\ 3) & \\ \langle a\ 1\ 0\rangle & \langle a\ 2\ 6\rangle & \langle a\ 3\ 1\rangle & \langle a\ 4\ 5\rangle & \langle a\ 5\ 7\rangle & \langle a\ 6\ 3\rangle & \langle a\ 7\ 2\rangle & \langle a\ b\ 4\rangle \\ \langle b\ 0\ 4\rangle & \langle b\ 1\ 7\rangle & \langle b\ 2\ 3\rangle & \langle b\ 3\ 5\rangle & \langle b\ 5\ 2\rangle & \langle b\ 6\ 1\rangle & \langle b\ 7\ 6\rangle & \langle b\ a\ 0\rangle \end{array}$$

and $\mathcal{A}_x^r = \mathcal{A}_0^r + x, x \in Z_8, r = 0,1,2,3$.

Observe Examples 1, 2 and 3, it is easy to see that if we replace every transitive triple (a,b,c) with cyclic triple $\langle a,b,c\rangle$ in DTS(v), then $\mathcal{B}_x^r (r = 1,2,3)$ turns into $\mathcal{B}_x$. i.e., three DTS(v)'s correspond to the same MTS(v). Similarly, if we replace every transitive triple (a,b,c) with cyclic triple $\langle a,b,c\rangle$ in HTS(v), then $\mathcal{A}_x^r (r = 0,1,2,3)$ turns into $\mathcal{B}_x$, i.e., four HTS(v)'s correspond to the same MTS(v). Here, the "correspondence relation" satisfies two properties: (1) these oriented triples have the same underlying points; (2) three relevant transitive triples correspond to three different cyclic shifts of a cyclic triple. Now, the problem is that whether the "correspondence relation" suits any MTS? If this is true, then the discussion about three types of oriented triple system may incorporate into Mendelsohn triple system. This will simplify the research of oriented triple systems. In this paper, we will study the "correspondence relation".

2. Theorems and conjecture

In what follows, we call the transitive triple (a, b, c) (or (b, c, a), or (c, a, b)) a cyclic shift of the cyclic triple $\langle a, b, c\rangle$. If the cyclic shifts of all blocks in MTS(v, λ) can be partitioned into three families, such that each family can form a block set of a DTS(v, λ), then we call the three DTS(v, λ)'s cyclic shifts of the MTS(v, λ).

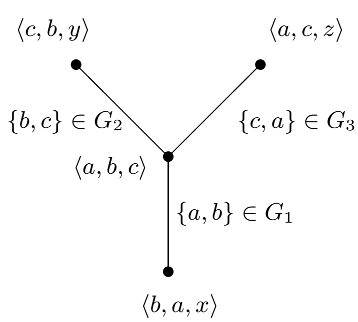
Given an MTS$(v, \lambda) = (X, \mathcal{B}), |X| = v$, we write the cyclic triples which can form some subsystems MTS(3) as $\overline{\mathcal{B}}$. Define a block-incident graph $G(\mathcal{B})$, where the vertex set is $\mathcal{B}\backslash\overline{\mathcal{B}}$, and the vertices B and B' are joint if and only if there are two common elements in B and B'. The edge joining $B = \langle a, b, x\rangle$ and $B' = \langle b, a, y\rangle$ is written as $\{a, b\}$. Evidently, $G(\mathcal{B})$ is a 3-regular graph. Suppose that the edges of $G(\mathcal{B})$ can be partitioned into three pairwise disjoint 1-factors G_1, G_2 and G_3, then $G(\mathcal{B})$ is 3-edge colorable. On the other hand, if $G(\mathcal{B})$ is 3-edge colorable, then $G(\mathcal{B})$ is 1-factorization. For the definitions of edge coloring, factor and factorization, we refer to [1].

Theorem 1 *If the block-incident graph of an* MTS(v, λ) *is 3-edge colorable, then there exist three cyclic shifts of the* MTS(v, λ) *(i.e., three pairwise disjoint* DTS(v, λ)*'s).*

Proof Let the block-incident graph $G(\mathcal{B})$ of an MTS(v, λ) be 3-edge colorable. Then the edges of $G(\mathcal{B})$ can be partitioned into three pairwise disjoint 1-factors G_1, G_2 and G_3. Now, we use the following "method" to construct three transitive triple sets $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$.

" Let the edge which joins the points $B = \langle a, b, x\rangle$ and $B' = \langle b, a, y\rangle$ be $\{a, b\}$ in $G(\mathcal{B})$. If $\{a, b\} \in G_i$, then the transitive triples (b, x, a) and $(a, y, b) \subseteq \mathcal{D}_i, i = 1, 2, 3$."

For each point $\langle a, b, c\rangle$ of the graph $G(\mathcal{B})$, there exist three elements $x, y, z \in X$ such that the three points joining $\langle a, b, c\rangle$ are $\langle b, a, x\rangle, \langle c, b, y\rangle, \langle a, c, z\rangle$ in $G(\mathcal{B})$. Without loss of generality, we let the three edges $\{a, b\}, \{b, c\}, \{c, a\}$ belong to G_1, G_2, G_3, respectively.



From the above "method", the three edges will derive that (b,c,a), $(a,x,b) \in \mathcal{D}_1$; (c,a,b), $(b,y,c) \in \mathcal{D}_2$; (a,b,c), $(c,z,a) \in \mathcal{D}_3$. So, we have the following conclusion:

$(*)$ The six ordered pairs contained in the two cyclic triples $\langle a,b,x\rangle$ and $\langle b,a,y\rangle$ are the same as those contained in the two transitive triples (b,x,a) and (a,y,b).

$(**)$ The three transitive triples corresponding to the same cyclic triple (a point in $G(\mathcal{B})$) are exactly the three cyclic shifts of the cyclic triple, and belong to three distinct $\mathcal{D}_i, i=1,2,3$, respectively.

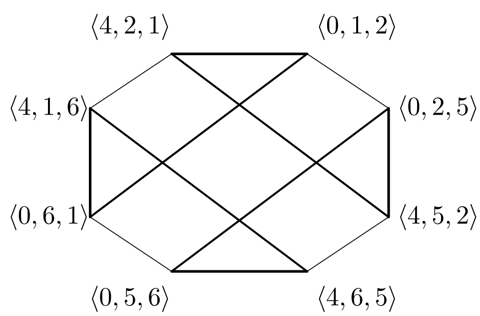
Further, when $\overline{\mathcal{B}}$ is nonempty, we define three subsystems DTS(3): $\{(u,v,w),(w,v,u)\}$, $\{(v,w,u),(u,w,v)\}$, $\{(w,u,v),(v,u,w)\}$ for each subsystem MTS(3) $=\{\langle u,v,w\rangle, \langle w,v,u\rangle\}$, and add them into $\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3$, respectively. By the conclusion $(*)$, the ordered pairs contained in the transitive triples of $\mathcal{D}_i$ are the same as those contained in the cyclic triples of $\mathcal{B}$. Hence, each $(X,\mathcal{D}_i)$ is a $DTS(v,\lambda)$ and $\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3$ are the three cyclic shifts of $\mathcal{B}$. □

Example 4 MTS(7) $=(Z_7,\mathcal{B})$, where

$$\begin{aligned}\mathcal{B}=\{&\langle 0\ 3\ 4\rangle, \ \langle 1\ 3\ 5\rangle, \ \langle 2\ 3\ 6\rangle, \ \langle 0\ 1\ 2\rangle, \ \langle 0\ 2\ 5\rangle, \ \langle 0\ 5\ 6\rangle, \ \langle 0\ 6\ 1\rangle,\\ &\langle 4\ 3\ 0\rangle, \ \langle 5\ 3\ 1\rangle, \ \langle 6\ 3\ 2\rangle, \ \langle 4\ 6\ 5\rangle, \ \langle 4\ 1\ 6\rangle, \ \langle 4\ 2\ 1\rangle, \ \langle 4\ 5\ 2\rangle\}.\end{aligned}$$

$$\overline{\mathcal{B}}=\{\langle 0\ 3\ 4\rangle, \ \langle 1\ 3\ 5\rangle, \ \langle 2\ 3\ 6\rangle, \ \langle 4\ 3\ 0\rangle, \ \langle 5\ 3\ 1\rangle, \ \langle 6\ 3\ 2\rangle\}.$$

The block-incident graph $G(\mathcal{B})$ is



Obviously, $G(\mathcal{B})$ is a 3-regular graph and its 3-edge coloring is displayed as above. The three 1-factors are:

G_1	G_2	G_3
$\langle 0,1,2\rangle \overset{\{0,2\}}{--} \langle 0,2,5\rangle$	$\langle 0,1,2\rangle \overset{\{1,2\}}{--} \langle 4,2,1\rangle$	$\langle 0,1,2\rangle \overset{\{0,1\}}{--} \langle 0,6,1\rangle$
$\langle 4,5,2\rangle \overset{\{4,5\}}{--} \langle 4,6,5\rangle$	$\langle 0,2,5\rangle \overset{\{2,5\}}{--} \langle 4,5,2\rangle$	$\langle 0,2,5\rangle \overset{\{0,5\}}{--} \langle 0,5,6\rangle$
$\langle 0,5,6\rangle \overset{\{0,6\}}{--} \langle 0,6,1\rangle$	$\langle 4,6,5\rangle \overset{\{5,6\}}{--} \langle 0,5,6\rangle$	$\langle 4,5,2\rangle \overset{\{4,2\}}{--} \langle 4,2,1\rangle$
$\langle 4,1,6\rangle \overset{\{4,1\}}{--} \langle 4,2,1\rangle$	$\langle 0,6,1\rangle \overset{\{1,6\}}{--} \langle 4,1,6\rangle$	$\langle 4,6,5\rangle \overset{\{4,6\}}{--} \langle 4,1,6\rangle$

At last, we get three cyclic shifts of $\mathcal{B}$:

$$\begin{array}{llllll|lll}
\mathcal{D}_1: & (0\ 1\ 2) & (5\ 2\ 4) & (0\ 5\ 6) & (1\ 6\ 4) & & (0\ 3\ 4) & (1\ 3\ 5) & (2\ 3\ 6) \\
& (2\ 5\ 0) & (4\ 6\ 5) & (6\ 1\ 0) & (4\ 2\ 1) & & (4\ 3\ 0) & (5\ 3\ 1) & (6\ 3\ 2)
\end{array}$$

$$\begin{array}{llllll|lll}
\mathcal{D}_2: & (2\ 0\ 1) & (5\ 0\ 2) & (5\ 4\ 6) & (1\ 0\ 6) & & (3\ 4\ 0) & (3\ 5\ 1) & (3\ 6\ 2) \\
& (1\ 4\ 2) & (2\ 4\ 5) & (6\ 0\ 5) & (6\ 4\ 1) & & (0\ 4\ 3) & (1\ 5\ 3) & (2\ 6\ 3)
\end{array}$$

$$\begin{array}{llllll|lll}
\mathcal{D}_3: & (1\ 2\ 0) & (0\ 2\ 5) & (4\ 5\ 2) & (6\ 5\ 4) & & (4\ 0\ 3) & (5\ 1\ 3) & (6\ 2\ 3) \\
& (0\ 6\ 1) & (5\ 6\ 0) & (2\ 1\ 4) & (4\ 1\ 6) & & (3\ 0\ 4) & (3\ 1\ 5) & (3\ 2\ 6)
\end{array}$$

It is easy to see that $(Z_7, \mathcal{D}_i)$ is a DTS(7),$1 \le i \le 3$, and they are pairwise disjoint.

Conjecture For any MTS(v,λ) $(X,\mathcal{B})$, the block-incident graph $G(\mathcal{B})$ is 3-edge colorable.

Theorem 2 *If the block-incident graph of an* MTS(v,λ) *is 3-edge colorable, then there exist four pairwise disjoint* HTS(v,λ)*'s,* $v \ge 6$.

Proof First, an MTS$(v,\lambda) = (X,\mathcal{B})$ exists if and only if[2]:

$$\begin{cases} \lambda \equiv 0 \pmod 3, \ v \ge 3; \\ \lambda \equiv 1,2 \pmod 3, \ v \equiv 0,1 \pmod 3, \ v \ge 3 \ \text{且} \ (v,\lambda) \ne (6,1). \end{cases}$$

So, $|\mathcal{B}| = \frac{\lambda v(v-1)}{3}$ is even. Secondly, from Theorem 1, if the block-incident graph of an MTS(v,λ) $(X,\mathcal{B})$ is 3-edge colorable, then we can get three disjoint DTS(v,λ) $(X,\mathcal{D}_i)$, $i = 1,2,3$. And the blocks of the DTS satisfy: "if $(a,x,b) \in \mathcal{D}_i$, then there exists $y \in X$ such that $(b,y,a) \in \mathcal{D}_i$", i.e., the ordered pairs $(a,b),(b,a)$ which consist of both side elements of the two transitive triples are of negative direction each other. Since $|\mathcal{B}|$ is even, the transitive triples are in pairs, and the six ordered pairs contained in transitive triples $(a,x,b),(b,y,a)$ are the same as those in cyclic triples $\langle a,x,b\rangle, \langle b,y,a\rangle$. Hence, if we replace a pair of transitive triples $(u,w,v),(v,z,u)$ in each $\mathcal{D}_i$ $(i = 1,2,3)$ with cyclic triples $\langle u,w,v\rangle, \langle v,z,u\rangle$ (the six transitive triples must correspond to different cyclic triples. Since $v \ge 6$, $|\mathcal{B}| \ge 10$, the number of blocks satisfies the condition), we will get three HTS(v,λ)'s. At last, the six transitive triples together with the other cyclic triples in MTS(v,λ) will form an HTS(v,λ). It is not difficult to check that the four HTS(v,λ)'s are pairwise disjoint. $\square$

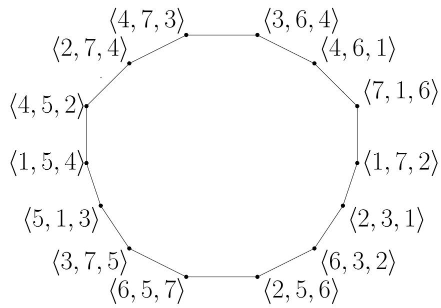
Example 5 Another MTS(7) $= (Z_7,\mathcal{B})$ (which is not isomorphic with Example 4), where

$$\begin{array}{llllllll}
\mathcal{B} = & \{\langle 4\ 7\ 3\rangle, & \langle 3\ 6\ 4\rangle, & \langle 4\ 6\ 1\rangle, & \langle 7\ 1\ 6\rangle, & \langle 1\ 7\ 2\rangle, & \langle 2\ 3\ 1\rangle, & \langle 6\ 3\ 2\rangle, \\
& \langle 2\ 5\ 6\rangle, & \langle 6\ 5\ 7\rangle, & \langle 3\ 7\ 5\rangle, & \langle 5\ 1\ 3\rangle, & \langle 1\ 5\ 4\rangle, & \langle 4\ 5\ 2\rangle, & \langle 2\ 7\ 4\rangle\}.
\end{array}$$

This MTS(7) has no MTS(3). The block-incident graph $G(\mathcal{B})$ and its 3-edge coloring are displayed as follows:

The three cyclic shifts of $\mathcal{B}$ are:

$$\begin{array}{llllllll}
\mathcal{D}_1: & (4\ 7\ 3) & (3\ 6\ 4) & (1\ 4\ 6) & (6\ 7\ 1) & (1\ 7\ 2) & (2\ 3\ 1) & (6\ 3\ 2) \\
& (2\ 5\ 6) & (7\ 6\ 5) & (5\ 3\ 7) & (1\ 3\ 5) & (5\ 4\ 1) & (4\ 5\ 2) & (2\ 7\ 4)
\end{array}$$



$$\begin{array}{lccccccc}
\mathcal{D}_2: & (7\ 3\ 4) & (4\ 3\ 6) & (6\ 1\ 4) & (1\ 6\ 7) & (7\ 2\ 1) & (3\ 1\ 2) & (2\ 6\ 3) \\
 & (6\ 2\ 5) & (5\ 7\ 6) & (3\ 7\ 5) & (5\ 1\ 3) & (4\ 1\ 5) & (5\ 2\ 4) & (4\ 2\ 7)
\end{array}$$

$$\begin{array}{lccccccc}
\mathcal{D}_3: & (3\ 4\ 7) & (6\ 4\ 3) & (4\ 6\ 1) & (7\ 1\ 6) & (2\ 1\ 7) & (1\ 2\ 3) & (3\ 2\ 6) \\
 & (5\ 6\ 2) & (6\ 5\ 7) & (7\ 5\ 3) & (3\ 5\ 1) & (1\ 5\ 4) & (2\ 4\ 5) & (7\ 4\ 2)
\end{array}$$

Furthermore, let

$$\begin{array}{lccccccc}
\mathcal{A}_0: & \underline{\langle 4\ 7\ 3\rangle} & \underline{\langle 3\ 6\ 4\rangle} & (1\ 4\ 6) & (6\ 7\ 1) & (1\ 7\ 2) & (2\ 3\ 1) & (6\ 3\ 2) \\
 & (2\ 5\ 6) & (7\ 6\ 5) & (5\ 3\ 7) & (1\ 3\ 5) & (5\ 4\ 1) & (4\ 5\ 2) & (2\ 7\ 4)
\end{array}$$

$$\begin{array}{lccccccc}
\mathcal{A}_1: & (7\ 3\ 4) & (4\ 3\ 6) & (6\ 1\ 4) & \underline{\langle 1\ 6\ 7\rangle} & \underline{\langle 7\ 2\ 1\rangle} & (3\ 1\ 2) & (2\ 6\ 3) \\
 & (6\ 2\ 5) & (5\ 7\ 6) & (3\ 7\ 5) & (5\ 1\ 3) & (4\ 1\ 5) & (5\ 2\ 4) & (4\ 2\ 7)
\end{array}$$

$$\begin{array}{lccccccc}
\mathcal{A}_2: & (3\ 4\ 7) & (6\ 4\ 3) & (4\ 6\ 1) & (7\ 1\ 6) & (2\ 1\ 7) & \underline{\langle 1\ 2\ 3\rangle} & (3\ 2\ 6) \\
 & (5\ 6\ 2) & (6\ 5\ 7) & (7\ 5\ 3) & \underline{\langle 3\ 5\ 1\rangle} & (1\ 5\ 4) & (2\ 4\ 5) & (7\ 4\ 2)
\end{array}$$

$$\begin{array}{lccccccc}
\mathcal{A}_3: & \underline{(4\ 7\ 3)} & \underline{(3\ 6\ 4)} & \langle 4\ 6\ 1\rangle & \underline{(1\ 6\ 7)} & \underline{(7\ 2\ 1)} & \underline{(1\ 2\ 3)} & \langle 6\ 3\ 2\rangle \\
 & \langle 2\ 5\ 6\rangle & \langle 6\ 5\ 7\rangle & \langle 3\ 7\ 5\rangle & \underline{(3\ 5\ 1)} & \langle 1\ 5\ 4\rangle & \langle 4\ 5\ 2\rangle & \langle 2\ 7\ 4\rangle
\end{array}$$

Then $(Z_7, \mathcal{D}_i)$ is a DTS(7), $i = 1, 2, 3$, and $(Z_7, \mathcal{A}_j)$ is an HTS(7), $j = 0, 1, 2, 3$.

3. Constructions for large set and overlarge set

In this part, the notion "some type" means: (1) resolvable; (2) almost resolvable; (3) pure; (4) no limitation.

Theorem 3 *If there exists "some type" large set or overlarge set of* MTS(v,λ), *and the block-incident graph corresponding to each* MTS(v,λ) *is 3-edge colorable, then there exists the same type large set or overlarge set of* DTS(v,λ) *and* HTS(v,λ).

Proof First, if the block-incident graph corresponding to each MTS(v,λ) is 3-edge colorable, then each MTS(v,λ) corresponds to three pairwise disjoint DTS(v,λ)'s from Theorem 1. And when some MTS(v,λ)'s are pairwise disjoint, the corresponding DTS(v,λ)'s are pairwise disjoint. Secondly, an LMTS(v,λ) has $(v-2)/\lambda$ MTS(v,λ)'s, each MTS(v,λ) has three cyclic shifts. So, there exist $3(v-2)/\lambda$ disjoint DTS(v,λ)'s. An OLMTS(v) has $v+1$ MTS(v)'s, and each MTS(v)

has three cyclic shifts. Then there exist $3(v+1)$ disjoint DTS(v)'s. The number of small design fits the demand. On the other hand, an cyclic triple corresponds to three transitive triples, the number of all blocks satisfies the condition. At last, when MTS(v,λ) is resolvable (or almost resolvable, or pure), its three cyclic shifts are also resolvable (or almost resolvable, or pure). Similarly, LHTS(v,λ) (or OLHTS(v)) exists from Theorem 2. □

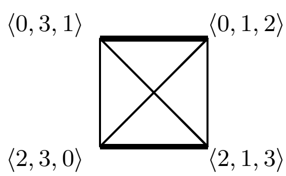
Corollary 4 *If "some type" large set or overlarge set of* MTS(v,λ) *can be generated from one or several base* MTS(v,λ) *under the action of a group of automorphisms, and the block-incident graphs corresponding to each base* MTS(v,λ) *are 3-edge colorable, then there exists the same type large set or overlarge set of* DTS(v,λ) *and* HTS(v,λ).

Example 6 LARMTS(4) $=\{(Z_4,\mathcal{B}_i):i=1,2\}$, where

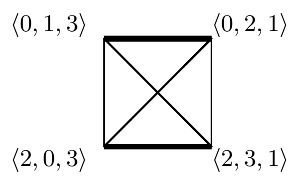
$$\mathcal{B}_1=\{\langle 0,1,2\rangle,\langle 2,1,3\rangle,\langle 2,3,0\rangle,\langle 0,3,1\rangle\};$$
$$\mathcal{B}_2=\{\langle 0,2,1\rangle,\langle 2,3,1\rangle,\langle 2,0,3\rangle,\langle 0,1,3\rangle\}.$$

The block-incident graph and 3-edge coloring are displayed as follows:

$G(\mathcal{B}_1)$:



$G(\mathcal{B}_2)$:



The three cyclic shifts of $\mathcal{B}_1$ are:

$$\mathcal{D}_1=\{(2,0,1),(1,3,2),(0,2,3),(3,1,0)\};$$
$$\mathcal{D}_2=\{(1,2,0),(0,3,1),(3,0,2),(2,1,3)\};$$
$$\mathcal{D}_3=\{(0,1,2),(2,3,0),(3,2,1),(1,0,3)\}.$$

The three cyclic shifts of $\mathcal{B}_2$ are:

$$\mathcal{C}_1=\{(1,0,2),(2,3,1),(3,2,0),(0,1,3)\};$$
$$\mathcal{C}_2=\{(0,2,1),(1,3,0),(2,0,3),(3,1,2)\};$$
$$\mathcal{C}_3=\{(2,1,0),(0,3,2),(1,2,3),(3,0,1)\}.$$

Then $\{(Z_4,\mathcal{D}_i):i=1,2,3\}\cup\{(Z_4,\mathcal{C}_i):i=1,2,3\}$ is an LARDTS(4).

Next, when we display the block-incident graph $G(\mathcal{B})$ and its 3-edge coloring of an MTS(v,λ) $(X,\mathcal{B})$, we can only write the 2-factor formed by some even cycles. The two sets which consist of adjacent edges in even cycles will give transitive triple sets $\mathcal{D}_1,\mathcal{D}_2$, and the remainder cyclic shift for all cyclic triples will give $\mathcal{D}_3$.

Example 7 OLRMTS(9) $=\{(Z_{10}\backslash\{x\},\mathcal{B}_x):\ x\in Z_{10}\}$, where $\mathcal{B}_x=\mathcal{B}_0+x,\ x\in Z_{10}$,

$$\begin{array}{llllllll}
\mathcal{B}_0: & \langle 2\ 4\ 1\rangle & \langle 5\ 4\ 7\rangle & \langle 8\ 4\ 5\rangle & \langle 6\ 7\ 3\rangle & \langle 3\ 7\ 4\rangle & \langle 6\ 5\ 2\rangle & \langle 2\ 3\ 8\rangle & \langle 8\ 7\ 2\rangle \\
 & \langle 3\ 5\ 6\rangle & \langle 6\ 9\ 8\rangle & \langle 2\ 7\ 6\rangle & \langle 9\ 4\ 2\rangle & \langle 8\ 1\ 6\rangle & \langle 4\ 8\ 3\rangle & \langle 4\ 9\ 6\rangle & \langle 6\ 1\ 4\rangle \\
 & \langle 7\ 8\ 9\rangle & \langle 1\ 3\ 2\rangle & \langle 3\ 1\ 9\rangle & \langle 5\ 1\ 8\rangle & \langle 2\ 5\ 9\rangle & \langle 9\ 1\ 7\rangle & \langle 7\ 1\ 5\rangle & \langle 9\ 5\ 3\rangle
\end{array}$$

The 2-factor of $G(\mathcal{B}_0)$ is:

$$(\langle 2\ 4\ 1\rangle \text{ - } \langle 1\ 3\ 2\rangle \text{ - } \langle 3\ 1\ 9\rangle \text{ - } \langle 9\ 5\ 3\rangle \text{ - } \langle 2\ 5\ 9\rangle \text{ - } \langle 9\ 4\ 2\rangle \text{ - } \langle 4\ 9\ 6\rangle \text{ - } \langle 6\ 1\ 4\rangle);$$
$$(\langle 6\ 9\ 8\rangle \text{ - } \langle 8\ 1\ 6\rangle \text{ - } \langle 5\ 1\ 8\rangle \text{ - } \langle 8\ 4\ 5\rangle \text{ - } \langle 5\ 4\ 7\rangle \text{ - } \langle 7\ 1\ 5\rangle \text{ - } \langle 9\ 1\ 7\rangle \text{ - } \langle 7\ 8\ 9\rangle);$$
$$(\langle 6\ 7\ 3\rangle \text{ - } \langle 3\ 5\ 6\rangle \text{ - } \langle 6\ 5\ 2\rangle \text{ - } \langle 2\ 7\ 6\rangle \text{ - } \langle 8\ 7\ 2\rangle \text{ - } \langle 2\ 3\ 8\rangle \text{ - } \langle 4\ 8\ 3\rangle \text{ - } \langle 3\ 7\ 4\rangle).$$

The three cyclic shifts of $\mathcal{B}_0$ are:

$$\begin{array}{lllllllll}
\mathcal{D}_0^1: & (2\ 4\ 1) & (5\ 4\ 7) & (8\ 4\ 5) & (6\ 7\ 3) & (3\ 7\ 4) & (6\ 5\ 2) & (2\ 3\ 8) & (8\ 7\ 2) \\
 & (3\ 5\ 6) & (6\ 9\ 8) & (2\ 7\ 6) & (9\ 4\ 2) & (8\ 1\ 6) & (4\ 8\ 3) & (4\ 9\ 6) & (6\ 1\ 4) \\
 & (7\ 8\ 9) & (1\ 3\ 2) & (3\ 1\ 9) & (5\ 1\ 8) & (2\ 5\ 9) & (9\ 1\ 7) & (7\ 1\ 5) & (9\ 5\ 3)
\end{array}$$

$$\begin{array}{lllllllll}
\mathcal{D}_0^2: & (4\ 1\ 2) & (7\ 5\ 4) & (4\ 5\ 8) & (7\ 3\ 6) & (4\ 3\ 7) & (2\ 6\ 5) & (3\ 8\ 2) & (7\ 2\ 8) \\
 & (5\ 6\ 3) & (9\ 8\ 6) & (6\ 2\ 7) & (2\ 9\ 4) & (6\ 8\ 1) & (8\ 3\ 4) & (6\ 4\ 9) & (1\ 4\ 6) \\
 & (8\ 9\ 7) & (2\ 1\ 3) & (9\ 3\ 1) & (1\ 8\ 5) & (5\ 9\ 2) & (1\ 7\ 9) & (5\ 7\ 1) & (3\ 9\ 5)
\end{array}$$

$$\begin{array}{lllllllll}
\mathcal{D}_0^3: & (1\ 2\ 4) & (4\ 7\ 5) & (5\ 8\ 4) & (3\ 6\ 7) & (7\ 4\ 3) & (5\ 2\ 6) & (8\ 2\ 3) & (2\ 8\ 7) \\
 & (6\ 3\ 5) & (8\ 6\ 9) & (7\ 6\ 2) & (4\ 2\ 9) & (1\ 6\ 8) & (3\ 4\ 8) & (9\ 6\ 4) & (4\ 6\ 1) \\
 & (9\ 7\ 8) & (3\ 2\ 1) & (1\ 9\ 3) & (8\ 5\ 1) & (9\ 2\ 5) & (7\ 9\ 1) & (1\ 5\ 7) & (5\ 3\ 9)
\end{array}$$

Then $\{(Z_{10}\backslash\{x\}, \mathcal{D}_x^r):\ x \in Z_{10},\ r = 1,2,3\}$ is an OLRDTS(9), where $\mathcal{D}_x^r = \mathcal{D}_0^r + x,\ x \in Z_{10},\ r = 1,2,3$.

Moreover, let

$$\begin{array}{lllllllll}
\mathcal{A}_0^0: & (1\ 2\ 4) & (4\ 7\ 5) & (5\ 8\ 4) & (3\ 6\ 7) & (7\ 4\ 3) & (5\ 2\ 6) & (8\ 2\ 3) & \langle 2\ 8\ 7\rangle \\
 & (6\ 3\ 5) & (8\ 6\ 9) & \langle 7\ 6\ 2\rangle & (4\ 2\ 9) & (1\ 6\ 8) & (3\ 4\ 8) & (9\ 6\ 4) & (4\ 6\ 1) \\
 & (9\ 7\ 8) & (3\ 2\ 1) & (1\ 9\ 3) & (8\ 5\ 1) & (9\ 2\ 5) & (7\ 9\ 1) & (1\ 5\ 7) & (5\ 3\ 9) \\
\mathcal{A}_0^1: & \langle 2\ 4\ 1\rangle & (5\ 4\ 7) & (8\ 4\ 5) & (6\ 7\ 3) & (3\ 7\ 4) & (6\ 5\ 2) & (2\ 3\ 8) & (8\ 7\ 2) \\
 & (3\ 5\ 6) & (6\ 9\ 8) & (2\ 7\ 6) & (9\ 4\ 2) & (8\ 1\ 6) & (4\ 8\ 3) & (4\ 9\ 6) & (6\ 1\ 4) \\
 & (7\ 8\ 9) & \langle 1\ 3\ 2\rangle & (3\ 1\ 9) & (5\ 1\ 8) & (2\ 5\ 9) & (9\ 1\ 7) & (7\ 1\ 5) & (9\ 5\ 3) \\
\mathcal{A}_0^2: & (4\ 1\ 2) & \langle 7\ 5\ 4\rangle & (4\ 5\ 8) & (7\ 3\ 6) & \langle 4\ 3\ 7\rangle & (2\ 6\ 5) & (3\ 8\ 2) & (7\ 2\ 8) \\
 & (5\ 6\ 3) & (9\ 8\ 6) & (6\ 2\ 7) & (2\ 9\ 4) & (6\ 8\ 1) & (8\ 3\ 4) & (6\ 4\ 9) & (1\ 4\ 6) \\
 & (8\ 9\ 7) & (2\ 1\ 3) & (9\ 3\ 1) & (1\ 8\ 5) & (5\ 9\ 2) & (1\ 7\ 9) & (5\ 7\ 1) & (3\ 9\ 5) \\
\mathcal{A}_0^3: & (2\ 4\ 1) & (7\ 5\ 4) & \langle 8\ 4\ 5\rangle & \langle 6\ 7\ 3\rangle & (4\ 3\ 7) & \langle 6\ 5\ 2\rangle & \langle 2\ 3\ 8\rangle & (2\ 8\ 7) \\
 & \langle 3\ 5\ 6\rangle & \langle 6\ 9\ 8\rangle & (7\ 6\ 2) & \langle 9\ 4\ 2\rangle & \langle 8\ 1\ 6\rangle & \langle 4\ 8\ 3\rangle & \langle 4\ 9\ 6\rangle & \langle 6\ 1\ 4\rangle \\
 & \langle 7\ 8\ 9\rangle & (1\ 3\ 2) & \langle 3\ 1\ 9\rangle & \langle 5\ 1\ 8\rangle & \langle 2\ 5\ 9\rangle & \langle 9\ 1\ 7\rangle & \langle 7\ 1\ 5\rangle & \langle 9\ 5\ 3\rangle
\end{array}$$

Then $\{(Z_{10}\backslash\{x\}, \mathcal{A}_x^r):\ x \in Z_{10},\ r = 0,1,2,3\}$ is an OLRHTS(9), where $\mathcal{A}_x^r = \mathcal{A}_0^r + x,\ x \in Z_{10},\ r = 0,1,2,3$.

References

[1] BONDY J A, MURTY U S R. *Graph Theory with Applications* [M]. American Elsevier Publishing Co., Inc., New York, 1976.

[2] SHEN Hao. *Embeddings of pure Mendelsohn triple systems and pure directed triple systems* [J]. J. Combin. Des., 1995, **3**(1): 41–50.

[3] LINDER C C, STREET A. *Construction of lar sets of pairwise disjoint transitive triple systems* [J]. European J. Combin., 1983, **4**: 335–346.

[4] KANG Qingde, LEI Jianguo, CHANG Yanxun. *The spectrum for large sets of disjoint Mendelsohn triple system with any index* [J]. J. Combin. Des., 1994, **2**(5): 351–358.

[5] KANG Qingde, CHANG Yanxun. *A completion of the spectrum for large set of disjoint transitive triple systems* [J]. J. Combin. Theory Ser. A, 1992, **60**(2): 287–294.

[6] KANG Qingde, LEI Jianguo. *On large sets of hybrid triple systems* [J]. J. Statist. Plann. Inference, 1996, **51**(2): 181–188.

[7] ZHANG Jie, TIAN Zihong. *Further results on overlarge sets of oriented triple systems* [J]. Gongcheng Shuxue Xuebao, 2004, **21**(3): 345–350. (in Chinese)

[8] KANG Qingde. *Large sets of three types of oriented triple systems* [J]. J. Statist. Plann. Inference, 1997, **58**(1): 151–176.

[9] BENNETT F E, KANG Qingde, LEI Jianguo. et al. *Large sets of disjoint pure Mendelsohn triple systems* [J]. J. Statist. Plann. Inference, 2001, **95**(1-2): 89–115.

[10] LEI Jianguo. *Further results on large set of disjoint pure Mendelsohn triple systems* [J]. J. Combin. Des., 2000, **8**(4): 274–290.