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Packings and Coverings of a Graph with 6 Vertices and 7 Edges

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Abstract Let λK_v be the complete multigraph with v vertices and G a finite simple graph. A G-design (G-packing design, G-covering design) of λK_v, denoted by (v, G, λ)-GD ((v, G, λ)-PD, (v, G, λ)-CD), is a pair $(X, \mathcal{B})$ where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called blocks, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly (at most, at least) λ blocks of $\mathcal{B}$. A packing (covering) design is said to be maximum (minimum) if no other such packing (covering) design has more (fewer) blocks. In this paper, a simple graph G with 6 vertices and 7 edges is discussed, and the maximum G-$PD(v)$ and the minimum G-$CD(v)$ are constructed for all orders v.

Keywords G-design; G-packing design; G-covering design.

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1. Introduction

A complete multigraph of order v and index λ, denoted by λK_v, is a graph with v vertices, where any two distinct vertices x and y are joined by λ edges $\{x, y\}$. Let G be a finite simple graph. A G-design (G-packing design, G-covering design) of λK_v, denoted by G-$GD_\lambda(v)$ (G-$PD_\lambda(v)$, G-$CD_\lambda(v)$), is a pair $(X, \mathcal{B})$ where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called blocks, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly (at most, at least) λ blocks of $\mathcal{B}$. The necessary conditions for the existence of a G-$GD_\lambda(v)$ are $v \geq |V(G)|$ and

$$\begin{cases} \lambda v(v-1) \equiv 0 \pmod{2|E(G)|} \\ \lambda(v-1) \equiv 0 \pmod{d} \end{cases} \tag{*}$$

where $V(G)$ and $E(G)$ denote the sets of vertices and edges of G respectively, and d is the greatest common divisor of the degrees of all vertices in G. A G-$PD_\lambda(v)$ (G-$CD_\lambda(v)$) is called maximum (minimum) if no other such packing (covering) has more (fewer) blocks. The number of blocks in a maximum packing (minimum covering), denoted by $p(v, G, \lambda)$ ($c(v, G, \lambda)$), is called

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the packing (covering) number. It is well known that

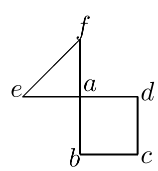
$$p(v,G,\lambda) \le \left\lfloor \frac{\lambda v(v-1)}{2|E(G)|} \right\rfloor \le \left\lceil \frac{\lambda v(v-1)}{2|E(G)|} \right\rceil \le c(v,G,\lambda). \tag{**}$$

A G-$PD_\lambda(v)$ (G-$CD_\lambda(v)$) is called optimal and denoted by G-$OPD_\lambda(v)$ (G-$OCD_\lambda(v)$) if the left (right) equality in $(**)$ holds.

The leave $L_\lambda(\mathcal{D})$ of a packing $\mathcal{D}$ is a subgraph of λK_v and its edges are the supplement of $\mathcal{D}$ in λK_v. The number of edges in $L_\lambda(\mathcal{D})$ is denoted by $|L_\lambda(\mathcal{D})|$. Especially, when $\mathcal{D}$ is maximum, $|L_\lambda(\mathcal{D})|$ is called leave-number and is denoted by $l_\lambda(v)$. Similarly, the excess $R_\lambda(\mathcal{D})$ of a covering D is a subgraph of λK_v and its edges are the supplement of λK_v in $\mathcal{D}$. When $\mathcal{D}$ is minimum, $|R_\lambda(\mathcal{D})|$ is called excess-number and denoted by $r_\lambda(v)$.

Let $X = \bigcup_{i=1}^t X_i$ be the vertex set of $K_{n_1,n_2,\ldots,n_t}$, a complete multipartite graph consisting of t parts with size $n_1, n_2, \ldots, n_t$ respectively, where the sets X_i $(1 \le i \le t)$ are disjoint. Denote $v = \sum_{i=1}^t n_i$ and $\mathcal{G} = \{X_1, X_2, \ldots, X_t\}$. For any given graph G, if the edges of $\lambda K_{n_1,n_2,\ldots,n_t}$, a t-partite graph with replication λ, can be decomposed into sub-graphs $\mathcal{A}$, each of which is isomorphic to G and is called block, then the system $(X, \mathcal{G}, \mathcal{A})$ is called a holey G-design with index λ, denoted by G-$HD_\lambda(T)$, where $T = n_1^1 n_2^1 \cdots n_t^1$ is the type of the holey G-design. Usually, the type is denoted by exponential form, for example, the type $n_1^{k_1} n_2^{k_2} \cdots n_m^{k_m}$ denotes k_1 occurrences of n_1, k_2 occurrences of n_2, etc. A G-$HD_\lambda(1^{v-w}w^1)$ is called an incomplete G-design, denoted by G-$ID_\lambda(v,w)$.

There is a quite long time to the reseach of the graph packing and covering designs, which involved the simple graphs with less vertices and less edges[1]−[4], and some special graphs[5]. But there is a few conclusions for the simple graphs with more than five vertices. In this paper, we will discuss the maximum packing and the minimum covering of the folowing graph G with six vertices and seven edges for $\lambda = 1$. For convenience, as a block in a design, the graph may be denoted by (a,b,c,d,e,f) according to the following vertex-labels.



Lemma 1.1[6] *There exists a G-$GD_\lambda(v)$ if and only if $v \ge 6$ and $\lambda = 1, v \equiv 1, 7 \bmod 14$; $\lambda = 2, v \equiv 0, 8 \bmod 14$; $\lambda = 7, v \equiv 3, 5, 9, 11, 13 \bmod 14$; $\lambda = 14, v \equiv 0 \bmod 2$.*

2. Recursive method

In what follows, the element (x,i) in $Z_m \times Z_n$ may be denoted by x_i briefly. And $x_i + y_j = (x+y)_{i+j}$, $\infty + x = \infty$, $\infty + x_i = \infty$, $\infty_l + x_i = \infty_{l+i}$. For the block $B = (x,y,z,u,v,w)$, $B + t = (x+t, y+t, z+t, u+t, v+t, w+t)$. In $Z_m \times Z_n$, a block mod (m,n) means that the first coordinate is mod m and the second is mod n, while mod $(m,-)$ means that the first coordinate is mod m and the second does not change. For $i, j, k \in Z$, $i < j$ and $i \equiv j \pmod{k}$, define $[i,j]_k = \{x : i \le x \le j,\ x \equiv i \pmod{k}\}$.

What's more, the symbols C_n, P_n and $St(n)$ denote the cycle with n vertices, the path with n vertices, and the star with n terminal vertices, respectively. The disjoint union of graphs G and H is denoted by $G \bigcup H$. Specially, the disjoint union of n graphs G is denoted by nG. And, $\triangle_2$ means two C_3 with one common vertex.

Our recursive constructions use the following standard "Filling in Holes" method.

Lemma 2.1[6] *For given graph G and positive integers h, w, m, λ, if there exist a G-$HD_\lambda(h^m)$, a G-$ID_\lambda(h+w, w)$ and a G-$OPD_\lambda(s)$ (G-$OCD_\lambda(s)$), where $s = w$ or $h+w$, then there exists a G-$OPD_\lambda(mh+w)$ (G-$OCD_\lambda(mh+w)$) too.*

Lemma 2.2 *There exists no G-$OPD(v)$ and $p(v, G, 1) \le \lfloor \frac{\binom{v}{2} - \frac{v}{2}}{7} \rfloor$ for even v. There exists no G-$OPD(v)$ for $v \equiv 9, 13 \pmod{14}$. The leave of a G-$OPD(v)$ must be C_3 for $v \equiv 3, 5 \pmod{14}$, or one among $\{C_6, 2C_3, \triangle_2\}$ for $v \equiv 11 \pmod{14}$.*

Proof For even v, since the degree of any vertex in G is even, each vertex in K_v must appear in the leave, so the leave-number $l \ge \frac{v}{2}$. However, for any OPD, the leave-number $l \le 6$. It is impossible that $l \ge \frac{v}{2}$ for $v > 12$. But, for the remaining even orders $v = 6, 8, 10$ and 12, the leave-numbers of corresponding OPD are $1, 0, 3$ and 3. Thus, there exists no G-$OPD(v)$ and $p(v, G, 1) \le \lfloor \frac{\binom{v}{2} - \frac{v}{2}}{7} \rfloor$.

For odd v, the degree of any vertex in the leave must be even. If a G-$OPD(v)$ exists, the leave-number will be the following three cases.

$l = 1$ for $v \equiv 9, 13$ mod 14, so there exists no G-$OPD(v)$ since the leave must be P_2;

$l = 3$ for $v \equiv 3, 5$ mod 14, so the leave can only be $St(3), 3P_2, P_4, P_2 \bigcup P_3$ or C_3, but the first four graphs cannot become a leave;

$l = 6$ for $v \equiv 11$ mod 14. Let the number of $2°$-vertices, $4°$-vertices and $6°$-vertices be a, b and c, respectively. Then $2a + 4b + 6c = 12$. The solutions are $(a, b, c) = (0, 0, 2), (1, 1, 1), (3, 0, 1), (0, 3, 0), (2, 2, 0), (4, 1, 0)$ and $(6, 0, 0)$. Obviously, the first five solutions have no corresponding graphs, while the graphs satisfying the last two solutions are C_6, $2C_3$ and $\triangle_2$.

Since there is no G-$GD(14)$ by Lemma 1.1, a new recursive construction is presented below. First, define two auxiliary designs for even w:

G-$IPD(14 + w, w)$ is a G-$HD(2^7 w^1)$;

G-$ICD(14 + w, w)$ is a union of a G-$ID(14 + w, w)$ and $7P_2$ on the 14-set.

In the view of PD and CD, the first one is with the leave $K_w \bigcup 7P_2$, and the second one is with the leave K_w and the excess $7P_2$.

Lemma 2.3 *For $t > 0$ and even $w \ge 0$, if there exist a G-$HD(14^t)$, a G-$IPD(14 + w, w)$ (G-$ICD(14+w, w)$) and a maximum G-$PD(w)$ (minimum G-$CD(w)$), then there exists a maximum G-$PD(14t + w)$ (minimum G-$CD(14t + w)$).*

Proof Let $X = (Z_{14} \times Z_t) \bigcup W$, where W is another w-set. Denote G-$HD(14^t) = (Z_{14} \times Z_t, \mathcal{A})$, G-$IPD(14 + w, w) = ((Z_{14} \times \{i\}) \bigcup W, \mathcal{B}_i)$, or G-$ICD(14 + w, w)$, $i \in Z_t$. And, $\mathcal{C}$ is a maximum

G-$PD(w)$ or minimum G-$CD(w)$ on the set W. Define $\Omega = \mathcal{A}\bigcup(\bigcup_{i=0}^{t-1}\mathcal{B}_i)\bigcup\mathcal{C}$. Then (X,Ω) is a maximum G-$PD(14+w)$ (minimum G-$CD(14+w)$). In fact, denote $v = 14t + w$. Then

$$|\mathcal{A}| = \frac{\binom{t}{2}14^2}{7} = 14t^2 - 14t,\ |\mathcal{B}_i| = \frac{\binom{14+w}{2} - \binom{w}{2} - 7}{7} = 12 + 2w,\ |\mathcal{C}| = \lfloor\frac{\binom{w}{2} - \frac{w}{2}}{7}\rfloor = \lfloor\frac{w(w-2)}{14}\rfloor.$$

But, by Lemma 2.2,

$$p(v,G,1) \le \lfloor\frac{\binom{v}{2} - \frac{v}{2}}{7}\rfloor = 14t^2 + (2w-2)t + \lfloor\frac{w(w-2)}{14}\rfloor.$$

Therefore, $|\mathcal{A}| + t|\mathcal{B}_i| + |\mathcal{C}| = 14t^2 + (2w-2)t + \lfloor\frac{w(w-2)}{14}\rfloor = p(v,G,1)$. Furthermore, there is no excess edge in Ω, so the lemma holds for PD.

As for CD, $|\mathcal{B}_i| = \frac{\binom{14+w}{2}-\binom{w}{2}+7}{7} = 14 + 2w,\ |\mathcal{C}| = \lceil\frac{\binom{w}{2}-\frac{w}{2}}{7}\rceil = \lceil\frac{w^2}{14}\rceil,\ c(v,G,1) \ge \lceil\frac{\binom{v}{2}+\frac{v}{2}}{7}\rceil = 14t^2 + 2wt + \lceil\frac{w^2}{14}\rceil$, so $c(v,G,1) = 14t^2 + 2wt + \lceil\frac{w^2}{14}\rceil$.

By the recursive constructions in Lemmas 2.1 and 2.3, our task in §3 and §4 is to construct G-$HD(14^t)$ and those small designs listed in the following Table.

v(mod14)	$ID(14+w,w)$	$IPD(14+w,w), ICD(14+w,w)$	maxPD, minCD
0		$w=0$	$0,28$
2		$w=2$	$2,30$
3	$w=3$		$3,31$
4		$w=4$	$4,32$
5	$w=5$		$5(19),33$
6		$w=6$	$6,34$
8		$w=8$	$8,36$
9	$w=9$		$9,37$
10		$w=10$	$10,38$
11	$w=11$		$11,39$
12		$w=12$	$12,40$
13	$w=13$		$13,41$

Table 2.1 The desired designs for the main results

3. HD, ID and IPD (ICD)

Lemma 3.1[6] *There exists a G-$HD(14^t)$ for $t \ge 3$.*

Lemma 3.2 *There exists a G-$ID(14+w,w)$ for $w = 3,5,9,11,13$, no one for any even w.*

Proof Take the vertex set $(Z_7 \times Z_2)\bigcup\{x_1,\ldots,x_w\}$, where $\{x_1,\ldots,x_w\}$ is the hole.

$\underline{w=3:}$ $(0_0,x_1,0_1,x_2,4_1,6_1),(0_1,2_0,x_3,3_1,1_1,0_0)$, mod $(7,-)$;

$(0_0,2_1,6_0,5_0,3_1,1_0),(1_0,6_0,1_1,5_0,4_1,2_0)$, $(2_0,0_0,6_0,4_0,5_1,3_0)$,

$(3_0,6_0,2_0,5_0,6_1,4_0),(4_0,0_0,3_0,1_0,0_1,5_0)$.

$\underline{w=5:}$ $(0_0,3_0,x_1,3_1,1_1,2_1),(0_1,0_0,x_2,2_1,x_4,3_0),(0_1,3_1,x_3,1_0,x_5,2_0)$ mod $(7,-)$;

$(0_0, 1_0, 3_0, 2_0, 6_0, 5_0), (4_0, 2_0, 1_0, 6_0, 3_0, 5_0)$.

<u>$w = 9$:</u> $(0_0, x_5, 2_1, x_1, 0_1, 1_1), (0_0, x_6, 0_1, x_2, 2_1, 4_1), (0_0, x_7, 0_1, x_3, 3_1, 6_1)$,
$(0_0, x_8, 0_1, x_4, x_9, 5_1)$ mod $(7, -)$;
$(3_0, 0_0, 1_0, 2_0, 4_0, 5_0), (6_0, 4_0, 0_0, 2_0, 1_0, 3_0), (5_0, 2_0, 4_0, 1_0, 6_0, 0_0)$.

<u>$w = 11$:</u> $(0_0, 1_0, x_1, 1_1, x_6, 0_1), (0_0, 2_0, x_2, 3_1, x_7, 2_1), (0_0, 3_0, x_3, 5_1, x_8, 4_1)$,
$(0_1, x_9, 0_0, x_4, 1_1, 3_1), (0_0, x_{10}, 0_1, x_5, x_{11}, 6_1)$. mod $(7, -)$.

<u>$w = 13$:</u> $(0_0, x_2, 1_1, x_1, x_3, 0_1), (0_0, x_5, 0_1, x_4, x_6, 3_1), (0_0, x_8, 0_1, x_7, x_9, 5_1)$,
$(0_0, x_{11}, 0_1, x_{10}, 2_0, 6_1)$, $(0_1, x_{13}, 0_0, x_{12}, 1_1, 3_1)$ mod $(7, -)$;
$(6_0, 5_0, 4_0, 3_0, 1_1, 0_0), (1_0, 4_0, 6_1, 5_0, 2_1, 0_0), (2_0, 5_0, 0_1, 6_0, 3_1, 1_0), (3_0, 5_1, 4_0, 0_0, 4_1, 2_0)$.

If G-$ID(14 + w, w)$ exists for even w, take the vertex set $Z_{14} \bigcup \{x_1, \ldots, x_w\}$, then the degree $13 + w$ of any vetex in Z_{14} is odd. It is a contrary since the degree of any vertex in G is even.

Lemma 3.3 *There exists a G-$IPD(14 + w, w)$ for $w \in [0, 12]_2$.*

Proof Each G-$IPD(14 + w, w)$ is constructed on the given vertex set.

<u>$w = 0$:</u> $Z_{12} \bigcup \{x_1, x_2\}$, $(0, 4, x_1, 5, 1, 3) + 2i$, $(0, 4, x_2, 5, 1, 3) + 2i + 1$ $0 \le i \le 5$.

<u>$w = 2$:</u> Z_{16}, $(0, 4, 11, 5, 1, 3)$ mod 16.

<u>$w = 4$:</u> $Z_{14} \bigcup \{a, b, c, d\}$, $(0, 4, a, 5, 1, 3) + 2i$, $(0, 4, b, 5, 1, 3) + 2i + 1$ $0 \le i \le 6$;
$(c, 6, 0, 8, 1, 7), (c, 10, 4, 12, 13, 5), (c, 0, d, 2, 3, 11)$,
$(d, 1, 9, 3, 12, 6), (d, 8, 2, 10, 7, 13), (d, 4, c, 9, 5, 11)$.

<u>$w = 6$:</u> $Z_{14} \bigcup \{x_1, \ldots, x_6\}$, $(0, 3, x_1, 4, 1, 6) + 2i$, $(0, 3, x_2, 4, 1, 6) + 2i + 1$ $0 \le i \le 6$;
$(0, x_4, 7, x_5, x_3, 2) + i$ $i = 0, 1, 4, 5$; $(0, x_4, 7, x_5, x_6, 2) + i$ $i = 2, 3, 6$;
$(x_6, 7, 9, 11, 0, 12)$, $(x_6, 10, x_3, 9, 1, 13)$, $(x_3, 12, 10, 8, 11, 13)$.

<u>$w = 8$:</u> $(Z_7 \times Z_2) \bigcup \{x_1, \ldots, x_8\}$, $(0_0, x_1, 3_1, 6_1, x_2, 5_1), (0_0, x_3, 1_1, 3_1, x_4, 4_1)$,
$(0_0, x_5, 0_1, 1_1, x_6, 2_1), (0_0, x_7, 0_1, x_8, 1_0, 3_0)$ mod $(7, -)$.

<u>$w = 10$:</u> $Z_{14} \bigcup \{x_1, \ldots, x_{10}\}$, $(0, x_3, 7, x_5, x_1, 1) + 2i$, $(0, x_4, 7, x_6, x_2, 1) + 2i + 1$,
$(0, x_7, 7, x_9, 2, 5) + i$, $(0, x_8, 7, x_{10}, 2, 5) + i + 7$ $0 \le i \le 6$;
$(6, 0, 4, 12, 2, 10) + i$, $(8, 0, 10, 4, 2, 12) + i$ $i = 0, 1$.

<u>$w = 12$:</u> $(Z_{12} \times Z_2) \bigcup \{x_1, x_2\}$, $(4_1, 2_0, x_1, 2_1, 0_0, 1_1), (5_1, 2_0, x_2, 1_1, 0_0, 0_1)$,
$(0_0, 9_1, 1_0, 11_1, 6_1, 7_1)$ mod $(12, -)$.

Lemma 3.4 *There exists a G-$ICD(14 + w, w)$ for $w \in [0, 12]_2$.*

Proof Take the vertex set Z_{14} for $w = 0$ or $(Z_7 \times Z_2) \bigcup \{x_1, \ldots, x_w\}$ for $w > 0$.

<u>$w = 0$:</u> $(0, 4, 11, 5, 1, 3)$ mod 14.

<u>$w = 2$:</u> $(0_1, 4_0, x_1, 1_1, 0_0, 2_0), (6_1, 6_0, x_2, 3_1, 0_0, 4_1)$ mod $(7, -)$;
$(6_0, 3_0, 4_0, 5_0, 0_0, 1_1), (1_0, 5_0, 6_1, 4_0, 0_0, 2_1), (2_0, 6_0, 0_1, 5_0, 1_0, 3_1), (3_0, 0_0, 4_0, 5_1, 2_0, 4_1)$.

<u>$w = 4$:</u> $(0_0, 3_0, x_3, 2_1, 0_1, x_4), (0_1, 2_0, 5_1, 1_0, 1_1, 3_1)$ mod $(7, -)$;
$(2_0, 4_0, x_1, 3_1, 2_1, x_2) + i_0$ $i = 0, 1, 3, 4$;
$(0_0, 2_0, 3_0, 1_0, 0_1, x_2), (6_0, x_1, 1_1, 0_0, 4_0, 5_0), (1_0, 2_0, x_1, 2_1, 1_1, x_2), (4_0, 3_0, x_1, 5_1, 4_1, x_2)$.

<u>$w = 6$:</u> $(0_1, 1_0, x_1, 1_1, 0_0, x_2), (5_1, 1_0, x_3, 3_1, 0_0, x_4), (1_1, 1_0, x_5, 4_1, 0_0, x_6)$ mod $(7, -)$;
$(0_0, 5_0, 6_0, 2_1, 1_0, 3_1), (1_0, 5_0, 1_1, 6_0, 2_0, 4_1), (2_0, 4_0, 6_0, 0_0, 3_0, 5_1)$,

$(3_0,5_0,2_0,6_0,4_0,6_1),(4_0,1_0,3_0,0_0,5_0,0_1)$.

$\underline{w=8:}$ $(0_0,3_0,x_1,1_1,0_1,x_2),(3_1,0_0,x_3,2_1,1_0,x_4),(5_1,0_0,x_5,3_1,1_0,x_6),$
$(6_1,0_0,x_7,3_1,6_0,x_8)$ mod $(7,-)$; $(0_0,6_0,4_0,5_0,1_0,2_0),(3_0,5_0,6_0,1_0,2_0,4_0)$.

$\underline{w=10}:$ $(0_0,x_1,1_1,x_2,0_1,x_3),(0_0,x_4,1_1,x_5,0_1,x_6),(4_1,2_0,x_7,2_1,0_0,x_8),$
$(6_1,1_0,x_9,3_1,0_0,x_{10})$ mod $(7,-)$; $(0_0,3_1,2_1,1_1,2_0,6_0),(2_0,5_1,4_1,3_1,1_0,3_0),$
$(4_0,0_1,6_1,5_1,5_0,2_0),(6_0,5_0,1_1,0_1,4_0,3_0),(5_0,1_0,4_0,0_0,3_0,6_1),(1_0,0_0,3_0,4_1,6_0,2_1)$.

$\underline{w=12:}$ $(0_0,x_1,1_1,x_2,0_1,x_3),(0_0,x_4,1_1,x_5,0_1,x_6),(2_1,1_0,x_7,1_1,0_0,x_8),$
$(4_1,1_0,x_9,2_1,0_0,x_{10}),(6_1,1_0,x_{11},3_1,0_0,x_{12})$ mod $(7,-)$;
$(3_0,0_0,1_0,2_0,4_0,5_0),(6_0,4_0,0_0,2_0,1_0,3_0),(5_0,2_0,4_0,1_0,6_0,0_0)$.

4. Packings and coverings

Theorem 4.1 *There exists a $(v,G,1)$-OPD for $v\equiv 3,5,11 \pmod{14}$ and $v\geq 6$. But, $p(14t+w,G,1)=14t^2+(2w-1)t+\lfloor\frac{w(w-1)}{14}\rfloor-1$ for $w=9,13$ and $t\geq 0$.*

Proof By Table 2.1, the *OPD*s for desired small orders are constructed as follows.

$\underline{G\text{-}OPD(31)}$ $(Z_7\times Z_4)\bigcup\{x_1,x_2,x_3\}$, $(0_2,0_0,x_3,5_1,2_3,2_0),(0_0,1_2,x_3,3_3,2_1,6_3),$
$(0_2,3_0,x_1,2_1,0_3,1_1),(0_1,3_2,x_1,0_3,0_0,2_3),(1_3,4_0,x_2,3_1,0_2,0_1),(0_0,3_2,x_2,1_3,1_1,2_2),$
$(0_1,3_0,6_1,1_0,3_1,1_1),(0_2,3_1,6_3,1_0,3_2,1_2),(0_3,3_2,6_3,1_2,3_3,1_3)$ mod $(7,-)$;
$(0_0,3_0,2_0,1_0,6_0,2_0),(4_0,6_0,1_0,3_0,2_0,5_0),(5_0,1_0,4_0,0_0,3_0,6_0)$.

$\underline{G\text{-}OPD(19)}$ $Z_{16}\bigcup\{x_1,x_2,x_3\}$, $(0,4,11,5,3,1)$ mod 16;
$(x_2,4,x_1,12,0,8)+i$, $(x_3,4,x_1,12,0,8)+i+4$, $0\leq i\leq 3$.

$\underline{G\text{-}OPD(33)}$ $(Z_{15}\times Z_2)\bigcup\{x_1,x_2,x_3\}$, $(0_0,7_0,x_1,3_1,6_0,7_1),(0_0,5_0,x_2,4_1,0_1,6_1),$
$(0_0,12_1,2_0,13_1,1_0,3_0),(0_0,4_0,x_3,5_1,2_1,9_1),(0_1,1_0,9_1,4_1,1_1,3_1)$, mod $(15,-)$.

$\underline{G\text{-}OPD(11)}$ $Z_7\bigcup\{x_0,x_1,x_2,x_3\}$, $(x_i,1,0,2,3,6)+i$ $0\leq i\leq 3$;
$(0,5,x_1,6,x_3,3),(1,x_1,x_0,x_3,4,6),(5,2,x_2,6,4,x_0)$.

$\underline{G\text{-}OPD(39)}$ $Z_{35}\bigcup\{x_1,\dots,x_4\}$, $(0,13,2,14,1,3),(0,8,1,10,20,4)$ mod 35;
$(0,5,x_3,6,17,x_1)+2i$, $(0,5,x_4,6,17,x_2)+2i+17$ $0\leq i\leq 8$;
$(0,5,x_4,6,17,x_1)+2i+1$, $(0,5,x_3,6,17,x_2)+2i+18$ $0\leq i\leq 7$;
$(x_1,x_4,5,34,x_2,x_3)$.

By Lemma 2.2, there exists no G-$OPD(v)$ for $v\equiv 9,13$ mod 14. In the following, we give the maximum G-$PD(v)$s for desired small orders v.

$\underline{p(9,G,1)=4}$ Z_9, $(0,7,4,5,3,1),(1,7,5,8,2,4),(2,7,8,0,5,3),(6,7,3,8,1,5)$.

$\underline{p(37,G,1)=94}$ $Z_{31}\bigcup\{x_1,\dots,x_6\}$, $(0,5,11,4,1,3),(0,11,23,10,17,8)$ mod 31;
$(x_1,15,x_5,0,22,7)+i$, $(x_2,15,x_5,0,22,7)+i+7$, $(x_4,15,x_6,0,22,7)+i+21$ $0\leq i\leq 6$;
$(x_3,15,x_6,0,22,7)+i+15$ $0\leq i\leq 5$; $(x_5,x_6,x_3,x_2,x_4,x_1),$
$(x_3,29,x_5,14,5,21),(x_6,12,x_3,13,x_1,29),(x_2,5,20,x_4,6,21),(x_4,x_3,28,x_6,4,19)$.

$\underline{p(13,G,1)=10}$ $(Z_5\times Z_2)\bigcup\{x_1,x_2,x_3\}$, $(0_0,2_0,x_1,1_1,0_1,x_2),(0_1,3_0,x_3,2_1,1_0,2_0)$ mod $(5,-)$.

$\underline{p(41,G,1)=116}$ $Z_{39}\bigcup\{x_1,x_2\}$, $(0,23,13,6,5,4),(0,12,1,14,17,19)$ mod 39;
$(0,8,x_1,9,15,18)+2i$, $(0,8,x_2,9,15,18)+2i+1$ $0\leq i\leq 18$.

Theorem 4.2 *There exists a G-$OCD(v)$ for $v \equiv 3, 5, 9, 13 \pmod{14}$. But, $c(14t + 11, G, 1) = 14t^2 + 21t + 9$ for $t \geq 0$.*

Proof By Table 2.1, the OCDs for desired small orders are constructed as follows.

(1) For $v = 3, 31, 19, 33$, the leave $L(\mathcal{B})$ of G-$OPD(v)$ is a subgraph of G, so we can obtain the G-$OCD(v)$ by adding a block containing this $L(\mathcal{B})$.

(2) For $v = 9, 37, 13, 41$, the leave $L(\mathcal{B})$ of the maximum $(v, G, 1)$-PD can be covered by two G, so we can obtain the $(v, G, 1)$-OCD by adding two blocks containing this $L(\mathcal{B})$.

(3) Suppose a G-$OCD(14t + 11)$ exists for $t \geq 0$. Then the excess must be $P_2 = \{a, b\}$, and the sum of degree of a in the OCD is $14t + 11$, an odd number. It is impossible since the degree of each vertex in G is even. Thus, there exists no G-$OCD(14t + 11)$. While for $v = 11, 39$, the leave of G-$OPD(v)$ can be covered by two G, so $c(14t + 11, G, 1) = 14t^2 + 21t + 9$.

Theorem 4.3 $p(14t + w, G, 1) = 14t^2 + (2w - 2)t + \lfloor \frac{w(w-2)}{14} \rfloor$ *for $t \geq 0$ and $w \in [0, 12]_2$.*

Proof By Lemma 2.3, we only need to construct those maximum PDs listed in Table 2.1.

$\underline{p(v, G, 1) = 0}$ for $v = 0, 2, 4$.

$\underline{p(28, G, 1) = 52}$ $(Z_{13} \times Z_2) \bigcup \{x_1, x_2\}$, $(0_0, 6_0, x_1, 4_1, 0_1, 6_1), (0_0, 5_0, x_2, 5_1, 2_1, 7_1)$, $(0_1, 4_1, 1_0, 5_0, 3_1, 1_1), (0_1, 4_0, 6_0, 3_0, 2_0, 1_0)$ mod $(13, -)$.

$\underline{p(30, G, 1) = 60}$ $Z_{15} \times Z_2$, $(0_0, 14_1, 7_1, 4_0, 7_0, 6_0), (0_0, 10_1, 8_1, 11_1, 12_1, 13_1)$, $(0_0, 2_1, 2_0, 8_1, 3_0, 5_0), (0_0, 5_1, 1_1, 7_1, 4_1, 9_1)$ mod $(15, -)$.

$\underline{p(32, G, 1) = 68}$ Z_{32}, $(0, 8, 16, 24, 3, 1) + i$, $(4, 0, 5, 11, 20, 12) + i$ $0 \leq i \leq 3$; $(0, 12, 26, 11, 19, 9)$ mod 32; $(4, 9, 15, 8, 7, 5) + i$ $0 \leq i \leq 27$.

$\underline{p(6, G, 1) = 1}$ Z_6, $(0, 1, 2, 3, 4, 5)$.

$\underline{p(34, G, 1) = 77}$ $(Z_{11} \times Z_3) \bigcup \{x\}$, $(0_0, 4_0, 3_1, 5_0, 1_0, 3_0)$ mod $(11, 3)$; $(0_0, 7_1, 1_0, 9_2, 8_1, 10_2), (0_0, 4_1, 1_0, 4_2, 5_1, 6_2)$, $(0_1, 5_2, 2_1, 8_2, 0_0, 7_2), (5_2, 5_0, x, 5_1, 0_0, 1_1)$ mod $(11, -)$.

$\underline{p(8, G, 1) = 3}$ $Z_7 \bigcup \{x\}$, $(0, 5, 4, x, 3, 1) + i$ $i = 0, 1, 2$.

$\underline{p(36, G, 1) = 87}$ $Z_{12} \times Z_3$, $(0_0, 6_0, 8_0, 10_0, 1_0, 2_0), (4_0, 5_0, 7_0, 6_0, 3_0, 2_0), (9_0, 3_0, 1_0, 11_0, 8_0, 7_0)$; $(0_0, 1_1, 9_0, 4_0, 3_0, 0_1), (5_2, 4_0, 4_2, 2_0, 0_2, 3_2), (0_2, 2_0, 10_2, 1_0, 1_2, 5_1), (6_2, 0_0, 7_2, 2_0, 10_2, 0_1)$, $(0_1, 3_1, 4_2, 4_1, 11_2, 2_1), (0_0, 5_1, 2_0, 8_1, 10_1, 11_1), (2_1, 5_2, 3_1, 7_2, 0_0, 7_1)$ mod $(12, -)$.

$\underline{p(10, G, 1) = 5}$ $Z_5 \times Z_2$, $(0_0, 3_1, 5_1, 1_0, 2_1, 1_1)$ mod $(5, -)$.

$\underline{p(38, G, 1) = 97}$ $Z_{32} \bigcup \{x_1, \ldots, x_6\}$ $(x_1, x_2, x_3, x_4, x_5, x_6)$; $(0, 13, x_5, 15, 8, 18) + 4i + k$, $(0, 13, x_6, 15, 8, 18) + 4i + 2 + k$ $0 \leq i \leq 7, k = 0, 1$; $(0, 6, x_1, 7, 1, 3) + 2i$, $(0, 6, x_2, 7, 1, 3) + 2i + 1$, $(0, 11, x_3, 12, 4, 9) + 2i$, $(0, 11, x_4, 12, 4, 9) + 2i + 1$ $0 \leq i \leq 15$.

$\underline{p(12, G, 1) = 8}$ $Z_4 \times Z_3$, $(0_0, 1_1, 1_0, 3_1, 0_2, 1_2), (3_2, 2_1, 2_2, 0_1, 1_0, 0_0)$ mod $(4, -)$.

$\underline{p(40, G, 1) = 108}$ $Z_{36} \bigcup \{x_1, \cdots x_4\}$, $(0, 15, 1, 17, 6, 13)$ mod 36; $(0, 11, x_1, 12, 1, 3) + 2i$, $(0, 11, x_2, 12, 1, 3) + 2i + 1$ $0 \leq i \leq 17$; $(0, 8, x_3, 10, 4, 9) + 4i + k$, $(0, 8, x_4, 10, 4, 9) + 4i + 2 + k$ $0 \leq i \leq 8$, $k = 0, 1$.

Theorem 4.4 $c(14t + w, G, 1) = 14t^2 + 2wt + \lceil \frac{w^2}{14} \rceil$ *for $t \geq 0$ and $w \in [0, 12]_2$.*

Proof By Lemma 2.3, we only need to construct those minimum CDs listed in Table 2.1.

(1) For $t=0$, by the proof of Theorem 3.1, it is obvious that the leave of the maximum G-$PD(w)$, where $w \in [0,12]_2$, can be covered by $\lceil \frac{w^2}{14} \rceil - \lfloor \frac{w(w-2)}{14} \rfloor$ blocks.

(2) For $t=2$, when $w=2,4,10$, it is easy to see that the leave of the maximum G-$PD(28+w)$ can be covered by $\lceil \frac{w^2}{14} \rceil - \lfloor \frac{w(w-2)}{14} \rfloor + 4$ blocks, so we only need to consider the cases for $w=0,6,8,12$.

$\underline{c(28,G,1)=56}$ $Z_7 \times Z_4$, $(0_0,5_1,2_0,6_1,1_0,3_0),(0_0,5_2,2_0,6_2,1_1,2_1)$,
$(0_2,5_3,2_2,6_3,1_1,3_1),(0_2,0_3,5_1,5_0,2_1,6_1),(0_2,2_3,1_2,0_0,2_2,3_2)$,
$(0_0,5_3,2_0,6_3,0_3,2_3),(0_1,2_2,2_1,3_2,0_3,1_3),(1_3,3_1,6_3,2_1,0_0,4_3)$ mod $(7,-)$.

$\underline{c(34,G,1)=83}$ $(Z_{16} \times Z_2)\bigcup\{x_1,x_2\}$, $(0_0,10_1,2_0,11_1,6_0,7_0),(0_0,x_1,0_1,x_2,3_0,5_0)$,
$(4_1,1_0,8_1,2_0,0_0,4_0),(0_1,15_0,4_1,8_1,6_1,7_1),(0_1,2_0,14_1,1_0,3_1,5_1)$ mod $(16,-)$;
$(2_0,4_0,12_0,10_0,0_0,8_0),(3_0,5_0,13_0,11_0,1_0,9_0),(7_0,x_1,x_2,15_0,6_0,14_0)$.

$\underline{c(36,G,1)=93}$ $Z_{18} \times Z_2$, $(0_0,2_1,16_0,1_1,2_0,5_0)$ mod $(18,2)$;
$(0_0,8_1,2_0,6_0,7_0,8_0),(0_1,6_0,16_1,5_0,7_1,8_1),(0_0,7_1,11_1,5_1,0_1,9_1)$ mod $(18,-)$;
$(3_0,6_0,15_0,12_0,0_0,9_0)+i_0$ $i=0,1,2$.

$\underline{c(40,G,1)=115}$ $Z_{38}\bigcup\{x_1,x_2\}$, $(0,15,2,14,4,7)$, $(0,19,2,18,5,6)$ mod 38;
$(0,8,19,9,2,x_1)+4i+k$, $(0,8,19,9,2,x_2)+4i+2+k$ $0 \le i \le 8$, $k=0,1$;
$(36,6,17,7,0,x_1),(37,7,18,8,1,x_2),(x_1,2,0,x_2,37,1)$.

References

[1] GE Gennian. *Existence of holey LSSOM of type 2^n with application to G_7-packings of K_v* [J]. J. Statist. Plann. Inference, 2001, **94**(2): 211–218.

[2] ZHANG Shucheng, YIN Jianxing. *Packing of K_v with certain graphs of five vertices* [J]. J. Statist. Plann. Inference, 2002, **106**(1-2): 387–408.

[3] KANG Qingde, WANG Zhiqin. *Optimal packings and coverings of λK_v with graph $K_5 - P_5$* [J]. Bull. Inst. Combin. Appl., 2004, **41**: 22–41.

[4] KANG Qingde, WANG Zhiqin. *Maximum $K_{2,3}$-packing designs and minimum $K_{2,3}$-covering designs of λK_v* [J]. J. Math. Res. Exposition, 2005, **25**(1): 1–16.

[5] KANG Qingde, ZHANG Yanfang, ZUO Huijuan. *Packings and coverings of λK_v by k-circuits with one chord* [J]. Discrete Math., 2004, **279**(1-3): 287–315.

[6] KANG Qingde, DU Yanke, TIAN Zihong. *Decomposition of λK_v into some graph with six vertices and seven edges* [J]. J. Statist. Plann. Inference, 2006, **136**(4): 1394–1409.