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Finite Groups Whose Nontrivial Normal Subgroups Have the Same Order

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Abstract In this paper, we study finite groups all of whose nontrivial normal subgroups have the same order. In the solvable case, the groups are determined. In the insolvable case, some characterizations are given.

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1. Introduction

Determining the structure of finite groups is quite important in the study of finite groups. The structure of finite groups are usually characterized by the property of normal subgroups. It is well known that if every subgroup of a finite group G is normal, then G is a Dedekind group, whose structure had been determined. In another extreme case, if a finite group G has no nontrivial normal subgroups, then G is a simple group. The classification of finite simple groups have been completed. It is easy to see that the number of nontrivial normal subgroup of a finite group has great influence on the structure of a finite group. This motivates us to study finite groups which have "many" normal subgroups or "few" normal subgroups. In this paper, we study finite groups all of whose nontrivial normal subgroups have the same order.

Our main results are:

Theorem 1.1 *All nontrivial normal subgroups of a finite group G have the same order if and only if G is one of the following:*

(1) G is a simple group;

(2) G has a unique nontrivial normal subgroup;

(3) $G \cong T \times T$, where T is a finite simple group;

(4) $G \cong A_8 \times L_3(4)$;

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(5) $G \cong B_n(q) \times C_n(q)$, *where* $n \geq 3$, q *is odd.*

Theorem 1.2 *Let* G *be a finite solvable group. If* G *has a unique nontrivial normal subgroup, then*

(1) G *is a cyclic* p*-group of order* p^2.

(2) G *is a semidirect product* $G = P \rtimes Q$, *where* P *is an elementary abelian* p*-group and* Q *is a cyclic group of order* q, *with* p *and* q *being distinct primes. Moreover, the action of* Q *on* P *is irreducible.*

Theorem 1.3 *Let* G *be a finite non-solvable group. If* G *has a unique nontrivial normal subgroup* K, *then*

(I) K *is solvable.*

(i) If $K \leq Z(G)$, *then* G *is a covering group of a finite simple group and* $Z(G) \cong C_p$.

(ii) If $K \nleq Z(G)$, *then* $G/K \cong D$ *is a finite non-abelian simple group.* D *acts irreducibly on* K, *furthermore,*

(ii-a) G *is a split extension of* K *by* D *if and only if* G *is a primitive permutation group of affine type, and the stabilizer of* G *is isomorphic to* D.

(ii-b) G *is a non-split extension of* K *by* D.

(II) K *is non-solvable.*

(i) Assume that T *is a non-abelian simple group. Then* $K \cong T$ *if and only if* G *is almost simple group, and* T *is a socle of* G *and* $G/T \cong C_p$.

(ii) If $K \cong T^n$, *where* T *is a non-abelian simple group and* $n > 1$, *then* $G/K \cong D$, *where* D *is a simple group. Furthermore,*

(ii-a) If D *is an abelian simple group, that is,* $D \cong C_p$ *(*p *a prime), then* $G = \langle T^p, x\rangle$ *and there exists* $\sigma_i \in \mathrm{Aut}(T)$ *such that* $\forall(1,1,\ldots,t,\ldots,1) \in T^p$ *satisfies* $(1,1,\ldots,t,\ldots,1)^x = (1,1,\ldots,t^{\sigma_i},\ldots,1) \in T^p$, *where* t *is the* i*th factor of* T^p, t^{σ_i} *is the* i^{δ}*th factor of* T^p, $\delta = (123\cdots p)$, $(\sigma_i\sigma_{i+1}\cdots\sigma_p\sigma_1\cdots\sigma_{i-1})^{p^{k_i}} = 1$, $\sigma_i\sigma_{i+1}\cdots\sigma_p\sigma_1\cdots\sigma_{i-1} \in \mathrm{Inn}(T)$, $o(x) = p^{\max\{k_i\}+1}$, $i = 1,2,\ldots,p$, $\langle x\rangle$ *acts transitively on set* $\{T_1, T_2, \ldots, T_n\}$. *Conversely, a group* G *constructed above has only one nontrivial normal subgroup.*

(ii-b) If D *is a non-abelian simple group, then* $K \cong T^n$ *is the unique nontrivial normal subgroup of* G *if and only if* D *acts transitively on set* $\{T_1, T_2, \ldots, T_n\}$, *where* $T_i \cong T$, $i = 1,2,\ldots,n$. *Especially* K *has a complement in* G *if and only if* $G = (T_1 \times T_2 \times \cdots \times T_n) \rtimes D$, *where* D *acts transitively on set* $\{T_1, T_2, \ldots, T_n\}$.

In this paper, only finite groups are considered. Our notation is mostly standard, which can be refered to [2]. We will give some preliminaries in section two. And in section three we will give the proof of main results.

2. Preliminaries

We introduce some concepts and lemmas.

Definition 2.1 *Let* G *be a non-abelian group. If all proper subgroups and proper quotient* *groups of G are abelian, then G is called minimal non-abelian group. If G is a non-nilpotent group and all proper subgroups of G are nilpotent, then G is called an inner nilpotent group.*

Definition 2.2 *Let G be a finite group and T is a finite non-abelian simple group. G is called an almost simple group if $T \le G \le \mathrm{Aut(T)}$.*

Definition 2.3 *G is called perfect group if $G' = G$.*

Definition 2.4 *Let G be a finite perfect group. A central extension of G is a group H satisfying $H/Z(H) \cong G$. A central extension which is perfect is called a covering group.*

Lemma 2.1[1,Th 1] *Let G be a simple group, $|G| = |M|$, where M is a known simple group. Then*

(1) If $|M| = |A_8| = |L_3(4)|$, then $G \cong A_8$ or $L_3(4)$;

(2) If $|M| = |B_n(q)| = |C_n(q)|$, where $n \ge 3$, q is odd, then $G \cong B_n(q)$ or $C_n(q)$;

(3) If $|M|$ is neither the case (1) nor (2), then $G \cong M$.

Lemma 2.2[2,Th 5.4] *Let G be inner nilpotent group. Then $|G| = p^a q^b$, where $p \neq q$ are prime, and there is normal Sylow subgroup. Without loss of generality, assume that $P \trianglelefteq G$, where $P \in \mathrm{Syl}_p(G)$. Q is cyclic, where $Q \in \mathrm{Syl}_q(G)$, $Q \ntrianglelefteq G$, and $\Phi(Q) \le Z(G)$.*

We give two lemmas which are useful in the proof of our theorems and easily proven.

Lemma 2.3 *Let $G = T_1 \times \cdots \times T_s$, where $T_1, \ldots, T_s$ are ismorphic non-abelian simple groups. Then every nontrivial normal subgroup of G has the form $T_{i_1} \times \cdots \times T_{i_t}$ $(1 \le i_1 \le \cdots \le i_t \le s)$.*

Lemma 2.4 *If $H \cong T^m$, where T is a non-abelian simple, then $\mathrm{Aut}(H) = A \ \mathrm{wr}_\Gamma \ S_\Gamma$, where $A = \mathrm{Aut}(T)$ and $\Gamma = \{1, 2, \ldots, m\}$.*

3. Proof of main results

The Proof of Theorem 1.1

Proof $\Longrightarrow$ If G has no nontrivial normal subgroup, then G is a simple group, i.e, case (1) holds; If G has a unique nontrivial normal subgroup, then case (2) holds; If G has more than one nontrivial normal subgroups, assume M, N are two different nontrivial normal subgroups of G which have the same order. By hypothesis, we have $M \cap N = 1, G = MN$ and M, N are all simple group. It follows that $G = M \times N$. By Lemma 2.1, G is one of the case (3), (4) and (5).

$\Longleftarrow$ For (1) and (2), the conclusion obviously holds. For (3), since $G/T \cong T$, where T is a finite simple group, we have that $G > T > 1$ is a chief series of G and T is the unique chief factor of G. By Jordan-Hölder theorem, the nontrivial normal subgroups of G are all isomorphic to T. Therefore all nontrivial normal subgroups of G have the same order. Case (4) and (5) can be proved in the same way as that of case (3). The proof is completed. $\square$

The proof of Theorem 1.2

Proof of (1) It is straightforward.

Proof of (2) $\Longrightarrow$ Since G is a non-abelian solvable group, G' is a nontrivial normal subgroup of G. By hypothesis, G' is the unique nontrivial normal subgroup of G. It follows that $G > G' > 1$ is a chief series of G. Since G is solvable, G' is an elementary abelian p-group for some prime p. Let $P = G'$ and $|P| = p^n$. Then $|G/P| = q$, where p, q are primes, $p \neq q$. Hence $G = P \rtimes Q$, where $Q \in \mathrm{Syl}_q(G)$ and Q acts irreducibly on P.

$\Longleftarrow$ Since Q acts irreducibly on P, P is a minimal normal subgroup of G, and hence it is the unique nontrivial normal subgroup. So, G satisfies the condition of the theorem. The proof is completed. $\square$

Remark The groups in Theorem 1.2(2) are minimal non-abelian groups. The structure of them can be found in [5, Theorem 1.5]. Note that in non-nilpotent case, a group being minimal non-abelian is equivalent to it being minimal non-nilpotent.

The Proof of Theorem 1.3

Proof of (I) Case (i).

$\Longrightarrow$ Since $1 \neq K \leq Z(G)$ and K is the unique nontrivial normal subgroup of G, we have $K = Z(G)$ and $|Z(G)| = p$. Let $G/Z(G) \cong S$. By hypothesis we get that S is a finite non-abelian simple group. We assert that G is perfect. If not, by hypothesis, $G' = K$. Since K is solvable, G is solvable, a contradiction. So G is a covering group of S and $Z(G) \neq 1$. Since any subgroup of $Z(G)$ is normal in G, by assumption, $Z(G) \cong C_p$ for a prime p.

$\Longleftarrow$ We only need to prove $Z(G)$ is the unique nontrivial normal subgroup of G. Assume N is a nontrivial normal subgroup of G and $N \neq Z(G)$. By hypothesis, $G/Z(G)$ is isomorphic to a finite non-abelian simple group. Therefore $N \bigcap Z(G) = 1$, and $NZ(G) = G$, i.e, $G = Z(G) \times N$. Since $Z(G) = \Phi(G)$, it means that $\Phi(G)$ has a complement in G, a contradiction. So $K = Z(G)$.

Case (ii) Since K is the unique nontrivial normal subgroup of G, $G/K \cong D$ is a finite non-abelian simple group. D acts irreducibly on K.

$\Longrightarrow$ It is straightforward to prove by the definition of primitive permutation group of affine type.

$\Longleftarrow$ Since G is a primitive permutation group of affine type, and the stabilizer of G is isomorphic to D. Let $G = M \rtimes D$ and D acts irreducibly on M, where $M \cong C_p^n$. We need to prove that M is the unique nontrivial normal subgroup of G. Assume that N is a nontrivial normal subgroup of G and $N \neq M$. Since $N \bigcap M \trianglelefteq G$ and D acts irreducibly on M, $N \bigcap M = 1$ or $N \bigcap M = M$. If $N \bigcap M = 1$, since $N \cong NM/M \trianglelefteq G/M = DM/M \cong D$ and D is a simple group, $N \cong D$. It follows that $G = NM$. Therefore, $G = N \times M$. Since $M \cong C_p^n$, $C_p \trianglelefteq G$, but D acts irreducibly on M, a contradiction. So $N \bigcap M = M$, i.e, $M \leq N$. Since $G/M \cong D$ is a simple group, $N = M$, a contradiction.

So, (ii-a) holds. If G is a non-split extension of K by D, then (ii-b) holds. The proof is completed. $\square$

Proof of (II) Since K is the unique nontrivial normal subgroup of G and K is non-solvable, $K \cong T^n$, where T is a finite non-abelian simple group.

Case (i) $\Longrightarrow$ If $K = T$, Since $C_G(T) \trianglelefteq G$, $C_G(T) = 1$. If not, we get that T is an abelian group, a contradiction. By N/C Theorem, $T \leq G \lesssim \mathrm{Aut}(T)$. Hence G is an almost simple group and T is a socle of G. Since $G/T \lesssim \mathrm{Aut}(T)/T \cong \mathrm{Out}(T)$ and $\mathrm{Out}(T)$ is solvable, G/T is solvable. By hypothesis, G/T is a simple group. Hence G/T is a cyclic p-group of order p.

$\Longleftarrow$ By hypothesis, T is a nontrivial normal subgroup of G. So we need to prove that T is the unique nontrivial normal subgroup of G. Assume N is a nontrivial normal subgroup of G and $N \neq T$. Since $N \bigcap T \trianglelefteq T$ and T is simple group, we have $N \bigcap T = 1$ or $N \bigcap T = T$. If $N \bigcap T = T$, then $N = G$, a contradiction. So $N \bigcap T = 1$. Thus $G = NT = N \times T$. It follows that $N \leq C_{\mathrm{Aut}(T)}(\mathrm{Inn}(T)) = 1$. So $N = 1$, a contradiction. Hence T is the unique nontrivial normal subgroup of G. That is, $K = T$.

Case (ii) If $K = T^n$, where $n > 1$. Since K is the unique nontrivial normal subgroup of G, we have $G/K \cong D$ is simple group. We distinguish two cases. First, we prove (ii-a).

$\Longrightarrow$ If D is an abelian simple group, i.e, $D \cong C_p$, (p is a prime) consider the conjugate action of G on set $\Omega = \{T_1, T_2, \ldots, T_n\}$. We denote it by f. We assert that f is transitive. If not, every orbit of f is corresponding to a nontrivial normal subgroup of G. This is contrary to hypothesis, so f is transitive. Since $\ker f$ is a nontrivial normal subgroup of G, we have $\ker f = T^n = K$. It follows that $|G/G_{T_i}| = n$, where G_{T_i} is the stabilizer of T_i in G. Since $|G/\ker f| = p$ and $n|p$, we have $n = 1$ or $n = p$. If $n = 1$, the case had been discussed. If $n = p$, take $x \in P \in \mathrm{Syl}_p(G)$ and $x \notin T^p = \ker f$, we have that $G = \langle T^p, x\rangle$. Consider the conjugate action of $\langle x\rangle$ on Ω. It follows that the length of orbits are powers of p. Since the length of Ω is p, the length of orbits is p. Hence the action of $\langle x\rangle$ on Ω is transitive.

Without lose of generality, assume that ${T_i}^x = T_{i^\delta}$, where $\delta = (12\cdots p)$. Then $\forall g \in T_i$, we have $g^x \in T_{i^\delta}$. Since $T_i \cong T_{i^\delta}$, $\forall (1, \ldots, t, \ldots, 1) \in T^p$, where t is ith factor of T^p, there exists $\sigma_i \in \mathrm{Aut}(T)$ such that $(1, \ldots, t, \ldots, 1)^x = (1, \ldots, t^{\sigma_i}, \ldots, 1) \in T^p$, where t^{σ_i} is i^δth factor of T^p, $i = 1, 2, \ldots, p$. Therefore, $(t_1, \ldots, t_p)^{x^p} = ({t_1}^{\sigma_1\cdots\sigma_p}, {t_2}^{\sigma_2\sigma_3\cdots\sigma_1}, \ldots, {t_p}^{\sigma_p\sigma_1\cdots\sigma_{p-1}})$. By Lemma 2.4 we have $\mathrm{Aut}(T^p) = (\mathrm{Aut}(T))^p \rtimes S_p$. Since $x^p \in T^p = \ker f \in \mathrm{Aut}(T^p)$, $x^p = (\sigma_1\sigma_2\cdots\sigma_p, \sigma_2\sigma_3\cdots\sigma_1, \ldots, \sigma_p\sigma_1\cdots\sigma_{p-1})$, and $\sigma_i\sigma_{i+1}\cdots\sigma_p\sigma_1\cdots\sigma_{i-1} \in \mathrm{Inn}(T)$. On the other hand, x^p is a p element. It follows that $(\sigma_i\sigma_{i+1}\cdots\sigma_p\sigma_1\cdots\sigma_{i-1})^{p^{k_i}} = 1$, $o(x) = p^{\max\{k_i\}+1}$.

$\Longleftarrow$ Since there exists $\sigma_i \in \mathrm{Aut}(T)$ such that $\forall (1, \ldots, t, \ldots, 1) \in T^p$ satisfies $(1, \ldots, t, \ldots, 1)^x = (1, \ldots, t^{\sigma_i}, \ldots, 1) \in T^p$. x can be regarded as an automorphism of T^p. Hence we can determine x by σ_i. Since x normalizes T^p, $T^p \trianglelefteq G$. We need to prove that T^p is the unique nontrivial normal subgroup of G.

Assume N is a nontrivial normal subgroup of G and $N \neq T^p$. If $N \bigcap T^p \neq 1$, then $N \bigcap T^p \trianglelefteq G$. Since $N \bigcap T^p \trianglelefteq T^p$, by Lemma 2.3, there exists $T_i \in N \bigcap T^p$. Since $\langle x\rangle$ acts transitively on set $\{T_1, T_2, \ldots, T_p\}$, $T^p \leq N \bigcap T^p$. It follows that $N = G$, a contradiction. Hence $N \bigcap T^p = 1$ and $G = NT^p = N \times T^p$. It follows that $T_i \trianglelefteq G$. Since $\langle x\rangle$ acts transitively on set $\{T_1, T_2, \ldots, T_p\}$, we have $x \notin N_G(T_i)$, a contradiction.

We prove (ii-b) as follows.

$\Longrightarrow$ Since every orbit of the action is corresponding to a normal subgroup of G and G has a unique nontrivial normal subgroup, D acts transitively on set $\{T_1, T_2, \ldots, T_n\}$, where $T_i \cong T$, $i = 1, 2, \ldots, n$.

$\Longleftarrow$ We need to prove T^n is the unique nontrivial normal subgroup of G. Assume M is a nontrivial normal subgroup of G and $M \neq T^n$. If $M \bigcap T^n \neq 1$, then $M \bigcap T^n \trianglelefteq T^n$. By Lemma 2.3, there exists $T_i \in M \bigcap T^n$. Since D acts transitively on set $\{T_1, T_2, \ldots, T_n\}$, $T^n \leq M \bigcap T^n$, i.e, $T^n \leq M$. On the other hand, since G/T^n is a simple group, it follows that $M = T^n$, a contradiction. If $M \bigcap T^n = 1$, since $MT^n \trianglelefteq G$ and G/T^n is a simple group, $MT^n = G$. Hence $G = M \times T^n$. It follows that $T_i \trianglelefteq G$. That means that the action of D on set $\{T_1, T_2, \ldots, T_n\}$ is not transitive, a contradiction.

The proof is completed. □

We will give an example to explain the case (I)(ii-a) in Theorem 1.3 exists.

Example 3.1 $\mathrm{AGL}(n, 2) = C_2^n \rtimes \mathrm{GL}(n, 2)$ has a unique nontrivial normal subgroup C_2^n. Also $G = C_5^3 \rtimes SL(3, 5)$ has a unique nontrivial normal subgroup C_5^3.

The case (I)(ii-b) in Theorem 1.3 exists, which can be found in [4, p48, Coro. 2.2.50].

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