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Uniqueness of Cycle Length Distributions of Certain Bipartite Graphs $K_{n,n+7} - A(|A| \leq 3)$

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Abstract The cycle length distribution of a graph of order n is denoted by $(c_1, c_2, \ldots, c_n)$, where c_i is the number of cycles of length i. In this paper, we obtain that a graph G is uniquely determined by its cycle distribution if: (1) $G = K_{n,n+7}$ $(n \geq 10)$; or (2) $G = K_{n,n+7} - A$ $(|A| = 1, n \geq 12)$; or (3) $G = K_{n,n+7} - A$ $(|A| = 2, n \geq 14)$; or (4) $G = K_{n,n+7} - A$ $(|A| = 3, n \geq 16)$, where $A \subseteq E(K_{n,n+7})$.

Keywords cycle; cycle length distribution; bipartite graph; uniqueness of cycle length distribution.

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1. Introduction

The cycle length distribution of a graph G of order n is denoted by $(c_1, c_2, \ldots, c_n)$, where c_i is the number of cycles of length i in G, $i = 1, \ldots, n$. In graph theory, there are many interesting problems which could be reduced to the cycle length distribution problem. In 1973, Entringer raised the problem how to determine those graphs which have their cycle distributions $(c_1, c_2, \ldots, c_n)$ satisfying $c_1 = c_2 = 0, c_i = 1$ $(i = 3, 4, \ldots, n)$. Moreover, in 1975, Erdös proposed another problem how to determine the maximum possible number of edges of a graph which has its cycle distribution $(c_1, c_2, \ldots, c_n)$ while $c_i \leq 1$ $(i = 1, 2, \ldots, n)$. Both of these two problems are very interesting, and have been studied in [2–5]. In general, there may exist more than one graph having the same cycle distribution $(c_1, c_2, \ldots, c_n)$. Therefore, a question followed is which kinds of graphs are uniquely determined by their cycle distributions[6]. In [7, 8], the authors have proved that the following graphs are uniquely determined by their distributions:

$$G = K_{n,r} - A(A \subseteq E(K_{n,r}), |A| \leq 1, n \leq r \leq \min(n+6, 2n-3));$$

$$G = K_{n,r} - A(A \subseteq E(K_{n,r}), |A| = 2, n \leq r \leq \min(n+6, 2n-5));$$

$$G = K_{n,r} - A(A \subseteq E(K_{n,r}), |A| = 3, n \leq r \leq \min(n+6, 2n-7)).$$

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We notice that, in [7,8], the authors discussed the uniqueness of cycle distributions for bipartite graphs $K_{n,r} - A$ ($n \le r \le n+6, |A| \le 3$) only. In this paper, we will study the uniqueness of cycle distributions for bipartite graphs $K_{n,r} - A$ where $r = n+7, |A| \le 3$. The main results obtained can be stated as: a graph G is uniquely determined by its cycle distribution if:

(1) $G = K_{n,n+7}$ ($n \ge 10$); or

(2) $G = K_{n,n+7} - A$ ($|A| = 1, n \ge 12$); or

(3) $G = K_{n,n+7} - A$ ($|A| = 2, n \ge 14$); or

(4) $G = K_{n,n+7} - A$ ($|A| = 3, n \ge 16$),

where $A \subseteq E(K_{n,n+7})$.

2. Some lemmas

Throughout this paper, we assume that G is a simple bipartite graph, $V(G) = X \bigcup Y$, and every edge of G joins a vertex in X to a vertex in Y. Let $|X| = n$, $|Y| = r$ ($n \le r$), and $X_j = \{G | G = K_{n,r} - A, |A| = j\}$. Moreover, let $c_4(G)$ denote the number of cycles of length 4, $m_j = \min_{G \in X_j} c_4(G)$, and $M_j = \max_{G \in X_j} c_4(G)$. We have

Lemma 1[8] *If $n > j \ge 2$, then*

$$m_j = \binom{n}{2}\binom{r}{2} - j\binom{n-1}{1}\binom{r-1}{1} + \binom{j}{2},$$

$$M_j = \binom{n}{2}\binom{r}{2} - j\binom{n-1}{1}\binom{r-1}{1} + \binom{j}{2}(r-1).$$

Lemma 2[9] *If $j \ge 2, n \ge \frac{1}{2}j(j+1)+2$, then $M_{j+1} < m_j$.*

Lemma 3[7] *Let $G = K_{n,r} - A, |A| = j$. If $r \ge n \ge j+2$, then $c_{2n}(G) \ne 0$.*

3. Main results

Theorem 1 *Suppose $n \ge 10$. G is complete bipartite graph $K_{n,n+7}$ if and only if its cycle distribution $(c_1, c_2, \ldots, c_{2n+7})$ satisfies the following conditions:*

$$c_i = \begin{cases} \frac{1}{2}\binom{n}{p}\binom{n+7}{p} p[(p-1)!]^2, & i = 2p,\ p = 2,3,\ldots,n; \\ 0, & \textit{otherwise}. \end{cases}$$

Proof "⇒" Since $K_{n,n+7}$ is a simple bipartite graph, $c_1 = c_2 = 0$, $c_{2p+1} = 0$ ($p = 1,2,\ldots,n$). If $i > 2n$, clearly $c_i = 0$. For each $i = 2p$ ($p = 2,3,\ldots,n$), G contains $\binom{n}{p}\binom{n+7}{p}$ complete bipartite subgraphs $K_{p,p}$ with order i. Furthermore, each complete bipartite subgraph $K_{p,p}$ must contain $\frac{1}{2}p[(p-1)!]^2$ cycles of length i. So $c_i = \frac{1}{2}\binom{n}{p}\binom{n+7}{p} p[(p-1)!]^2$.

"⇐" If $G \ne K_{n,n+7}$, we then must have $G = K_{n,n+7} - A$ ($|A| \ge 1$) or $G = K_{n+k,7+n-k} - A$ ($|A| \ge 0, 1 \le k \le 3$). Consider the following cases.

Case 1 If $G = K_{n,n+7} - A, |A| \geq 1$, then we have $c_4(G) < \binom{n}{2}\binom{n+7}{2}$. Clearly, this contradicts the assumption.

Case 2 Suppose $G = K_{n+k,n+7-k} - A$, where $|A| = j \geq 0$ and $1 \leq k \leq 3$. By Lemma 3, if $n+k \geq j+2$, equivalently, $0 \leq j \leq n+k-2$, then $c_{2n+2k} \neq 0$. It is a contradiction. Hence, $G \in \{K_{n+k,n+7-k} - A \| A| = j \geq n+k-1\}$. Obviously, $c_4(G) \leq \max_{|A|=j=n+k-1} c_4(K_{n+k,n+7-k} - A)$. According to Lemma 1, we have

$$c_4(G) \leq \binom{n+k}{2}\binom{n+7-k}{2} - (n+k-1)\binom{n+k-1}{1}\binom{n+7-k-1}{1} + \binom{n+k-1}{2}(n+7-k-1).$$

Let $f(k)$ denote the right hand side of the above inequality. The inequality can be rewritten as $c_4(G) \leq f(k)$. In what follows, we will prove $f(k) < \binom{n}{2}\binom{n+7}{2}$. Denote $H(k) = f(k) - \binom{n}{2}\binom{n+7}{2}$ and consider the following cases:

(a) $k = 1$.

$$\begin{aligned} H(1) &= f(1) - \binom{n}{2}\binom{n+7}{2} \\ &= \binom{n+1}{2}\binom{n+6}{2} - n\binom{n}{1}\binom{n+5}{1} + \binom{n}{2}(n+5) - \binom{n}{2}\binom{n+7}{2} \\ &= -\frac{1}{2}n(n^2-31). \end{aligned}$$

Thus, if $n \geq 10$, then $H(1) < 0$.

(b) $k = 2$.

$$\begin{aligned} H(2) &= f(2) - \binom{n}{2}\binom{n+7}{2} \\ &= \binom{n+2}{2}\binom{n+5}{2} - (n+1)\binom{n+1}{1}\binom{n+4}{1} + \binom{n+1}{2}(n+4) - \\ &\quad \binom{n}{2}\binom{n+7}{2} \\ &= -\frac{1}{2}(n^3 - 3n^2 - 46n - 12) = -\frac{1}{2}[n(n^2 - 3n - 46) - 12]. \end{aligned}$$

That is to say, $H(2) < 0$ if $n \geq 10$.

(c) $k = 3$.

$$
\begin{aligned}
H(3) &= f(3) - \binom{n}{2}\binom{n+7}{2} \\
&= \binom{n+3}{2}\binom{n+4}{2} - (n+2)\binom{n+2}{1}\binom{n+3}{1} + \binom{n+2}{2}(n+3) - \\
&\quad \binom{n}{2}\binom{n+7}{2} \\
&= -\frac{1}{2}(n^3 - 4n^2 - 51n - 18) = -\frac{1}{2}[n(n^2 - 4n - 51) - 18].
\end{aligned}
$$

So $H(3) < 0$ when $n \geq 10$.

In summary, we obtain that if $n \geq 10$, then must have $f(k) < \binom{n}{2}\binom{n+7}{2}$, or equivalently, $c_4(G) < \binom{n}{2}\binom{n+7}{2}$. This is a contradiction. So $G \notin \{K_{n+k,n+7-k} - A||A| = j \geq n + k - 1\}$, that is, $G = K_{n,n+7}$. □

Theorem 2 *Suppose $n \geq 12$ and $A \subseteq E(K_{n,n+7})$. G is complete bipartite graph $K_{n,n+7} - A$ $(|A| = 1)$ if and only if its cycle distribution $(c_1, c_2, \ldots, c_{2n+7})$ satisfies the following conditions:*

$$
c_i = \begin{cases} \frac{1}{2}\binom{n}{p}\binom{n+7}{p}p[(p-1)!]^2 - \binom{n-1}{p-1}\binom{n+6}{p-1}[(p-1)!]^2, & i = 2p,\ p = 2, 3, \ldots, n; \\ 0, & \textit{otherwise}. \end{cases}
$$

Proof "⇒" Let $A = \{e\}$ and $G = K_{n,n+7} - e$. Since $K_{n,n+7} - e$ is a simple bipartite graph, $c_1 = c_2 = 0$, $c_{2p+1} = 0$ $(p = 1, 2, \ldots, n)$, and $c_i = 0$ if $i > 2n$. For each $i = 2p$ $(p = 2, 3, \ldots, n)$, $K_{n,n+7}$ contains $\frac{1}{2}\binom{n}{p}\binom{n+7}{p}p[(p-1)!]^2$ cycles of length i. Moreover, $K_{n,n+7}$ contains $\binom{n-1}{p-1}\binom{n+6}{p-1}$ complete bipartite subgraphs $K_{p,p}$ of order i which include edge e and each of these complete bipartite subgraphs $K_{p,p}$ contains $[(p-1)!]^2$ cycles of length i which include edge e. Hence, $K_{n,n+7}$ contains $\binom{n-1}{p-1}\binom{n+6}{p-1}[(p-1)!]^2$ cycles of length i which include edge e. Therefore, $K_{n,n+7} - e$ contains $\frac{1}{2}\binom{n}{p}\binom{n+7}{p}p[(p-1)!]^2 - \binom{n-1}{p-1}\binom{n+6}{p-1}[(p-1)!]^2$ cycles of length i.

"⇐" Suppose $G \neq K_{n,n+7} - e$. Then we must have $G = K_{n,n+7} - A$ $(|A| \geq 2)$ or $G = K_{n+k,n+7-k} - A$ $(|A| \geq 0, 1 \leq k \leq 3)$. By Theorem 1, we know that if $n \geq 10$, $K_{n,n+7}$ is determined by its cycle distribution. So $G \neq K_{n,n+7}$. Consider the following cases.

Case 1 $G = K_{n,n+7} - A$, where $|A| \geq 2$. In this case, we have $c_4(G) < c_4(K_{n,n+7} - e)$. This is a contradiction.

Case 2 $G = K_{n+k,n+k-7}$, where $|A| = j \ge 0$ and $1 \le k \le 3$. From Lemma 3, if $n+k \ge j+2$, that is, $0 \le j \le n+k-2$, then $c_{2n+2k} \ne 0$. It contradicts $c_{2n+2k} = 0$. So, $G \in \{K_{n+k,n+7-k} - A||A| = j \ge n+k-1\}$. Clearly, $c_4(G) \le \max_{|A|=j=n+k-1} c_4(K_{n+k,n+7-k} - A)$. According to Lemma 1,

$$c_4(G) \le \binom{n+k}{2}\binom{n+7-k}{2} - (n+k-1)\binom{n+k-1}{1}\binom{n+7-k-1}{1} + \binom{n+k-1}{2}(n+7-k-1).$$

Let $f(k)$ denote the right hand side of the above inequality. The inequality can be rewritten as $c_4(G) \le f(k)$. Now we will prove $f(k) < \binom{n}{2}\binom{n+7}{2} - \binom{n-1}{1}\binom{n+6}{1}$, or equivalently, $f(k) - \binom{n}{2}\binom{n+7}{2} + \binom{n-1}{1}\binom{n+6}{1} < 0$.

Denote $H(k) = f(k) - \binom{n}{2}\binom{n+7}{2} + \binom{n-1}{1}\binom{n+6}{1}$ and consider:

(a) $k = 1$.

$$\begin{aligned} H(1) &= f(1) - \binom{n}{2}\binom{n+7}{2} + \binom{n-1}{1}\binom{n+6}{1} \\ &= -\frac{1}{2}(n^3 - 2n^2 - 41n + 12) = -\frac{1}{2}[n(n^2 - 2n - 41) + 12]. \end{aligned}$$

Thus, if $n \ge 12$, $H(1) < 0$.

(b) $k = 2$.

$$\begin{aligned} H(2) &= f(2) - \binom{n}{2}\binom{n+7}{2} + \binom{n-1}{1}\binom{n+6}{1} \\ &= -\frac{1}{2}n(n^2 - 5n - 56). \end{aligned}$$

That is to say, if $n \ge 12$, $H(2) < 0$.

(c) $k = 3$.

$$\begin{aligned} H(3) &= f(3) - \binom{n}{2}\binom{n+7}{2} + \binom{n-1}{1}\binom{n+6}{1} \\ &= -\frac{1}{2}(n^3 - 6n^2 - 61n - 6) = -\frac{1}{2}[n(n^2 - 6n - 61) - 6]. \end{aligned}$$

So, if $n \ge 12$, $H(3) < 0$.

In summary, we have $f(k) < \binom{n}{2}\binom{n+7}{2} - \binom{n-1}{1}\binom{n+6}{1}$, or equivalently,

$$c_4(G) < \binom{n}{2}\binom{n+7}{2} - \binom{n-1}{1}\binom{n+6}{1}.$$

This is a contradiction. So $G \notin \{K_{n+k,n+7-k} - A||A| = j \ge n+k-1\}$, that is, $G = K_{n,n+7} - e$.

Theorem 3 *If $n \geq 14$, then $G = K_{n,n+7} - A(|A| = 2)$ is determined by its cycle distribution.*

Proof From [8], we know that $K_{n,n+7}[A]$ must be one of three types of graphs as shown in Figure 1.

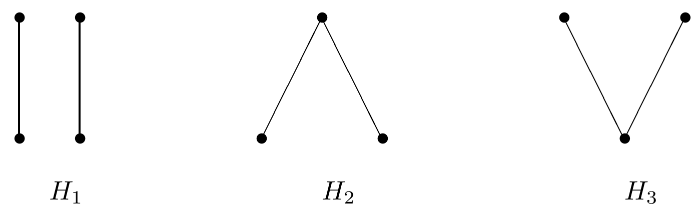


Figure 1 $K_{n,n+7} - |A|$ $(|A| = 2)$

Denote these three types of graphs by H_1, H_2 and H_3, respectively. Let $c_4(G_i)$ denote the number of cycles of length 4 in $G_i = K_{n,n+7} - E(H_i)$. Clearly, we have

$$c_4(G_1) = \binom{n}{2}\binom{n+7}{2} - 2\binom{n-1}{1}\binom{n+6}{1} + 1,$$
$$c_4(G_2) = \binom{n}{2}\binom{n+7}{2} - 2\binom{n-1}{1}\binom{n+6}{1} + n - 1,$$
$$c_4(G_3) = \binom{n}{2}\binom{n+7}{2} - 2\binom{n-1}{1}\binom{n+6}{1} + n + 6.$$

Obviously, $c_4(G_1) < c_4(G_2) < c_4(G_3)$. It shows that $G_i (i = 1, 2, 3)$ have different cycle distributions.

Now we will prove that if $G' \neq G = K_{n,n+7} - A(|A| = 2)$, G' and G have different cycle distributions. Clearly, G' must be in one of the following three cases.

Case 1 $G' \in \{K_{n,n+7} - A | 0 \leq |A| \leq 1 \text{ or } |A| \geq 3\}$. By Theorems 1 and 2, $G' = K_{n,n+7}$ or $G' = K_{n,n+7} - A(|A| = 1)$, both are determined by their cycle distributions. Moreover, if $G' = K_{n,n+7} - A(|A| \geq 3)$, according to Lemma 2, there must have $\max c_4(G') < \min c_4(G)$. Therefore, if $G' \in \{G | G = K_{n,n+7} - A, |A| \neq 2\}$, G' and G have different cycle distributions.

Case 2 $G' \in \{K_{n+k,n+7-k} - A | |A| = j, 0 \leq j \leq n+k-2, 1 \leq k \leq 3\}$. Obviously, $n+k \geq j+2$. By Lemma 3, $c_{2n+2k}(G') \neq 0$ and $c_{2n+2k}(G) = 0$. Thus, G' and G have different cycle distributions too.

Case 3 $G' \in \{K_{n+k,n+7-k} - A | |A| = j, j \geq n+k-1, 1 \leq k \leq 3\}$. Let G'' be any graph in $\{K_{n+k,n+7-k} - A | |A| = j, j \geq n+k-1, 1 \leq k \leq 3\}$. Then $c_4(G') \leq \max c_4(G'')$. Denote $f(k) = \max c_4(G'')$. From Lemma 1, we know

$$f(k) = \binom{n+k}{2}\binom{n+7-k}{2} - (n+k-1)\binom{n+k-1}{1}\binom{n+7-k-1}{1} +$$

$$\binom{n+k-1}{2}(n+7-k-1).$$

In what follows, we need to prove $f(k) < c_4(G_1) < c_4(G_2) < c_4(G_3)$ only, or equivalently, to prove

$$f(k) < c_4(G_1) = \binom{n}{2}\binom{n+7}{2} - 2\binom{n-1}{1}\binom{n+6}{1} + 1.$$

Denote $H(k)$ be $f(k) - \binom{n}{2}\binom{n+7}{2} + 2\binom{n-1}{1}\binom{n+6}{1} - 1$. Consider the following cases:

(a) $k = 1$.

$$H(1) = f(1) - \binom{n}{2}\binom{n+7}{2} + 2\binom{n-1}{1}\binom{n+6}{1} - 1 = -\frac{1}{2}(n^3 - 4n^2 - 51n + 26)$$
$$= -\frac{1}{2}[n(n^2 - 4n - 51) + 26].$$

Thus, we have $H(1) < 0$ if $n \ge 14$.

(b) $k = 2$.

$$H(2) = f(2) - \binom{n}{2}\binom{n+7}{2} + 2\binom{n-1}{1}\binom{n+6}{1} - 1 = -\frac{1}{2}(n^3 - 7n^2 - 66n + 14)$$
$$= -\frac{1}{2}[n(n^2 - 7n - 66) + 14].$$

Thus, if $n \ge 14$, $H(2) < 0$.

(c) $k = 3$.

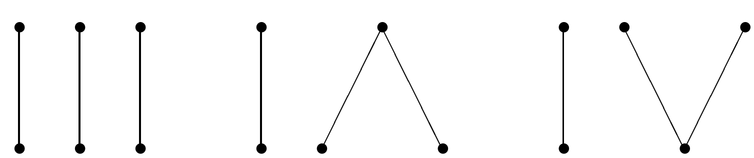
$$H(3) = f(3) - \binom{n}{2}\binom{n+7}{2} + 2\binom{n-1}{1}\binom{n+6}{1} - 1$$
$$= -\frac{1}{2}(n^3 - 8n^2 - 71n + 8) = -\frac{1}{2}[n(n^2 - 8n - 71) + 8].$$

Therefore, $H(3) < 0$ when $n \ge 14$.

In summary, we have $f(k) < c_4(G_1) < c_4(G_2) < c_4(G_3)$. So $c_4(G') < c_4(G)$, where $G \in \{G_1, G_2, G_3\}$. Therefore, G' and G have different cycle distributions. Thus we have completed the proof. □

Theorem 4 *$G = K_{n,n+7} - A(|A| = 3)$ is determined by its cycle distribution if $n \ge 16$.*

Proof From [8], $K_{n,n+7}[A]$ must be one of six types of graphs as shown in Figure 2.



H_1 H_2 H_3

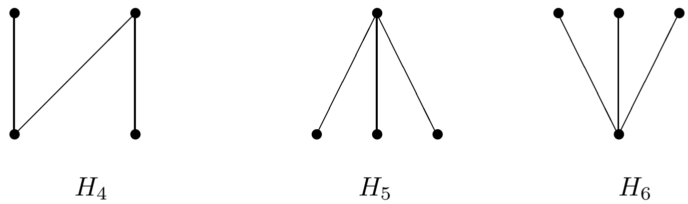


Figure 2 $K_{n,n+7}-|A|$ ($|A|=3$)

Denote these six types of graphs by H_1, H_2, H_3, H_4, H_5, and H_6, respectively. Let $c_4(G_i)$ denote the number of cycles of length 4 of $G_i=K_{n,n+7}-E(H_i)$. Clearly, we have

$$\begin{aligned}
c_4(G_1) &= \binom{n}{2}\binom{n+7}{2}-3\binom{n-1}{1}\binom{n+6}{1}+3,\\
c_4(G_2) &= \binom{n}{2}\binom{n+7}{2}-3\binom{n-1}{1}\binom{n+6}{1}+(n-1)+2,\\
c_4(G_3) &= \binom{n}{2}\binom{n+7}{2}-3\binom{n-1}{1}\binom{n+6}{1}+(n+6)+2,\\
c_4(G_4) &= \binom{n}{2}\binom{n+7}{2}-3\binom{n-1}{1}\binom{n+6}{1}+(2n+6)-1,\\
c_4(G_5) &= \binom{n}{2}\binom{n+7}{2}-3\binom{n-1}{1}\binom{n+6}{1}+3(n-1),\\
c_4(G_6) &= \binom{n}{2}\binom{n+7}{2}-3\binom{n-1}{1}\binom{n+6}{1}+3(n+6).
\end{aligned}$$

Obviously, there are $c_4(G_1)<c_4(G_2)<c_4(G_3)<c_4(G_4)<c_4(G_5)<c_4(G_6)$. It shows that $G_i(i=1,\ldots,6)$ have different cycle distributions.

Now we will prove that if $G'\neq G=K_{n,n+7}-A(|A|=3)$, G' and G have different cycle distributions. We know that G' must be in one of the following three cases.

Case 1 $G'\in\{K_{n,n+7}-A|0\leq|A|\leq 2$ or $|A|\geq 4\}$. By Theorems 1, 2 and 3, $G'=K_{n,n+7}$, $G'=K_{n,n+7}-A(|A|=1)$, and $G'=K_{n,n+7}-A(|A|=2)$ are determined by their cycle distributions. Moreover, if $G'=K_{n,n+7}-A(|A|\geq 4)$, by Lemma 2, $\max c_4(G')<\min c_4(G)$. Thus, if $G'\in\{G|G=K_{n,n+7}-A,|A|\neq 3\}$, G' and G have different cycle distributions.

Case 2 $G'\in\{K_{n+k,n+7-k}-A||A|=j,0\leq j\leq n+k-2,1\leq k\leq 3\}$. As $n+k\geq j+2$, by Lemma 3, $c_{2n+2k}(G')\neq 0$ and $c_{2n+2k}(G)=0$. So G' and G have different cycle distributions.

Case 3 $G'\in\{K_{n+k,n+7-k}-A||A|=j\geq n+k-1,1\leq k\leq 3\}$. Let G'' be any graphs in $\{K_{n+k,n+7-k}-A||A|=j=n+k-1,1\leq k\leq 3\}$. So $c_4(G')\leq\max c_4(G'')$. Denote

$f(k) = \max c_4(G'')$. By Lemma 1, we have

$$f(k) = \binom{n+k}{2}\binom{n+7-k}{2} - (n+k-1)\binom{n+k-1}{1}\binom{n+7-k-1}{1} + \binom{n+k-1}{2}(n+7-k-1).$$

We now prove $f(k) < c_4(G_1) < c_4(G_2) < c_4(G_3) < c_4(G_4) < c_4(G_5) < c_4(G_6)$. In fact, we only need to prove $f(k) < c_4(G_1) = \binom{n}{2}\binom{n+7}{2} - 3\binom{n-1}{1}\binom{n+6}{1} + 3$.

Let $H(k) = f(k) - \binom{n}{2}\binom{n+7}{2} + 3\binom{n-1}{1}\binom{n+6}{1} - 3$ and consider the following cases:

(a) $k = 1$.

$$\begin{aligned} H(1) &= f(1) - \binom{n}{2}\binom{n+7}{2} + 3\binom{n-1}{1}\binom{n+6}{1} - 3 \\ &= -\frac{1}{2}(n^3 - 6n^2 - 61n + 42) = -\frac{1}{2}[n(n^2 - 6n - 61) + 42]. \end{aligned}$$

Thus, if $n \geq 16$, $H(1) < 0$.

(b) $k = 2$.

$$\begin{aligned} H(2) &= f(2) - \binom{n}{2}\binom{n+7}{2} + 3\binom{n-1}{1}\binom{n+6}{1} - 3 \\ &= -\frac{1}{2}(n^3 - 9n^2 - 76n + 30) = -\frac{1}{2}[n(n^2 - 9n - 76) + 30]. \end{aligned}$$

It shows that $H(2) < 0$ if $n \geq 16$.

(c) $k = 3$.

$$\begin{aligned} H(3) &= f(3) - \binom{n}{2}\binom{n+7}{2} + 3\binom{n-1}{1}\binom{n+6}{1} - 3 \\ &= -\frac{1}{2}(n^3 - 10n^2 - 81n + 24) = -\frac{1}{2}[n(n^2 - 10n - 81) + 24]. \end{aligned}$$

Thus, if $n \geq 16$, $H(3) < 0$.

In summary, $f(k) < c_4(G_1) < c_4(G_2) < c_4(G_3) < c_4(G_4) < c_4(G_5) < c_4(G_6)$. So $c_4(G') < c_4(G)$, where $G \in \{G_1, G_2, G_3, G_4, G_5, G_6\}$. Therefore, G' and G have different cycle distributions. Thus, the proof is completed. □

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