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Hilbert Coefficients of Filtrations with Almost Maximal Depth

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Abstract Let $\mathcal{F}$ be a Hilbert filtration with respect to a Cohen-Macaulay R-module M. When $G(\mathcal{F},M)$ and $F_K(\mathcal{F},M)$ have almost maximal depths, we show that the length $\lambda(KI_nM/KJI_{n-1}M)$ and the reduction number $r_J^K(\mathcal{F},M)$ are independent of J. Lower bounds for the first and second Hilbert coefficients are obtained.

Keywords Hilbert coefficients; fiber cones; depth.

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1. Introduction

Let $(R,\mathfrak{m})$ be a Noetherian local ring with infinite residue field. We say that $\mathcal{F}=\{I_n\}_{n\geq 0}$ is a filtration if $I_0=R\supseteq I_1\supseteq I_2\supseteq\cdots$ is a chain of ideals of R such that $I_1\neq R$ and $I_mI_n\subseteq I_{m+n}$ for all m,n. For any filtration $\mathcal{F}=\{I_n\}_{n\geq 0}$, let $R(\mathcal{F})=\bigoplus_{n\geq 0}I_n$ and $G(\mathcal{F})=\bigoplus_{n\geq 0}I_n/I_{n+1}$ be the Rees ring and associated graded ring of $\mathcal{F}$. If $\mathcal{F}$ is an I_1-adic filtration, we write $R(I_1)$ and $G(I_1)$ for $R(\mathcal{F})$ and $G(\mathcal{F})$, respectively. Further, let M be a finitely generated R-module. We say that $\mathcal{F}$ is a Hilbert filtration with respect to M if $\lambda(M/I_1M)<\infty$ and $I_1I_nM=I_{n+1}M$ for large n. Hilbert $\mathcal{F}$-filtration with respect to M satisfies $\bigcap_{n\geq 0}I_nM=0$.

Throughout the paper, let $(R,\mathfrak{m})$ be a commutative Noetherian local ring with infinite residue field, M a finitely generated Cohen-Macaulay R-module of dimension $d>0$ and $\mathcal{F}$ a Hilbert filtration with respect to M. Let K be an $\mathfrak{m}$-primary ideal of R such that $I_{n+1}\subseteq KI_n$ for all $n\geq 0$.

Let $F_K(\mathcal{F})=\bigoplus_{n\geq 0}I_n/KI_n$ be the fiber cone of $\mathcal{F}$ with respect to K, and $G(\mathcal{F},M)=\bigoplus_{n\geq 0}I_nM/I_{n+1}M$, $F_K(\mathcal{F},M)=\bigoplus_{n\geq 0}I_nM/KI_nM$. Then $G(\mathcal{F},M)$ is a finitely generated $G(\mathcal{F})$-module and $F_K(\mathcal{F},M)$ is a finitely generated $F_K(\mathcal{F})$-module.

Let $H_K(\mathcal{F},M,n)=\lambda(M/KI_nM)$ be the Hilbert-Samuel function of $\mathcal{F}$ with respect to M and K, and $P_K(\mathcal{F},M,n)$ the corresponding polynomial. Then

$$P_K(\mathcal{F},M,n)=g_0(\mathcal{F},M)\binom{n+d-1}{d}-g_1(\mathcal{F},M)\binom{n+d-2}{d-1}+\cdots+(-1)^dg_d(\mathcal{F},M).$$

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In this paper, we will discuss the properties of Hilbert coefficients $g_i(\mathcal{F}, M)$ and related problems under the assumption that $G(\mathcal{F}, M)$ and $F_K(\mathcal{F}, M)$ have almost maximal depths, i.e., depth $G(\mathcal{F}, M) \geq d-1$ and depth $F_K(\mathcal{F}, M) \geq d-1$.

When $(R, \mathfrak{m})$ is Cohen-Macaulay of dimension $d > 0$, I is an $\mathfrak{m}$-primary ideal such that depth $G(I) \geq d-1$ and J is any minimal reduction of I, Corso, Polini and Pinto[2] ([5, Corollary 2.6]) showed that $\lambda(I^n/JI^{n-1})$ is independent of J for all $n \geq 1$. To generalize this result, we suppose that depth $G(\mathcal{F}, M) \geq d-1$ and depth $F_K(\mathcal{F}, M) \geq d-1$. Let J be a minimal reduction of $\mathcal{F}$ with respect to M. We will show that $\lambda(KI_nM/KJI_{n-1}M)$ does not depend on J for all $n \geq 1$.

Under the above assumptions on R and depth of $G(I)$, reduction numbers have also nice properties. It is proved in [5] that $r_J(I)$ is independent of J. Furthermore, Marley[10] showed that $r(I) = n(I) + d$, where $n(I)$ is the postulation number of I. We can also generalize these results to the filtration case. Assume that depth $G(\mathcal{F}, M) \geq d-1$ and depth $F_K(\mathcal{F}, M) \geq d-1$, and let J be a minimal reduction of $\mathcal{F}$ with respect to M. We will show that the K-reduction number $r_J^K(\mathcal{F}, M)$ is independent of J and $r^K(\mathcal{F}, M) = n^K(\mathcal{F}, M) + d$.

In Section 5, we will give lower bounds for $g_1(\mathcal{F}, M)$ and $g_2(\mathcal{F}, M)$. Our results are the following

$$g_1(\mathcal{F}, M) \geq \sum_{n\geq 1} \lambda(KI_nM + JM/JM) - \lambda(M/KM)$$

$$g_2(\mathcal{F}, M) \geq \sum_{n\geq 1} n\lambda(KI_{n+1}M + JM/JM) + \lambda(M/\bigcup_{k\geq 1}(KI_kM + J_{d-2}M) : I_1^k).$$

The above formula for the lower bound of $g_1(\mathcal{F}, M)$ generalize the formula for $g_1(\mathcal{F}, R)$ obtained by Jayanthan and Verma[7]. One simply puts $M = R$, then

$$g_1(\mathcal{F}, R) \geq \sum_{n\geq 1} \lambda(KI_n + J/J) - \lambda(R/K).$$

2. Preliminaries

Let $H(\mathcal{F}, M, n) = \lambda(M/I_nM)$ be the Hilbert-Samuel function of $\mathcal{F}$ with respect to M and $P(\mathcal{F}, M, n)$ the corresponding polynomial. We have that

$$P(\mathcal{F}, M, n) = e_0(\mathcal{F}, M)\binom{n+d-1}{d} - e_1(\mathcal{F}, M)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(\mathcal{F}, M).$$

Then $g_0(\mathcal{F}, M) = e_0(\mathcal{F}, M)$.

An ideal $J \subseteq I_1$ is said to be a reduction of $\mathcal{F}$ with respect to M if there exists an integer $r > 0$ such that $I_{n+1}M = JI_nM$ for all $n \geq r$. By [9, Lemma 1], there exist $x_1, \ldots, x_d \in I_1$ such that $(x_1, \ldots, x_d)$ is a minimal reduction of $\mathcal{F}$ with respect to M. Then, since M is Cohen-Macaulay, we have that $e_0(\mathcal{F}, M) = \lambda(M/(x_1, \ldots, x_d)M)$.

Let J be a minimal reduction of $\mathcal{F}$ with respect to M. The K-reduction number $r_J^K(\mathcal{F}, M)$ of $\mathcal{F}$ with respect to M and J is defined by

$$r_J^K(\mathcal{F}, M) = \min\{n \mid KI_{m+1}M = KJI_mM \text{ for all } m \geq n\},$$

and the K-reduction number $r^K(\mathcal{F}, M)$ of $\mathcal{F}$ with respect to M is defined as

$$r^K(\mathcal{F}, M) = \min\{r_J^K(\mathcal{F}, M) \mid J \text{ is a minimal reduction of } \mathcal{F} \text{ with respect to } M\}.$$

Set $n^K(\mathcal{F}, M) = \min\{n \mid P_K(\mathcal{F}, M, k) = H_K(\mathcal{F}, M, k) \text{ for all } k > n\}$, which we call as the K-postulation number of $\mathcal{F}$ with respect to M. Note that, as $P_K(\mathcal{F}, M, n)$ is a polynomial of degree d, $n^K(\mathcal{F}, M) \geq -d$.

Let $x \in I_1 \setminus I_2$ and x^* the initial form of x in $G(\mathcal{F})$. x^* is said to be superficial for $G(\mathcal{F}, M)$ if there exists an integer $c > 0$ such that $(I_{n+1}M : x) \cap I_c M = I_n M$ for all $n > c$. Similarly, for any $x \in I_1 \setminus KI_1$, let x^0 be the initial form of x in $F_K(\mathcal{F})$, x^0 is said to be superficial for $F_K(\mathcal{F}, M)$ if there exists an integer $c > 0$ such that $(KI_{n+1}M : x) \cap I_c M = KI_n M$ for all $n > c$. Superficial sequences are defined inductively.

Suppose that x^0 is superficial for $F_K(\mathcal{F}, M)$. Let " $-$ " denote images modulo (x). Thus, $\overline{\mathcal{F}} = \mathcal{F}/(x) = \{I_n + (x)/(x)\}_{n\geq 0}$, $\overline{J} = J/(x)$, $\overline{K} = K/(x)$, $\overline{M} = M/xM$. Since

$$H_{\overline{K}}(\overline{\mathcal{F}}, \overline{M}, n+1) = H_K(\mathcal{F}, M, n+1) - H_K(\mathcal{F}, M, n) + \lambda((KI_{n+1}M : x)/KI_n M),$$

it follows that

$$g_i(\overline{\mathcal{F}}, \overline{M}) = g_i(\mathcal{F}, M), i = 0, 1, \ldots, d-1.$$

The following lemma can be shown by similar arguments in [7] (cf. [3]).

Lemma 2.1 *Let J be a minimal reduction of $\mathcal{F}$ with respect to M. Then there exist $x_1, \ldots, x_d \in I_1 \setminus KI_1$ such that $J = (x_1, \ldots, x_d)$, $x_1^*, \ldots, x_d^*$ is a superficial sequence for $G(\mathcal{F}, M)$ and $x_1^0, \ldots, x_d^0$ is a superficial sequence for $F_K(\mathcal{F}, M)$.*

Furthermore, if depth $G(\mathcal{F}, M) \geq k$ and depth $F_K(\mathcal{F}, M) \geq k$ for an integer $k > 0$, then one may choose the above $x_1, \ldots, x_d$ such that $x_1^*, \ldots, x_k^*$ is a regular $G(\mathcal{F}, M)$-sequence and $x_1^0, \ldots, x_k^0$ is a regular $F_K(\mathcal{F}, M)$-sequence. In this case, for all $n \geq 0$,

$$(KI_{n+1}M + (x_1, \ldots, x_{i-1})M) : x_i = KI_n M + (x_1, \ldots, x_{i-1})M, \ i = 1, 2, \ldots, k.$$

The same arguments as in [8, Lemma 3.8] and [10, Lemma 1] can be applied to get the following

Proposition 2.2 *Let $J = (x_1, \ldots, x_d)$ be a minimal reduction of $\mathcal{F}$ with respect to M with generators chosen as in Lemma 2.1. Suppose that x_1^* is $G(\mathcal{F}, M)$-regular and x_1^0 is $F_K(\mathcal{F}, M)$-regular. Let " $-$ " denote images modulo (x_1). Then $r_{\overline{J}}^{\overline{K}}(\overline{\mathcal{F}}, \overline{M}) = r_J^K(\mathcal{F}, M)$ and $n^{\overline{K}}(\overline{\mathcal{F}}, \overline{M}) = n^K(\mathcal{F}, M) + 1$.*

The Hilbert coefficients $g_i(\mathcal{F}, M)$ can be calculated by Hilbert series. Write

$$P_K(\mathcal{F}, M, n) = g_0'(\mathcal{F}, M)\binom{n+d}{d} - g_1'(\mathcal{F}, M)\binom{n+d-1}{d-1} + \cdots + (-1)^d g_d'(\mathcal{F}, M).$$

Then $g_0'(\mathcal{F}, M) = g_0(\mathcal{F}, M)$ and $g_i'(\mathcal{F}, M) = g_i(\mathcal{F}, M) + g_{i-1}(\mathcal{F}, M)$, $i = 1, \ldots, d$. Let $H_{\mathcal{F}}(M, t) = \sum_{n\geq 0} H_K(\mathcal{F}, M, n)t^n$ be the Hilbert series of $\mathcal{F}$ with respect to M and K. Then there exists a unique polynomial $f(t) \in \mathbb{Z}[t]$ such that $H_{\mathcal{F}}(M, t) = \frac{f(t)}{(1-t)^{d+1}}$. Then $g_i'(\mathcal{F}, M) = \frac{f^{(i)}(1)}{i!}$, $i = 0, 1, \ldots, d$, [1, Chapt. 4].

Using the same arguments as in [4, Proposition 1.5], we have

Proposition 2.3 *Let $J = (x_1, \ldots, x_d)$ be a minimal reduction of $\mathcal{F}$ with respect to M with x_1^* being superficial for $G(\mathcal{F}, M)$ and x_1^0 being superficial for $F_K(\mathcal{F}, M)$. Let "$-$" denote images modulo (x_1). Then*

$$g_d(\mathcal{F}, M) = g_d(\overline{\mathcal{F}}, \overline{M}) - \sum_{n\geq 0}(-1)^d \lambda((KI_{n+1}M : x_1)/KI_nM).$$

Furthermore, if x_1^* is $G(\mathcal{F}, M)$-regular and x_1^0 is $F_K(\mathcal{F}, M)$-regular, then

$$\sum_{n\geq 0} H_{\overline{K}}(\overline{\mathcal{F}}, \overline{M}, n)t^n = (1-t)\sum_{n\geq 0} H_K(\mathcal{F}, M, n)t^n.$$

3. Independence of lengths

In this section, we prove the independence of the length $\lambda(KI_nM/KJI_{n-1}M)$ on J. We need a lemma.

Lemma 3.1 *Suppose that depth $G(\mathcal{F}, M) \geq d-1$ and depth $F_K(\mathcal{F}, M) \geq d-1$. Let J be a minimal reduction of $\mathcal{F}$ with respect to M and $r = r_J^K(\mathcal{F}, M)$. Then*

$$\sum_{n\geq 0} H_K(\mathcal{F}, M, n)t^n = \frac{\lambda(\frac{M}{KM}) + [\lambda(\frac{KM}{JM}) - \lambda(\frac{KI_1M}{KJM})]t + \sum\limits_{n=2}^{r+1}[\lambda(\frac{KI_{n-1}M}{KJI_{n-2}M}) - \lambda(\frac{KI_nM}{KJI_{n-1}M})]t^n}{(1-t)^{d+1}}.$$

Proof Choose generators $x_1, \ldots, x_d$ for J as in Lemma 2.1. Let "$-$" denote images modulo $(x_1, \ldots, x_{d-1})$. Then from Proposition 2.3, we have

$$\sum_{n\geq 0} H_K(\mathcal{F}, M, n)t^n = \frac{\sum\limits_{n\geq 0} H_{\overline{K}}(\overline{\mathcal{F}}, \overline{M}, n)t^n}{(1-t)^{d-1}}.$$

It is clear that $\lambda(M/KM) = \lambda(\overline{M}/\overline{KM})$, $\lambda(KM/JM) = \lambda(\overline{KM}/\overline{JM})$. For all $n \geq 0$, by Lemma 2.1,

$$(KI_{n+1}M + (x_1, \ldots, x_{i-1})M) : x_i = KI_nM + (x_1, \ldots, x_{i-1})M,\ \ i = 1, 2, \ldots, d-1,$$

it follows that

$$KI_{n+1}M \cap (x_1, \ldots, x_i)M \subseteq KJI_nM, i = 1, 2, \ldots, d-1.$$

Hence

$$\lambda(KI_nM/KJI_{n-1}M) = \lambda(\overline{KI}_n\overline{M}/\overline{KJI}_{n-1}\overline{M}), n = 1, 2, \ldots.$$

Thus, we may assume that $d = 1$ and it is enough to show that

$$\sum_{n\geq 0} H_K(\mathcal{F}, M, n)t^n = \frac{\lambda(\frac{M}{KM}) + [\lambda(\frac{KM}{x_1M}) - \lambda(\frac{KI_1M}{x_1KM})]t + \sum\limits_{n=2}^{r+1}[\lambda(\frac{KI_{n-1}M}{x_1KI_{n-2}M}) - \lambda(\frac{KI_nM}{x_1KI_{n-1}M})]t^n}{(1-t)^2}.$$

Set

$$\sum_{n\geq 0} H_K(\mathcal{F}, M, n)t^n = \frac{\sum\limits_{n\geq 0} a_n t^n}{(1-t)^2}.$$

Then, we get that

$$\begin{aligned} a_0 &= \lambda(M/KM) \\ a_1 &= \lambda(M/KI_1M) - 2\lambda(M/KM) \\ a_n &= \lambda(M/KI_nM) - 2\lambda(M/KI_{n-1}M) + \lambda(M/KI_{n-2}M), \quad n = 2, 3, \ldots, \end{aligned}$$

and, $a_n = 0$ for $n \gg 0$. Let $n \geq 2$. Note that

$$\lambda(M/KI_nM) - 2\lambda(M/KI_{n-1}M) + \lambda(M/KI_{n-2}M) = \lambda(\frac{KI_{n-1}M}{KI_nM}) - \lambda(\frac{KI_{n-2}M}{KI_{n-1}M}).$$

Thus

$$a_n = \lambda(KI_{n-1}M/KI_nM) - \lambda(KI_{n-2}M/KI_{n-1}M), n = 2, 3, \ldots.$$

Since x_1 is M-regular and

$$\lambda(KM/KI_1M) + \lambda(KI_1M/x_1KM) = \lambda(KM/x_1M) + \lambda(x_1M/x_1KM),$$

we have that

$$\begin{aligned} a_1 &= \lambda(KM/KI_1M) - \lambda(M/KM) \\ &= \lambda(KM/KI_1M) - \lambda(x_1M/x_1KM) \\ &= \lambda(KM/x_1M) - \lambda(KI_1M/x_1KM). \end{aligned}$$

Similarly, for $n \geq 2$, since

$$\begin{aligned} &\lambda(KI_{n-1}M/KI_nM) + \lambda(KI_nM/x_1KI_{n-1}M) \\ &\quad = \lambda(KI_{n-1}M/x_1KI_{n-2}M) + \lambda(x_1KI_{n-2}M/x_1KI_{n-1}M), \end{aligned}$$

it follows that

$$\begin{aligned} a_n &= \lambda(KI_{n-1}M/KI_nM) - \lambda(x_1KI_{n-2}M/x_1KI_{n-1}M) \\ &= \lambda(KI_{n-1}M/x_1KI_{n-2}M) - \lambda(KI_nM/x_1KI_{n-1}M). \end{aligned}$$

As $KI_nM = x_1KI_{n-1}M$ for all $n \geq r+1$, we see that $a_n = 0$ for all $n > r+1$. The result follows. □

Theorem 3.2 *Suppose that depth $G(\mathcal{F}, M) \geq d-1$ and depth $F_K(\mathcal{F}, M) \geq d-1$. Let J be a minimal reduction of $\mathcal{F}$ with respect to M. Then $\lambda(KI_nM/KJI_{n-1}M)$ does not depend on J for all $n \geq 0$.*

Proof Let $r = r_J^K(\mathcal{F}, M)$. Then, from Lemma 3.1, we have

$$\sum_{n\geq 0} H_K(\mathcal{F}, M, n)t^n = \frac{\lambda(\frac{M}{KM}) + [\lambda(\frac{KM}{JM}) - \lambda(\frac{KI_1M}{KJM})]t + \sum\limits_{n=2}^{r+1}[\lambda(\frac{KI_{n-1}M}{KJI_{n-2}M}) - \lambda(\frac{KI_nM}{KJI_{n-1}M})]t^n}{(1-t)^{d+1}}.$$

Denote the right side as $\frac{\sum_{n=0}^{r+1} a_n t^n}{(1-t)^{d+1}}$. Since $\sum_{n\geq 0} H_K(\mathcal{F}, M, n)t^n = \sum_{n\geq 0} \lambda(M/KI_nM)t^n$ is a series with coefficients independent of J, we see that a_n is independent of J.

Note that $\lambda(M/JM) = e_0(\mathcal{F}, M)$ is independent of J. Then $\lambda(KM/JM) = \lambda(M/JM) - \lambda(M/KM)$ is also independent of J. It follows that $\lambda(KI_1M/KJM) = \lambda(KM/JM) - a_1$ and $\lambda(KM/KJM) = \lambda(M/KI_1M) - e(\mathcal{F}, M) - a_1$ are independent of J. Inductively, suppose that $\lambda(KI_nM/KJI_{n-1}M)$ does not depend on J. Then

$$\lambda(KI_{n+1}M/KJI_nM) = \lambda(KI_nM/KJI_{n-1}M) - a_{n+1}$$

is also independent of J. The proof is completed. □

4. Independence of reduction numbers

We first show that $r_J^K(\mathcal{F}, M)$ is independent of J.

Theorem 4.1 *Assume that depth $G(\mathcal{F}, M) \geq d-1$ and depth $F_K(\mathcal{F}, M) \geq d-1$. Let J be a minimal reduction of $\mathcal{F}$ with respect to M. Then $r_J^K(\mathcal{F}, M)$ is independent of J.*

Proof If $d > 1$, then, by Lemma 2.1, we may choose $x_1, \ldots, x_d$ such that $J = (x_1, \ldots, x_d)$, $x_1^*, \ldots, x_{d-1}^*$ is $G(\mathcal{F}, M)$-regular and $x_1^0, \ldots, x_{d-1}^0$ is $F_K(\mathcal{F}, M)$-regular. Then $r_J^K(\mathcal{F}, M) = r_{\overline{J}}^{\overline{K}}(\overline{\mathcal{F}}, \overline{\mathcal{M}})$ by Proposition 2.2, where " $-$ " denote images modulo $(x_1, \ldots, x_{d-1})$. Hence we may assume that $d = 1$.

Suppose that $d = 1$. Let $J_1 = (x)$ and $J_2 = (y)$ be two minimal reductions of $\mathcal{F}$ with respect to M. Set $r_1 = r_{J_1}^K(\mathcal{F}, M)$ and $r_2 = r_{J_2}^K(\mathcal{F}, M)$. If $r_1 \neq r_2$, say $r_1 > r_2$, then

$$yKI_{r_1}M = KI_{r_1+1}M = xKI_{r_1}M = xyKI_{r_1-1}M.$$

It follows that $KI_{r_1}M = xKI_{r_1-1}M$, which contradicts the minimality of r_1. Hence $r_1 = r_2$.

Further, we calculate the reduction number.

Theorem 4.2 *Assume that depth $G(\mathcal{F}, M) \geq d-1$ and depth $F_K(\mathcal{F}, M) \geq d-1$. Then $r^K(\mathcal{F}, M) = n^K(\mathcal{F}, M) + d$.*

Proof Let J be a minimal reduction of $\mathcal{F}$ with respect to M. We want to show that $r_J^K(\mathcal{F}, M) = n^K(\mathcal{F}, M) + d$.

If $d > 1$, then, by Lemma 2.1, we may choose $x_1, \ldots, x_d$ such that $J = (x_1, \ldots, x_d)$, $x_1^*, \ldots, x_{d-1}^*$ is $G(\mathcal{F}, M)$-regular and $x_1^0, \ldots, x_{d-1}^0$ is $F_K(\mathcal{F}, M)$-regular. Then $r_J^K(\mathcal{F}, M) = r_{\overline{J}}^{\overline{K}}(\overline{\mathcal{F}}, \overline{\mathcal{M}})$ and $n^K(\mathcal{F}, M) = n^{\overline{K}}(\overline{\mathcal{F}}, \overline{\mathcal{M}}) + d - 1$ by Proposition 2.2, where " $-$ " denote images modulo $(x_1, \ldots, x_{d-1})$. Hence we may assume that $d = 1$. Let $r = r^K(\mathcal{F}, M)$.

If $n^K(\mathcal{F}, M) = -1$, then $P_K(\mathcal{F}, M, 0) = H_K(\mathcal{F}, M, 0)$. But $P_K(\mathcal{F}, M, 0) = -g_1(\mathcal{F}, M)$ and $H_K(\mathcal{F}, M, 0) = \lambda(M/KM)$. Note that

$$g_1(\mathcal{F}, M) = \sum_{n=1}^{r} \lambda(KI_nM/xKI_{n-1}M) - \lambda(M/KM).$$

It follows that

$$\sum_{n=1}^{r} \lambda(KI_nM/xKI_{n-1}M) = 0,$$

hence, $KI_nM = xKI_{n-1}M$ holds for all $n \geq 1$. Thus $r_J^K(\mathcal{F}, M) = 0$.

Now assume $n^K(\mathcal{F}, M) \geq 0$. Since $KI_{n+1}M = xKI_nM$ for all $n \geq r$, we have that $KI_nM = x^{n-r}KI_rM$ for all $n \geq r$. Thus

$$\begin{aligned} H_K(\mathcal{F}, M, n) = \lambda(M/KI_nM) &= \lambda(M/x^{n-r}M) + \lambda(x^{n-r}M/x^{n-r}KI_rM) \\ &= (n-r)\lambda(M/xM) + \lambda(M/KI_rM), \text{ for all } n \geq r. \end{aligned}$$

On the other hand, since

$$\begin{aligned} g_1(\mathcal{F}, M) =& \sum_{n=1}^{r} \lambda(KI_nM/xKI_{n-1}M) - \lambda(M/KM) \\ =& \lambda(KI_rM/xKI_{r-1}M) + \lambda(xKI_{r-1}M/x^2KI_{r-2}M) + \\ & \cdots + \lambda(x^{r-1}KI_1M/x^rKM) - \lambda(M/KM) \\ =& \lambda(KI_rM/x^rKM) - \lambda(M/KM), \end{aligned}$$

we get that, for all $n \geq r$,

$$\begin{aligned} & P_K(\mathcal{F}, M, n) = ng_0(\mathcal{F}, M) - g_1(\mathcal{F}, M) \\ & = n\lambda(M/xM) - \lambda(KI_rM/x^rKM) + \lambda(M/KM) \\ & = (n-r)\lambda(M/xM) + r\lambda(M/xM) - \lambda(KI_rM/x^rKM) + \lambda(M/KM) \\ & = (n-r)\lambda(M/xM) + \lambda(M/x^rM) - \lambda(KI_rM/x^rKM) + \lambda(x^rM/x^rKM) \\ & = (n-r)\lambda(M/xM) + \lambda(M/KI_rM) \\ & = H_K(\mathcal{F}, M, n). \end{aligned}$$

Therefore $n^K(\mathcal{F}, M) \leq r - 1$.

It remains to show that $r \leq n^K(\mathcal{F}, M) + 1$. For all $n \geq n^K(\mathcal{F}, M) + 1$, from $P_K(\mathcal{F}, M, n) = H_K(\mathcal{F}, M, n)$, i.e., $\lambda(M/KI_nM) = n\lambda(M/xM) - g_1(\mathcal{F}, M)$, we obtain $g_1(\mathcal{F}, M) = -\lambda(M/KI_nM) + \lambda(M/x^nM)$. It follows that, for all $n \geq n^K(\mathcal{F}, M) + 1$,

$$\begin{aligned} \lambda(M/KI_{n+1}M) &= (n+1)\lambda(M/xM) - g_1(\mathcal{F}, M) \\ &= \lambda(M/x^{n+1}M) + \lambda(M/KI_nM) - \lambda(M/x^nM) \\ &= \lambda(x^nM/x^{n+1}M) + \lambda(M/KI_nM) \\ &= \lambda(M/xM) + \lambda(xM/xKI_nM) \\ &= \lambda(M/xKI_nM). \end{aligned}$$

Thus $KI_{n+1}M = xKI_nM$ for all $n \geq n^K(\mathcal{F}, M) + 1$. Hence $r \leq n^K(\mathcal{F}, M) + 1$ as required. □

5. Lower bounds for $g_1(\mathcal{F}, M)$ and $g_2(\mathcal{F}, M)$

In this section, we will give lower bounds of the Hilbert coefficients $g_1(\mathcal{F}, M)$ and $g_2(\mathcal{F}, M)$.

Let us first give a lower bound for $g_1(\mathcal{F}, M)$.

Proposition 5.1 *Let J be a minimal reduction of $\mathcal{F}$ with respect to M. Then*

$$g_1(\mathcal{F}, M) \geq \sum_{n\geq 1} \lambda(KI_nM + JM/JM) - \lambda(M/KM).$$

Proof Choose $x_1, \ldots, x_d$ as in Lemma 2.1 such that $J = (x_1, \ldots, x_d)$. Let " $\overline{}$ " denote images modulo $(x_1, \ldots, x_{d-1})$. Then $\overline{KI}_{n+1}\overline{M} + \overline{JM}/\overline{JM} \cong KI_{n+1}M + JM/JM$, $M/KM \cong \overline{M}/\overline{KM}$ and $g_1(\mathcal{F}, M) = g_1(\overline{\mathcal{F}}, \overline{\mathcal{M}})$. Thus we may assume that $d = 1$. In this case, we have

$$g_1(\mathcal{F}, M) = \sum_{n\geq 1} \lambda(KI_nM/x_1KI_{n-1}M) - \lambda(M/KM).$$

But $KI_nM + x_1M/x_1M \cong KI_nM/KI_nM \cap x_1M$ is a factor module of $KI_nM/x_1KI_{n-1}M$, we get that $\lambda(KI_nM/x_1KI_{n-1}M) \geq \lambda(KI_nM + x_1M/x_1M)$. Hence

$$g_1(\mathcal{F}, M) \geq \sum_{n\geq 1} \lambda(KI_nM + x_1M/x_1M) - \lambda(M/KM).$$

For the second Hilbert coefficient, we need to generalize the definition of the Ratliff-Rush closure of a filtration introduced in [8].

Definition 5.2 *The Ratliff-Rush closure of $\mathcal{F}$ with respect to M and K is defined as $rr_K(\mathcal{F}, M) = \{rr_K(I_n, M)\}_{n\geq 0}$ with $rr_K(I_n, M) = \bigcup_{k\geq 1}(KI_{n+k}M : I_1^k)$.*

We will need the following properties of Ratliff-Rush closure, whose proof is similar to that of [8, Proposition 2.3].

Lemma 5.3 *$rr_K(I_n, M) = KI_nM$ for $n \gg 0$ and, if J is a minimal reduction of I_1, then, for all $n \geq 1$,*

$$rr_K(I_n, M) : J = rr_K(I_{n-1}, M).$$

Theorem 5.4 *Suppose that $d \geq 2$. Let J be a minimal reduction of I_1 and $x_1, \ldots, x_d$ as in Lemma 2.1 such that $J = (x_1, \ldots, x_d)$. Set $J_{d-2} = (x_1, \ldots, x_{d-2})$. Then*

$$g_2(\mathcal{F}, M) \geq \sum_{n\geq 1} n\lambda(\frac{KI_{n+1}M + JM}{JM}) + \lambda(\frac{M}{\bigcup_{k\geq 1}(KI_kM + J_{d-2}M) : I_1^k}).$$

Proof Firstly, let us show that J is also a minimal reduction of $\mathcal{F}$ with respect to M. Since $\mathcal{F}$ is Hilbert, there exists some $s \geq 1$ such that $I_1I_nM = I_{n+1}M$ for all $n \geq s$. As J is a minimal reduction of I_1, we have some $r \geq 1$ such that $I_1^{r+1} = JI_1^r$. Then, for any $n \geq r + s$,

$$I_{n+1}M = I_1^{r+1}I_{n-r}M = JI_1^rI_{n-r}M = JI_nM.$$

Hence, J is a minimal reduction of $\mathcal{F}$ with respect to M.

If $d > 2$, let " $\overline{}$ " denote images modulo J_{d-2}. Then $g_2(\mathcal{F}, M) = g_2(\overline{\mathcal{F}}, \overline{M})$, $\overline{KI}_{n+1}\overline{M} + \overline{JM}/\overline{JM} \cong KI_{n+1}M + JM/JM$ and $\overline{M}/rr_{\overline{K}}(\overline{I}_0, \overline{M}) \cong M/\cup_{k\geq 1}(KI_kM + J_{d-2}M) : I_1^k$. Thus we may assume that $d = 2$ and it is enough to show that

$$g_2(\mathcal{F}, M) \geq \sum_{n\geq 1} n\lambda(\frac{KI_{n+1}M + JM}{JM}) + \lambda(\frac{M}{rr_K(I_0, M)}).$$

Because that $rr_K(I_n, M) = KI_nM$ holds for $n \gg 0$, we can use $\sum_{n\geq 0} \lambda(\frac{M}{rr_K(I_n,M)})t^n$ to calculate $g_1^{'}(\mathcal{F}, M)$ and $g_2^{'}(\mathcal{F}, M)$.

Consider the exact sequence

$$0 \to \frac{M}{rr_K(I_{n-1}, M) : J} \xrightarrow{\beta} (\frac{M}{rr_K(I_{n-1}, M)})^2 \xrightarrow{\alpha} \frac{JM}{Jrr_K(I_{n-1}, M)} \to 0$$

where the map α and β are defined as, $\alpha(\overline{r}, \overline{s}) = \overline{x_1r + x_2s}$ and $\beta(\overline{r}) = (\overline{x_2r}, \overline{-x_1r})$. It follows that for all $n \geq 2$,

$$\begin{aligned}
2\lambda(\frac{M}{rr_K(I_{n-1}, M)}) &= \lambda(\frac{M}{rr_K(I_{n-1}, M) : J}) + \lambda(\frac{JM}{Jrr_K(I_{n-1}, M)}) \\
&= \lambda(\frac{M}{rr_K(I_{n-1}, M) : J}) + \lambda(\frac{M}{Jrr_K(I_{n-1}, M)}) - \lambda(\frac{M}{JM}) \\
&= \lambda(\frac{M}{rr_K(I_{n-1}, M) : J}) + \lambda(\frac{M}{Jrr_K(I_{n-1}, M)}) - e_0(\mathcal{F}, M).
\end{aligned}$$

Therefore

$$\begin{aligned}
&e_0(\mathcal{F}, M) + 2\lambda(\frac{M}{rr_K(I_{n-1}, M)}) - \lambda(\frac{M}{rr_K(I_n, M)}) - \lambda(\frac{M}{rr_K(I_{n-2}, M)}) \\
&= \lambda(\frac{M}{Jrr_K(I_{n-1}, M)}) - \lambda(\frac{M}{rr_K(I_n, M)}) + \lambda(\frac{M}{rr_K(I_{n-1}, M) : J}) - \lambda(\frac{M}{rr_K(I_{n-2}, M)}) \\
&= \lambda(\frac{rr_K(I_n, M)}{Jrr_K(I_{n-1}, M)}) - \lambda(\frac{rr_K(I_{n-1}, M) : J}{rr_K(I_{n-2}, M)}) \\
&= \lambda(\frac{rr_K(I_n, M)}{Jrr_K(I_{n-1}, M)}),
\end{aligned}$$

where the last equality holds because of $rr_K(I_n, M) : J = rr_K(I_{n-1}, M)$ for all $n \geq 1$, by Lemma 5.3.

Let $\sum_{n\geq 0} \lambda(\frac{M}{rr_K(I_n,M)})t^n = \frac{f(t)}{(1-t)^3}$. Since

$$\begin{aligned}
&\frac{e_0(\mathcal{F}, M) - f(t)}{1-t} = \frac{e_0(\mathcal{F}, M) - (1-t)^3 \sum_{n\geq 0} \lambda(\frac{M}{rr_K(I_n,M)})t^n}{1-t} \\
&= \sum_{n\geq 2}[e_0(\mathcal{F}, M) + 2\lambda(\frac{M}{rr_K(I_{n-1}, M)}) - \lambda(\frac{M}{rr_K(I_n, M)}) - \lambda(\frac{M}{rr_K(I_{n-2}, M)})]t^n + \\
&\quad e_0(\mathcal{F}, M)(1+t) - \lambda(\frac{M}{rr_K(I_0, M)})(1-2t) - \lambda(\frac{M}{rr_K(I_1, M)})t \\
&= \sum_{n\geq 2} \lambda(\frac{rr_K(I_n, M)}{Jrr_K(I_{n-1}, M)})t^n + e_0(\mathcal{F}, M)(1+t) - \lambda(\frac{M}{rr_K(I_0, M)})(1-2t) - \\
&\quad \lambda(\frac{M}{rr_K(I_1, M)})t,
\end{aligned}$$

it follows that

$$\begin{aligned}
f(t) =&e_0(\mathcal{F}, M) - (1-t)[\sum_{n\geq 2} \lambda(\frac{rr_K(I_n, M)}{Jrr_K(I_{n-1}, M)})t^n + e_0(\mathcal{F}, M)(1+t) - \\
&\lambda(\frac{M}{rr_K(I_0, M)})(1-2t) - \lambda(\frac{M}{rr_K(I_1, M)})t].
\end{aligned}$$

Then

$$f^{'}(t) = \sum_{n\geq 2} \lambda(\frac{rr_K(I_n, M)}{Jrr_K(I_{n-1}, M)})t^n + e_0(\mathcal{F}, M)(1+t) - \lambda(\frac{M}{rr_K(I_0, M)})(1-2t) - \lambda(\frac{M}{rr_K(I_1, M)})t-$$
$$(1-t)[\sum_{n\geq 2} n\lambda(\frac{rr_K(I_n, M)}{Jrr_K(I_{n-1}, M)})t^{n-1} + e_0(\mathcal{F}, M) + 2\lambda(\frac{M}{rr_K(I_0, M)}) - \lambda(\frac{M}{rr_K(I_1, M)})]$$

and

$$f^{''}(t) = 2[\sum_{n\geq 2} n\lambda(\frac{rr_K(I_n, M)}{Jrr_K(I_{n-1}, M)})t^{n-1} + e_0(\mathcal{F}, M) + 2\lambda(\frac{M}{rr_K(I_0, M)}) - \lambda(\frac{M}{rr_K(I_1, M)})]-$$
$$(1-t)\sum_{n\geq 2} n(n-1)\lambda(\frac{rr_K(I_n, M)}{Jrr_K(I_{n-1}, M)})t^{n-2}.$$

Hence

$$g_1^{'}(\mathcal{F}, M) = f^{'}(1) = \sum_{n\geq 2} \lambda(\frac{rr_K(I_n, M)}{Jrr_K(I_{n-1}, M)}) + 2e_0(\mathcal{F}, M) + \lambda(\frac{M}{rr_K(I_0, M)}) - \lambda(\frac{M}{rr_K(I_1, M)})$$
$$g_2^{'}(\mathcal{F}, M) = \frac{f^{''}(1)}{2} = \sum_{n\geq 2} n\lambda(\frac{rr_K(I_n, M)}{Jrr_K(I_{n-1}, M)}) + e_0(\mathcal{F}, M) + 2\lambda(\frac{M}{rr_K(I_0, M)}) - \lambda(\frac{M}{rr_K(I_1, M)}).$$

Therefore

$$\begin{aligned}
g_2(\mathcal{F}, M) &= g_2^{'}(\mathcal{F}, M) - g_1(\mathcal{F}, M) = g_2^{'}(\mathcal{F}, M) - g_1^{'}(\mathcal{F}, M) + e_0(\mathcal{F}, M) \\
&= \sum_{n\geq 1} n\lambda(\frac{rr_K(I_{n+1}, M)}{Jrr_K(I_n, M)}) + \lambda(\frac{M}{rr_K(I_0, M)}) \\
&= \sum_{n\geq 1} n[\lambda(\frac{rr_K(I_{n+1}, M)}{KI_{n+1}M + Jrr_K(I_n, M)}) + \lambda(\frac{KI_{n+1}M + Jrr_K(I_n, M)}{Jrr_K(I_n, M)})] + \lambda(\frac{M}{rr_K(I_0, M)}) \\
&= \sum_{n\geq 1} n[\lambda(\frac{rr_K(I_{n+1}, M)}{KI_{n+1}M + Jrr_K(I_n, M)}) + \lambda(\frac{KI_{n+1}M}{KI_{n+1}M \cap Jrr_K(I_n, M)})] + \lambda(\frac{M}{rr_K(I_0, M)}) \\
&= \sum_{n\geq 1} n[\lambda(\frac{rr_K(I_{n+1}, M)}{KI_{n+1}M + Jrr_K(I_n, M)}) + \lambda(\frac{KI_{n+1}M}{JM \cap KI_{n+1}M}) + \\
&\quad \lambda(\frac{JM \cap KI_{n+1}M}{KI_{n+1}M \cap Jrr_K(I_n, M)})] + \lambda(\frac{M}{rr_K(I_0, M)}) \\
&= \sum_{n\geq 1} n\lambda(\frac{KI_{n+1}M + JM}{JM}) + \lambda(\frac{M}{rr_K(I_0, M)}) + \\
&\quad \sum_{n\geq 1} n\lambda(\frac{rr_K(I_{n+1}, M)}{KI_{n+1}M + Jrr_K(I_n, M)}) + \sum_{n\geq 1} n\lambda(\frac{JM \cap KI_{n+1}M}{KI_{n+1}M \cap Jrr_K(I_n, M)}) \\
&\geq \sum_{n\geq 1} n\lambda(\frac{KI_{n+1}M + JM}{JM}) + \lambda(\frac{M}{rr_K(I_0, M)}).
\end{aligned}$$

The proof is completed. □

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