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$\aleph$-Products of Relative Flat Modules

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Abstract Let M be a right R-module and $\aleph$ an infinite cardinal number. A right R-module N is called $\aleph$-M-coherent if for any $0 \leq A < B \leq N$, such that $B/A \hookrightarrow mR$ for some $m \in M$, if B/A is finitely generated, then B/A is $\aleph$-fp. A ring R is called $\aleph$-M-coherent if R_R is $\aleph$-M-coherent. It is proved under some additional conditions that the $\aleph$-product of any family of M-flat left R-modules is M-flat if and only if R is $\aleph$-M-coherent. We also give some characterizations of $\aleph$-M-coherent modules and rings.

Keywords $\aleph$-M-coherent module; $\aleph$-M-coherent ring; M-flat module.

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1. Introduction

The concept of $\aleph$-products of modules was introduced by Dauns in [1] and has been studied by many authors (see, for example, [1, 2, 4, 5]).

Given right R-module M, so-called M-flat left R-modules were introduced in [3], which are flat relative to short exact sequences in the subcategory $\sigma[M]$ subgenerated by M, and where they also characterized the rings in which the direct products of M-flat modules remain M-flat. In this paper, following the direction of Dauns[1,2], Loustaunau[4] and Oyonarte[5], we characterize those rings for which $\aleph$-products of M-flat R-modules are M-flat, they will be called $\aleph$-M-coherent rings. These results generalize some results of Dauns[3] on direct products.

2. Preliminaries

We use $|X|$ to denote the cardinality of a set X. If I is a set and $\{M_i \mid i \in I\}$ is a family of right R-modules. Let $x = (x_i)_{i\in I} \in \prod_{i\in I} M_i$. We define the support of x, denoted by supp(x), as supp(x)=$\{i \in I \mid x_i \neq 0\}$. For an infinite cardinal number $\aleph$, define the $\aleph$-product of the $M_i's$ as

$$\prod_{i\in I}^{\aleph} M_i = \{x \in \prod_{i\in I} M_i \mid |\text{supp}(x)| < \aleph\}.$$

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Clearly one may view the direct sum and the direct product of a family of modules as two special cases of the same object, namely, the $\aleph$-product of a family of modules.

Let $\aleph$ be an infinite cardinal number and A a right R-module. Following Loustaunau[4], A is said to be $\aleph$-finitely generated, denoted by $\aleph$-fg, if every subset S of A, with $|S| < \aleph$, is contained in a finitely generated submodule of A. For example, every right R-module is $\aleph_0$-fg, and every finitely generated module is $\aleph$-fg for all $\aleph \geq \aleph_0$. If $\aleph > |A|$ and A is $\aleph$-fg, then A is finitely generated. If $0 \to A \to B \to C \to 0$ is an exact sequence with A and C $\aleph$-fg, then B is $\aleph$-fg. If $f : A \to B$ is an epimorphism with A $\aleph$-fg, then so is B. The following result appeared in [4, Lemma 1.2].

Lemma 2.1 *Let $\aleph$ be an infinite cardinal number and A a right R-module. Then the following statements are equivalent:*

(1) A is $\aleph$-fg.

(2) If $\{L_i \mid i \in I\}$ is any family of left R-modules and if $\alpha_A : A \otimes_R \prod_{i\in I}^{\aleph} L_i \longrightarrow \prod_{i\in I}^{\aleph}(A \otimes_R L_i)$ is defined via $\alpha_A[a \otimes (x_i)_{i\in I}] = (a \otimes x_i)_{i\in I}$, then α_A is an epimorphism.

(3) If I is any index set and if $\alpha_A : A \otimes_R \prod_{i\in I}^{\aleph} R \longrightarrow \prod_{i\in I}^{\aleph} A$ is defined via $\alpha_A[a \otimes (r_i)_{i\in I}] = (ar_i)_{i\in I}$, then α_A is an epimorphism.

Let $\aleph$ be an infinite cardinal number and A a finitely generated right R-module. Following Loustaunau[4], A is said to be $\aleph$-finitely presented, denoted by $\aleph$-fp, if there exists an exact sequence $0 \to K \to F \to A \to 0$ with F free of $\aleph$-fg and K $\aleph$-fg.

For example, every finitely generated right R-module is $\aleph_0$-fp, and every finitely presented right R-module is $\aleph$-fp. The following result appeared in [4, Lemma 1.4].

Lemma 2.2 *Let $\aleph$ be an infinite cardinal number and A a finitely generated right R-module. Then the following statements are equivalent:*

(1) A is $\aleph$-fp.

(2) α_A in Lemma 2.1 (2) is an isomorphism.

(3) α_A in Lemma 2.1 (3) is an isomorphism.

Lemma 2.3 *Let C be a finitely generated right R-module. Then C is $\aleph$-fp if and only if for every exact sequence $0 \to A \to B \to C \to 0$ in mod-R, if B is $\aleph$-fg, then A is $\aleph$-fg.*

Proof $\Rightarrow$) Since C is $\aleph$-fp, there exists an exact sequence $0 \to K \to F \to C \to 0$ with F free of $\aleph$-fg and K $\aleph$-fg. We get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\mu} & F & \xrightarrow{\nu} & C & \longrightarrow & 0 \\
 & & \downarrow \kappa & & \downarrow \lambda & & \| & & \\
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0
\end{array}$$

where λ is obtained by mapping the basis element $e_i \in F$ to an element $x_i \in B$ such that $\beta(x_i) = \nu(e_i)$. Then $\beta\lambda\mu = \nu\mu = 0$, so $\mathrm{Im}\lambda\mu \subseteq \mathrm{Ker}\beta = \mathrm{Im}\alpha$, and therefore $\lambda\mu$ can be factored over α by means of a homomorphism $\kappa : K \to A$. Thus by the Kernel Cokernel Lemma, we have $A/\mathrm{Im}\kappa \cong B/\mathrm{Im}\lambda$.

The module $B/\mathrm{Im}\lambda$ is $\aleph$-fg since B is $\aleph$-fg. So is then also the module $A/\mathrm{Im}\kappa$. Also $\mathrm{Im}\kappa$ is $\aleph$-fg since K is $\aleph$-fg. Then, by the exact sequence $0 \to \mathrm{Im}\kappa \to A \to A/\mathrm{Im}\kappa \to 0$, we have A is $\aleph$-fg.

$\Leftarrow$) Since C is finitely generated, there exists an exact sequence $0 \to \mathrm{Ker}\alpha \to F \xrightarrow{\alpha} C \to 0$ with F free and finitely generated. Then $\mathrm{Ker}\alpha$ is $\aleph$-fg. Therefore, C is $\aleph$-fp.

Lemma 2.4 *For any exact sequence $0 \to A \to B \to C \to 0$ of right R-modules, the following hold:*

(1) Let B, C be finitely generated. If A is $\aleph$-fg and B is $\aleph$-fp, then C is $\aleph$-fp.

(2) Let A, B, C be finitely generated. If A, C is $\aleph$-fp, then B is $\aleph$-fp.

Proof By analogy with the proof of [6, 25.1], the results follow from Lemma 2.3.

3. Main results

Throughout, M will be an arbitrary but fixed right R-module. For $K \leq N \in \text{Mod-}R$ and $x \in N$, set $x^{\perp} = \{r \in R \mid xr = 0\}$ and let $x^{-1}K = (x+K)^{\perp} = \{r \in R \mid xr \in K\}$. Following [6], a left R-module F is M-flat if for any submodule $0 \leq K < M$, the sequence $0 \to K \otimes_R F \to M \otimes_R F$ is monic. Following [3], a left R-module F is called $\sigma[M]$-flat if for any monomorphism $0 \to X \to Y$ in $\sigma[M]$ (with $X, Y \in \sigma[M]$), $0 \to X \otimes_R F \to Y \otimes_R F$ is exact, where $\sigma[M]$ is the full subcategory of Mod-R subgenerated by the given right R-module M. It is also shown in [3] that $\sigma[M]$-flat is equivalent to the simpler definition M-flat as the above. The following is a characterization of the property that a left R-module F is M-flat which was given in [3].

Lemma 3.1 *For a right R-module M and a left R-module F, the following are equivalent:*

(1) F is M-flat.

(2) For any finitely generated $L/m^{\perp}$, where $m \in M$ and $m^{\perp} \leq L \leq R$, $0 \to L/m^{\perp} \otimes_R F \to R/m^{\perp} \otimes_R F$ is exact.

Following [4], a right R-module N is called $\aleph$-coherent if N is finitely generated and every finitely generated submodule of N is $\aleph$-fp.

Following [3], a right R-module N is called M-coherent if for any $0 \leq A < B \leq N$ such that $B/A \hookrightarrow mR$ for some $m \in M$, if B/A is finitely generated, then B/A is finitely presented. A ring R is called right $\aleph$-coherent (resp. M-coherent) if R_R is $\aleph$-coherent (resp. M-coherent).

Definition 3.2 *A right R-module N is called $\aleph$-M-coherent if for any $0 \leq A < B \leq N$ such that $B/A \hookrightarrow mR$ for some $m \in M$, if B/A is finitely generated, then B/A is $\aleph$-fp.*

Define the ring R to be $\aleph$-M-coherent if R_R is $\aleph$-M-coherent. An equivalent definition is that R is $\aleph$-M-coherent if for any $m \in M$ and $m^{\perp} \subseteq L \leq R$ such that $L/m^{\perp}$ is finitely generated, it is also $\aleph$-fp.

Remarks (1) Every ring is $\aleph_0$-M-coherent.

(2) If a ring is M-coherent, then it is $\aleph$-M-coherent for all infinite cardinal number $\aleph$.

(3) If $\aleph \leq \aleph'$, then $\aleph'$-M-coherent implies $\aleph$-M-coherent.

Proposition 3.3 *For any right R-module M, R is $\aleph$-M-coherent if and only if for any $m \in M$, mR is an $\aleph$-coherent right R-module.*

Proof Every submodule of $mR \cong R/m^\perp$ is isomorphic to $L/m^\perp$ for some $m^\perp \leq L \leq R$.

Thus R is $\aleph$-M-coherent immediately translates into the logically equivalent statement that mR is $\aleph$-coherent.

The following proposition clarifies the connection between $\aleph$-coherent and $\aleph$-M-coherent rings.

Proposition 3.4 *(1) Let R be an right $\aleph$-coherent ring. If for every $m \in M$, $m^\perp$ is finitely generated, then R is $\aleph$-M-coherent.*

(2) Let R be an $\aleph$-M-coherent ring. If there exists an $m_0 \in M$ such that $m_0^\perp = 0$, then R is right $\aleph$-coherent.

(3) $\aleph$-R-coherent implies right $\aleph$-coherent. The converse holds when $\aleph > |R|$.

(4) If for any $m \in M$, $m^\perp$ is finitely generated and $\aleph$-fp, and for any finitely generated $L < R$, there exists $m \in M$ with $m^\perp \subseteq L$, then R is $\aleph$-M-coherent if and only if R is right $\aleph$-coherent.

Proof (1) Suppose that $m \in M$ with $m^\perp \subseteq L \leq R$ such that $L/m^\perp$ is finitely generated. Since $m^\perp$ is finitely generated, then from the exact sequence $0 \to m^\perp \to L \to L/m^\perp \to 0$ we can get L is finitely generated. Thus L is $\aleph$-fp since R is a right $\aleph$-coherent ring. Then by Lemma 2.4 (1), $L/m^\perp$ is $\aleph$-fp. Therefore, R is $\aleph$-M-coherent.

Conclusion (2) follows immediately.

(3) Let R be right $\aleph$-coherent. Suppose that $a \in R$ with $a^\perp \subseteq L \leq R$ such that $L/a^\perp$ is finitely generated. Since R is right $\aleph$-coherent, $a^\perp$ is $\aleph$-fg by [4, Theorem 1.7]. Thus $a^\perp$ is finitely generated since $|a^\perp| \leq |R| < \aleph$. Thus by the exact sequence $0 \to a^\perp \to L \to L/a^\perp \to 0$, L is finitely generated. Thus L is $\aleph$-fp since R is right $\aleph$-coherent. Thus $L/a^\perp$ is $\aleph$-fp by Lemma 2.4 (1). The other direction follows immediately by (2).

(4) Suppose that R is $\aleph$-M-coherent. Let $L < R$ be finitely generated. Then there exists $m \in M$ such that $m^\perp \subseteq L$ with $m^\perp$ finitely generated and $\aleph$-fp. Then $L/m^\perp$ is finitely generated. Thus $L/m^\perp$ is $\aleph$-fp. Then by the exact sequence $0 \to m^\perp \to L \to L/m^\perp \to 0$ and Lemma 2.4 (2), we can get L is $\aleph$-fp.

The other direction follows from (1).

Dauns's Theorem 2.6 in [3] is the special case $\aleph > |R|$ of the following much more general Theorem.

Theorem 3.5 *Let M be a right R-module, assume that all $m^\perp$ for $m \in M$ are $\aleph$-fg. Then the following are equivalent:*

(1) The $\aleph$-product of any family of M-flat left R-module is M-flat.

(2) The $\aleph$-product of any family of copies of R is M-flat as a left R-module.

(3) R is $\aleph$-M-coherent.

Proof (1) $\Rightarrow$ (2) Since R as a left R-module is flat, R is trivially M-flat for all modules M, and now $\prod_{i\in I}^{\aleph} R$ is M-flat for any index set I by (1).

(2) $\Rightarrow$ (3) Let $m \in M$, $m^{\perp} \subseteq L \leq R$ such that $L/m^{\perp}$ is finitely generated. By (2) and Lemma 3.1, we have the following commutative diagram for any index set I:

$$\begin{array}{ccccc} 0 & \longrightarrow & L/m^{\perp} \otimes_R \prod_{i\in I}^{\aleph} R & \longrightarrow & R/m^{\perp} \otimes_R \prod_{i\in I}^{\aleph} R \\ & & \Big\downarrow \alpha_{L/m^{\perp}} & & \Big\downarrow \alpha_{R/m^{\perp}} \\ 0 & \longrightarrow & \prod_{i\in I}^{\aleph} L/m^{\perp} & \longrightarrow & \prod_{i\in I}^{\aleph} R/m^{\perp} \end{array},$$

where $\alpha_{L/m^{\perp}}$ and $\alpha_{R/m^{\perp}}$ are as in Lemma 2.1. Since $m^{\perp}$ is $\aleph$-fg, by the exact sequence $0 \longrightarrow m^{\perp} \longrightarrow R \longrightarrow R/m^{\perp} \longrightarrow 0$ and Lemma 2.4 (1), we have $R/m^{\perp}$ is $\aleph$-fp. Thus $\alpha_{R/m^{\perp}}$ is monic by Lemma 2.2. Hence $\alpha_{L/m^{\perp}}$ is monic. Since $L/m^{\perp}$ is finitely generated, $\alpha_{L/m^{\perp}}$ is epic by Lemma 2.1. Hence $\alpha_{L/m^{\perp}}$ is an isomorphism. Thus $L/m^{\perp}$is $\aleph$-fp by Lemma 2.2.

(3) $\Rightarrow$ (1) Let $\{L_i \mid i \in I\}$ be a family of M-flat left R-modules. Let $m \in M$, $m^{\perp} \subseteq L \leq R$ such that $L/m^{\perp}$ is finitely generated. Then by (3) and Lemma 2.2, the map $\alpha_{L/m^{\perp}} : L/m^{\perp} \otimes_R \prod_{i\in I}^{\aleph} L_i \rightarrow \prod_{i\in I}^{\aleph}(L/m^{\perp} \otimes_R L_i)$ is monic. Also, for each $i \in I$, L_i is M-flat, so that the map $L/m^{\perp} \otimes_R L_i \rightarrow R/m^{\perp} \otimes_R L_i$ is monoic. Thus from the following commutative diagram:

$$\begin{array}{ccccc} & & L/m^{\perp} \otimes_R \prod_{i\in I}^{\aleph} L_i & \xrightarrow{f} & R/m^{\perp} \otimes_R \prod_{i\in I}^{\aleph} L_i \\ & & \Big\downarrow \alpha_{L/m^{\perp}} & & \Big\downarrow \alpha_{R/m^{\perp}} \\ 0 & \longrightarrow & \prod_{i\in I}^{\aleph}(L/m^{\perp} \otimes_R L_i) & \xrightarrow{g} & \prod_{i\in I}^{\aleph}(R/m^{\perp} \otimes_R L_i) \end{array}$$

we can get f is monic. Thus $\prod_{i\in I}^{\aleph} L_i$ is M-flat by Lemma 3.1.

Proposition 3.6 *For an R-module M, assume that all $m^{\perp}$ are finitely generated for $m \in M$, and that there exists an element $0 \neq m_0 \in M$ such that $m_0^{\perp} \subseteq m^{\perp}$ for all $m \in M$. If $R/m_0^{\perp}$ is an $\aleph$-coherent right R-module, then R is a $\aleph$-M-coherent ring.*

Proof Let $m \in M$, with $m^{\perp} \subseteq L \leq R$ such that $L/m^{\perp}$ is finitely generated. Since $m^{\perp}$ is finitely generated, from the exact sequence $0 \longrightarrow m^{\perp} \longrightarrow L \longrightarrow L/m^{\perp} \longrightarrow 0$ it follows that L, and hence $L/m_0^{\perp}$ are finitely generated. Hence $L/m_0^{\perp}$ is $\aleph$-fp since $R/m_0^{\perp}$ is an $\aleph$-coherent right R-module. But then Lemma 2.4 (1) and the exact sequence $0 \longrightarrow m^{\perp}/m_0^{\perp} \longrightarrow L/m_0^{\perp} \longrightarrow L/m^{\perp} \longrightarrow 0$ show that $L/m^{\perp}$ is $\aleph$-fp.

Hence R is $\aleph$-M-coherent.

Now we give some characterizations of $\aleph$-M-coherent modules and obtain a generalization of Dauns's Theorem 2.10 in [3].

Theorem 3.7 *For a ring R and right R-modules M and N, the following are equivalent:*

(1) N is $\aleph$-M-coherent.

(2) For any $m \in M$, $x \in N$, $0 \leq A \leq B \leq N$ such that $(B + xR)/A \hookrightarrow mR$ and such that B/A is finitely generated, it follows that $x^{-1}B$ is $\aleph$-fg.

(3) For any $m \in M$, $x \in N$, and $0 \leq A \leq N$ such that $(A + xR)/A \hookrightarrow mR$, it follows

that $x^{-1}A$ is $\aleph$-fg. And for any $m \in M$, $0 \le A < B \le N$, and $0 \le A < C \le N$ such that $B/A \hookrightarrow mR$, $C/A \hookrightarrow mR$, and also B/A, C/A are finitely generated, it follows that $(B \cap C)/A$ is $\aleph$-fg.

Proof (1) $\Rightarrow$ (2) Let $m \in M$, $x \in N$, $0 \le A \le B \le N$, $(B + xR)/A \hookrightarrow mR$ and B/A is finitely generated. Then $(B + xR)/A$ is finitely generated. Hence, by (1), $(B + xR)/A$ is $\aleph$-fp. Then by the exact sequence $0 \longrightarrow B/A \longrightarrow (B + xR)/A \longrightarrow R/x^{-1}B \longrightarrow 0$ and Lemma 2.4 (1), $R/x^{-1}B$ is $\aleph$-fp. Therefore by the exact sequence $0 \longrightarrow x^{-1}B \longrightarrow R \longrightarrow R/x^{-1}B \longrightarrow 0$ and Lemma 2.3, $x^{-1}B$ is $\aleph$-fg.

(2) $\Rightarrow$ (1) Let $0 \le A < B \le N$, $m \in M$, $B/A \hookrightarrow mR$, and also B/A is finitely generated, we will show that B/A is $\aleph$-fp. Suppose that B/A is n-generated, we use induction on n to show that B/A is $\aleph$-fp. Let $n = 1$. Then there exists $x \in N$ such that $B/A \simeq (A + xR)/A \simeq R/x^{-1}A$. By (2) and the exact sequence $0 \longrightarrow x^{-1}A \longrightarrow R \longrightarrow R/x^{-1}A \longrightarrow 0$ it follows that $R/x^{-1}A$ is $\aleph$-fp. Thus B/A is $\aleph$-fp. Let $n > 1$ and assume that every $n - 1$ generated submodule of mR is $\aleph$-fp. Since B/A is n-generated, there exist $A \le C \le N$ and $x \in N$ such that $B/A \simeq (C + xR)/A$ with C/A being $n - 1$ generated. By induction, C/A is $\aleph$-fp, while by hypothesis (2), $R/x^{-1}C$ is $\aleph$-fp. Now, by the exact sequence $0 \longrightarrow C/A \longrightarrow (C + xR)/A \longrightarrow R/x^{-1}C \longrightarrow 0$ and Lemma 2.4 (2), $(C + xR)/A$ is $\aleph$-fp.

(1) $\Rightarrow$ (3) Let $m \in M$, $x \in N$, $0 \le A \le N$ such that $(A + xR)/A \hookrightarrow mR$. By (1), $(A + xR)/A$ is $\aleph$-fp. Hence, $x^{-1}A$ is $\aleph$-fg. Let $0 \le A < B \le N$, $0 \le A < C \le N$, $m \in M$ such that $B/A \hookrightarrow mR$, $C/A \hookrightarrow mR$ and also B/A, C/A are finitely generated. Now, considering the exact sequence

$$0 \to (B \cap C)/A \xrightarrow{f} B/A \oplus C/A \xrightarrow{g} (B + C)/A \to 0$$

where $f : \overline{d} \mapsto (\bar{d}, -\bar{d})$, $g : (\bar{b}, \bar{c}) \mapsto \bar{b} + \bar{c}$. Since B/A, C/A are finitely generated, $B/A \oplus C/A$ is finitely generated. Thus $(B + C)/A$ is finitely generated. By (1), $(B + C)/A$ is $\aleph$-fp. Thus $(B \cap C)/A$ is $\aleph$-fg by Lemma 2.3.

(3) $\Rightarrow$ (1) Let $0 \le A < B \le N$, $m \in M$, $B/A \hookrightarrow mR$, with B/A finitely generated. We will show that B/A is $\aleph$-fp. Assume that B/A is n-generated. We use induction on n to show that B/A is $\aleph$-fp. Let $n = 1$, then there exists $x \in N$ such that $B/A \simeq (A + xR)/A \simeq R/x^{-1}A$. By (3), $R/x^{-1}A$ is $\aleph$-fp. Let $n > 1$ and assume that every $n - 1$ generated submodule of mR is $\aleph$-fp. Since B/A is n-generated, there exist $A < C \le N$ and $x \in N$ such that $B/A \simeq (C + xR)/A$ with C/A being $n - 1$ generated. By induction, C/A is $\aleph$-fp. By (3), $(C \cap (A + xR))/A$ is $\aleph$-fg. Then by the exact sequence

$$0 \longrightarrow (C \cap (A + xR))/A \longrightarrow C/A \oplus (A + xR)/A \longrightarrow (C + xR)/A \longrightarrow 0$$

and Lemma 2.4 (1), $(C + xR)/A$ is $\aleph$-fp. Thus B/A is $\aleph$-fp. Hence (1) holds.

Corollary 3.8 *For a right R-module M and a ring R, the following are equivalent:*

(1) R is $\aleph$-M-coherent.

(2) For any $m \in M$, $A \le R$ with $m^{\perp} \le A$, and any $b \in R$, if $A/m^{\perp}$ is finitely generated, then $b^{-1}A$ is $\aleph$-fg.

(3) For any $m \in M$ *and* $b \in R$, $b^{-1}m^{\perp}$ *is* $\aleph$*-fg; and for any* $m^{\perp} \leq A \leq R$, $m^{\perp} \leq B \leq R$ *such that* $A/m^{\perp}$ *and* $B/m^{\perp}$ *are finitely generated,* $(A \cap B)/m^{\perp}$ *is* $\aleph$*-fg.*

References

[1] DAUNS J. *Uniform dimensions and subdirect products* [J]. Pacific J. Math., 1987, **126**(1): 1–19.
[2] DAUNS J. *Subdirect products of injectives* [J]. Comm. Algebra, 1989, **17**(1): 179–196.
[3] DAUNS J. *Relative flatness and coherence* [J]. Comm. Algebra, 2006, **34**(1): 303–312.
[4] LOUSTAUNAU P. *Large subdirect products of projective modules* [J]. Comm. Algebra, 1989, **17**(1): 197–215.
[5] OYONARTE L, TORRECILLAS B. *Large subdirect products of flat modules* [J]. Comm. Algebra, 1996, **24**(4): 1389–1407.
[6] WISBAUER R. *Foundations of Module and Ring Theory* [M]. Philadelphia: Gordon and Breach Science Publishers, 1991.