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Schur Formal Ordering Inequalities Involving Parameter

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Abstract In this paper, to make an analogy to the classical Schur inequalities, we establish several ordering inequalities of Schur type with a parameter. As applications, some generalizations of Schur type with parameter are obtained.

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1. Introduction

Let $x, y, z \in R^+, \alpha \in R$. Then

$$\sum x^\alpha (x-y)(x-z) \geq 0. \tag{1}$$

This is the famous Schur inequality. The ternary fundamental Schur forms were established in [1, 2] to generalize the Schur inequality. We denote ternary fundamental symmetric forms as

$$\sigma_1 = x_1 + x_2 + x_3, \sigma_2 = x_1x_2 + x_2x_3 + x_3x_1, \sigma_3 = x_1x_2x_3.$$

Then ternary fundamental Schur forms (which are formally different from the original forms in [1, 2]) are

$$\begin{aligned}
f_{0,1}^{(m)} &= \sum x_1^{m-2}(x_1-x_2)(x_1-x_3) \quad (m \geq 2),\\
f_{0,2}^{(m)} &= \sum x_1^{m-3}(x_2+x_3)(x_1-x_2)(x_1-x_3) \quad (m \geq 3),\\
f_{0,j}^{(m)} &= \sigma_1^{m-2j}\sigma_2^{j-3}\sum x_1^2(x_2-x_3)^2(x_1-x_2)(x_1-x_3) \quad (3 \leq j \leq [\frac{m}{2}]),\\
f_{0,[\frac{m+2}{2}]}^{(m)} &= \sum (x_2x_3)^{\frac{2m-5+(-1)^m}{4}}(x_2+x_3)^{\frac{1-(-1)^m}{2}}(x_1-x_2)(x_1-x_3) \quad (m \geq 4),\\
f_{i,k}^{(m)} &= \sigma_3^i f_{0,k}^{(m-3i)} \quad (5 \leq 3i+2 \leq m, 1 \leq k \leq [\frac{m-3i+2}{2}]),
\end{aligned}$$

where $\sum$ denotes the cyclic sum for the variables (x_1, x_2, x_3).

There are two fundamental lemmas which are already proved in [1, 2]:

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Lemma 1[1,2] *Let $x_1, x_2, x_3 \geq 0$. Then*

$$f_{i,j}^{(n)} \geq 0. \tag{2}$$

Lemma 2[1,2] *Let $f(x_1, x_2, x_3)$ be a ternary symmetric form with degree n satisfying $f(1,1,1) = 0$. Then f can be uniquely written in the form of linear combination of $\{f_{i,j}^{(n)}\}$.*

A lot of difficult inequalities have been proved in [1, 2] by utilizing these two lemmas. We will extend the fundamental Schur formal inequalities farther in order to prove the inequalities with broader types.

2. Main results

Considering the inequality involving parameter u:

$$y_1(x_1 - ux_2)(x_1 - ux_3) + y_2(x_2 - ux_1)(x_2 - ux_3) + y_3(x_3 - ux_1)(x_3 - ux_2) \geq 0. \tag{3}$$

Let $a_k,\ y_k\ , x_k \in R^+$, $k = 1, 2, 3$. Then we have the following

Theorem 1 *Let $u \leq 1$. If the sequence $Y = [y_1, y_2, y_3]$ and the sequence $X = [x_1, x_2, x_3]$ are similarly ordered, namely, $y_i \geq y_j \geq y_k$, $x_i \geq x_j \geq x_k$ (where (i, j, k) is a permutation of $(1, 2, 3)$), then*

$$\begin{aligned} S_1(Y, X, u) =& y_1(x_1 - ux_2)(x_1 - ux_3) + y_2(x_2 - ux_3)(x_2 - ux_1) + \\ & y_3(x_3 - ux_1)(x_3 - ux_2) \geq 0, \end{aligned} \tag{4}$$

$$\begin{aligned} S_2(Y, X, u) =& \frac{y_1(x_1 - ux_2)(x_1 - ux_3)}{x_1} + \frac{y_2(x_2 - ux_3)(x_2 - ux_1)}{x_2} + \\ & \frac{y_3(x_3 - ux_1)(x_3 - ux_2)}{x_3} \geq 0, \end{aligned} \tag{5}$$

$$\begin{aligned} S_3(Y, X, u) =& y_1(x_2 + x_3)(x_1 - ux_2)(x_1 - ux_3) + y_2(x_1 + x_3)(x_2 - ux_3) \times \\ & (x_2 - ux_1) + y_3(x_1 + x_2)(x_3 - ux_1)(x_3 - ux_2) \geq 0, \end{aligned} \tag{6}$$

$$\begin{aligned} S_4(Y, X, u) =& \frac{y_1(x_1 - ux_2)(x_1 - ux_3)}{x_2 + x_3} + \frac{y_2(x_2 - ux_3)(x_2 - ux_1)}{x_1 + x_3} + \\ & \frac{y_3(x_3 - ux_1)(x_3 - ux_2)}{x_1 + x_2} \geq 0, \end{aligned} \tag{7}$$

where equalities hold when $u = 1, x_1 = x_2 = x_3$.

Proof Suppose that $y_1 \geq y_2 \geq y_3$, $x_1 \geq x_2 \geq x_3$ (otherwise we can do a repermutation to the sequences Y and X) and let

$$y_1 = a + b + c,\ \ y_2 = a + b, y_3 = a,\ \ x_1 = x + y + z,\ \ x_2 = x + y,\ \ x_3 = x,\ \ u = 1 - v$$

$(a, b, c, x, y, z, v \geq 0)$.

Thus $S_1(Y, X, u)$ transforms into a polynomial in a, b, c

$$\begin{aligned} S_1(Y, X, u) =& [z^2 + y^2 + zy + (3x^2 + 4yx + 2xz + y^2 + zy)v^2]a + [x(2y + 2x + z)v^2 + \\ & (xz + 2zy + 2y^2 + 2yx)v + z^2]b + [x(x + y)v^2 + z(y + z) + \\ & (2xz + yx + zy + y^2)v]c \geq 0. \end{aligned}$$

Removing the denominators, $S_2(Y,X,u)$ is also transformed into a polynomial in a,b,c, denoted as $\tilde{S}_2(Y,X,u)$:

$$\begin{aligned}\tilde{S}_2(Y,X,u) =&[8v^2x^3y+(6v^2-6v+3)zy^2x+3v^2x^4+(v-1)^2z^2y^2+4v^2x^3z+\\&(8v^2-2v+1)y^2x^2+(2v^2-2v+1)z^2x^2+2(v-1)^2y^3z+(v-1)^2y^4+\\&(2v^2-2v+1)z^2xy+(4v^2-4v+2)y^3x+(8v^2-2v+1)zyx^2]a+\\&[2v^2x^4+3xy^2zv+xz^2yv+2v(2+v)y^2x^2+v(1+2v)zx^3+2v(2+v)\times\\&zyx^2+2xy^3v+(v^2-v+1)z^2x^2+2v(1+2v)x^3y)]b+[v(2+v)y^2x^2+\\&2vx^3z+v(1+2v)x^3y+vxy^3+xz^2y+z^2x^2+(v+1)zy^2x+v^2x^4+\\&(1+3v)zyx^2]c.\end{aligned}$$

And $S_3(Y,X,u)$ is transformed into a polynomial in a,b,c.

$$\begin{aligned}S_3(Y,X,u) =&[12v^2x^2y+(3v^2-3v+3)y^2z+(2v^2-2v+2)z^2x+(v^2-v+1)z^2y+\\&(8v^2-2v+2)zyx+6v^2x^2z+(2v^2-2v+2)y^3+(8v^2-2v+2)xy^2+\\&6v^2x^3]a+[2v(3v+2)x^2y+2v(3+v)zyx+vz^2y+3vy^2z+2v(3+\\&v)xy^2+4v^2x^3+2vy^3+(v^2-v+2)z^2x+v(3v+2)x^2z]b+[v(3+\\&v)xy^2+(v+1)y^2z+2v^2x^3+2z^2x+4vzx^2+(4v+2)zyx+yz^2+\\&vy^3+v(3v+2)x^2y]c.\end{aligned}$$

The above expressions in every $'[\,]'$ can be regarded as the polynomials in x,y,z, whose coefficients are expressions in v with degree less than 2. Obviously, these expressions are nonnegative. Then $S_2(Y,X,u)\geq 0$ and $S_3(Y,X,u)\geq 0$.

4) It is clear that after removing the denominators, $S_4(Y,X,u)$ transforms into a polynomial in a,b,c and then $S_4(Y,X,u)\geq 0$.

Thus Theorem 1 is verified.

Remark During the proof, we use the order 'expand' and 'collect' in Maplesoft.

Theorem 2 *Let $u\geq 1$. If the sequence $Y=[y_1,y_2,y_3]$ and the sequence $X=[x_1,x_2,x_3]$ are oppositely ordered, namely, $y_i\geq y_j\geq y_k$, $x_i\leq x_j\leq x_k$, where (i,j,k) is a permutation of $(1,2,3)$, then*

$$\begin{aligned}S_1(Y,X,u) =&y_1(x_1-ux_2)(x_1-ux_3)+y_2(x_2-ux_3)(x_2-ux_1)+\\&y_3(x_3-ux_1)(x_3-ux_2)\geq 0,\end{aligned}\tag{8}$$

$$\begin{aligned}S_2(Y,X,u) =&\frac{y_1(x_1-ux_2)(x_1-ux_3)}{x_1}+\frac{y_2(x_2-ux_3)(x_2-ux_1)}{x_2}+\\&\frac{y_3(x_3-ux_1)(x_3-ux_2)}{x_3}\geq 0,\end{aligned}\tag{9}$$

$$\begin{aligned}S_3(Y,X,u) =&y_1(x_2+x_3)(x_1-ux_2)(x_1-ux_3)+y_2(x_1+x_3)(x_2-ux_3)\times\\&(x_2-ux_1)+y_3(x_1+x_2)(x_3-ux_1)(x_3-ux_2)\geq 0,\end{aligned}\tag{10}$$

$$\begin{aligned} S_4(Y,X,u) =& \frac{y_1(x_1-ux_2)(x_1-ux_3)}{x_2+x_3} + \frac{y_2(x_2-ux_3)(x_2-ux_1)}{x_1+x_3} + \\ & \frac{y_3(x_3-ux_1)(x_3-ux_2)}{x_1+x_2} \geq 0, \end{aligned} \tag{11}$$

where equalities hold when $u = 1, x_1 = x_2 = x_3$.

Proof Suppose that $y_1 \geq y_2 \geq y_3$, $x_1 \leq x_2 \leq x_3$ (otherwise the orders of sequences Y, X can be repermuted). Let

$$y_1 = a+b+c,\ y_2 = a+b, y_3 = a,\ x_3 = x+y+z,\ x_2 = x+y,\ x_1 = x,\ u = 1+v$$

$(a, b, c, x, y, z, v \geq 0)$.

And by removing the denominators, $S_1(Y,X,u), S_2(Y,X,u)$ and $S_3(Y,X,u)$ transform into polynomials in a, b, c. Analogously, we can easily have $S_1(Y,X,u) \geq 0$, $S_2(Y,X,u) \geq 0$, $S_3(Y,X,u) \geq 0$. The most difficult part is to prove $S_4(Y,X,u) \geq 0$. By reducing the fractions to a common denominator and removing the denominators in $S_4(Y,X,u)$, we get a new expression denoted as $\tilde{S}_4(Y,X,u)$:

$$\begin{aligned} \tilde{S}_4(Y,X,u) =& [(29v^2-4v+4)zx^2y + (v-1)^2y^4 + (15v^2-9v+6)zxy^2 + 32v^2x^3y + \\ & 16v^2x^3z + (29v^2-4v+4)y^2x^2 + 2(v-1)^2zy^3 + (5v^2-7v+10)z^2xy + \\ & 12v^2x^4 + (10v^2-6v+4)xy^3 + z^2((z-vy/2-xv)^2 + 5(y-3vy/10)^2 + \\ & (2x-xv)^2 + 30(vy/10-vx/6)^2 + 13v^2x^2/6 + 4zx + 4zy)]a + (y+ \\ & 2x)\cdot[4v^2x^3 + (1+v^2)y^3 + v(2v+1)y^2z + v(2+9v)x^2y + 2v(v+1)xz^2 + \\ & (6v^2+2v+2)y^2x + 2v(2+3v)x^2z + 2v(4v+3)xyz + v(v+1)z^2y]b + \\ & [(y+2x)(2x+y+z)(y+vy+xv)(xv+vy+vz+y+z)]c. \end{aligned}$$

It is clear that $S_4(Y,X,u) \geq 0$.

Thus Theorem 2 is verified.

It is obvious that Theorems 1 and 2 are the generalizations of Schur inequality (1). We call them Schur formal ordering inequalities involving parameter.

When the sequence $Y = [y_1, y_2, y_3]$ and the sequence $X = [x_1, x_2, x_3]$ are similarly ordered, the sequence $A = [y_2+y_3, y_3+y_1, y_1+y_2]$ and the sequence $X = [x_1, x_2, x_3]$ are oppositely ordered. When the sequence $Y = [y_1, y_2, y_3]$ and the sequence $X = [x_1, x_2, x_3]$ are oppositely ordered, the sequence $A = [y_2+y_3, y_3+y_1, y_1+y_2]$ and the sequence $X = [x_1, x_2, x_3]$ are similarly ordered. Then we have

Corollary *If $u \leq 1$, and the sequence $Y = [y_1, y_2, y_3]$ and the sequence $X = [x_1, x_2, x_3]$ have the reverse order, or if $u \geq 1$, and the sequence $Y = [y_1, y_2, y_3]$ and the sequence $X = [x_1, x_2, x_3]$ are similarly ordered, then*

$$\begin{aligned} C_1(Y,X,u) =& (y_2+y_3)(x_1-ux_2)(x_1-ux_3) + (y_1+y_3)(x_2-ux_3)(x_2-ux_1) + \\ & (y_1+y_2)(x_3-ux_1)(x_3-ux_2) \geq 0, \end{aligned} \tag{12}$$

$$C_2(Y,X,u) = \frac{(y_2+y_3)(x_1-ux_2)(x_1-ux_3)}{x_1} + \frac{(y_1+y_3)(x_2-ux_3)(x_2-ux_1)}{x_2} + \frac{(y_1+y_2)(x_3-ux_1)(x_3-ux_2)}{x_3} \geq 0, \tag{13}$$

$$C_3(Y,X,u) = (y_2+y_3)(x_2+x_3)(x_1-ux_2)(x_1-ux_3) + (y_1+y_3)(x_1+x_3)(x_2-ux_3)\cdot(x_2-ux_1) + (y_1+y_2)(x_1+x_2)(x_3-ux_1)(x_3-ux_2) \geq 0, \tag{14}$$

$$C_4(Y,X,u) = \frac{(y_2+y_3)(x_1-ux_2)(x_1-ux_3)}{x_2+x_3} + \frac{(y_1+y_3)(x_2-ux_3)(x_2-ux_1)}{x_1+x_3} + \frac{(y_1+y_2)(x_3-ux_1)(x_3-ux_2)}{x_1+x_2} \geq 0, \tag{15}$$

where equalities hold when $u=1, x_1=x_2=x_3$.

Theorem 3[3] *Let* $u=1$. *If* $x_1 \geq x_2 \geq x_3$, *and* $\sqrt{y_1}+\sqrt{y_3} \geq \sqrt{y_2}$, *then*

$$S_1(Y,X,1) = y_1(x_1-x_2)(x_1-x_3) + y_2(x_2-x_3)(x_2-x_1) + y_3(x_3-x_1)(x_3-x_2) \geq 0, \tag{16}$$

$$S_2(Y,X,1) = \frac{y_1(x_1-x_2)(x_1-x_3)}{x_1} + \frac{y_2(x_2-x_3)(x_2-x_1)}{x_2} \; \frac{y_3(x_3-x_2)(x_3-x_1)}{x_3} \geq 0, \tag{17}$$

$$S_4(Y,X,1) = \frac{y_1(x_1-x_2)(x_1-x_3)}{x_2+x_3} + \frac{y_2(x_2-x_3)(x_2-x_1)}{x_1+x_3} + \frac{y_3(x_3-x_1)(x_3-x_2)}{x_1+x_2} \geq 0, \tag{18}$$

where equalities hold when $x_1=x_2=x_3$.

Proof Let

$$y_1=a^2, y_3=c^2, y_2=(a+c)^2-b, x_1=x+y+z, x_2=x+y, x_3=x, (a,b,c,x,y,z \geq 0).$$

Substituting $y_1, y_3, y_2, x_1, x_2, x_3$ into $S_1(Y,X,1)$, $S_2(Y,X,1)$, $S_4(Y,X,1)$ gives

$$S_1(Y,X,1) = (az-cy)^2 + byz \geq 0;$$

$$S_2(Y,X,1) = \frac{(axz-y^2c-xyc-ycz)^2 + (zyx^2+(yz^2+zy^2)x)b}{x_1x_2x_3} \geq 0;$$

$$S_4(Y,X,1) = \frac{[z(z+2x+2y)a-y(y+2x)c]^2 + zy(y+2x)(z+2x+2y)b}{(x_1+x_2)(x_2+x_3)(x_3+x_1)} \geq 0.$$

The proof is completed.

3. Applications

In this section, we utilize the results in Section 2 to construct a series of beautiful inequalities.

Proposition 1 *If* $u=1$, $k \in R$, *or* $u<1$, $k \geq -1$, *or* $u>1$, $k \leq 0$, *then*

$$\sum x^k(x-uy)(x-uz) \geq 0. \tag{19}$$

The equality holds when $u = 1$, $x = y = z$.

Proof When $u = 1,\ k \in R$, (19) is Schur inequality (1). And when $k \geq 0$, $[x^k, y^k, z^k]$ and $[x, y, z]$ have the same order. Otherwise, when $k \leq 0$, $[x^k, y^k, z^k]$ and $[x, y, z]$ are oppositely ordered. Then according to (4), (8) in Theorems 1 and 2, (19) holds when $u < 1,\ k \geq -1$ or $u > 1, k \leq 0$.

The proofs of the following propositions are all similar to the one of Proposition 1, and will be omitted.

Proposition 2 *If* $u = 1,\ k \in R$, *or* $u < 1,\ k \geq 1$,*or* $u > 1,\ k \leq 0$, *then*

$$\sum y^k z^k (x - uy)(x - uz) \geq 0. \tag{20}$$

Proposition 3 *If* $u = 1,\ k \in R$, *or* $u < 1,\ k \geq 0$, *or* $u > 1,\ k \leq 0$, *then*

$$\sum x^k (y + z)(x - uy)(x - uz) \geq 0. \tag{21}$$

Proposition 4 *If* $u = 1,\ k \in R$, *or* $u < 1,\ k \leq 0$, *or* $u > 1,\ k \geq 0$, *then*

$$\sum y^k z^k (x + y)(x - uy)(x - uz) \geq 0. \tag{22}$$

Proposition 5 *If* $u = 1,\ k \in R$, *or* $u < 1,\ k \leq 0$, *or* $u > 1,\ k \geq 0$, *then*

$$\sum (y^k + z^k)(x - uy)(x - uz) \geq 0. \tag{23}$$

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