Journal of Mathematical Research & Exposition
Nov., 2008, Vol. 28, No. 4, pp. 898–904
DOI:10.3770/j.issn:1000-341X.2008.04.020
Http://jmre.dlut.edu.cn

A Note of Order Congruences on Ordered Semigroups

SHI Xiao Ping[1,2], *XIE Xiang Yun*[3]
(1. Department of Mathematics, Nanjing Normal University, Jiangsu 210046, China;
2. Department of Mathematics, Southeast University, Jiangsu 210096, China;
3. Department of Mathematics and Physics, Wuyi University, Guangdong 529020, China)
(E-mail: xpshinj@gmail.com)

Abstract Which subset of an ordered semigroup S can serve as a congruence class of certain order-congruence on S is an important problem. XIE Xiangyun proved that if every ideal C of an ordered semigroup S is a congruence class of one order-congruence on S, then C is convex and when C is strongly convex, the reverse statement is true in 2001. In this paper, we give an alternative constructing order congruence method, and we prove that every ideal B is a congruence class of one order congruence on S if and only if B is convex. Furthermore, we show that the order relation defined by this method is "the least" order congruence containing B as a congruence class.

Keywords ordered semigroup; order congruence; convex ideal.

Document code A
MR(2000) Subject Classification 06F05; 20M10
Chinese Library Classification O152.7

1. Introduction

A semigroup S is a set together with a binary operation: $S \times S \longrightarrow S$ which satisfies the associative property: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in S$. An ideal of semigroup S is a non-empty subset I of S such that $SI \subseteq I$ and $IS \subseteq I$[2].

A semigroup S that is also a partially ordered set, in which the binary operation and the order relation are compatible, is called an ordered semigroup, that is, semigroup S is called an ordered semigroup if $s, t, x, y \in S$ and $x \leq y, s \leq t$, then $sx \leq ty$. A non-empty subset I of S is called an ordered ideal of an ordered semigroup S if (i) $SI \subseteq I$, $IS \subseteq I$ and (ii) $a \leq b \in I$ implies $a \in I$, for all $a, b \in S$[6].

A nonempty subset C of poset A is called convex if $a \leq b \leq c$ implies $b \in C$ for all $a, c \in C, b \in A$. A nonempty subset B of poset A is called strongly convex if $B = (B] := \{a \in A | (\exists\, b \in B)\, a \leq b\}$. Therefore, I is an ordered ideal of ordered semigroup S if and only if I is an ideal of semigroup S and I is a strongly convex subset of S.

Let S be a semigroup or an ordered semigroup. By a *congruence* we mean an equivalence relation ρ on S which is compatible with the binary operation on S . If S is a semigroup and ρ is

Received date: 2006-08-31; **Accepted date**: 2007-03-23
Foundation item: the National Natural Science Foundation of China (Nos. 10626012; 103410020); the Jiangsu Planned for Postdoctoral Research Found (No. 0502022B); the Natural Science Foundation of Guangdong Province (No. 0501332); the Educational Department Natural Science Foundation of Guangdong Province (No. Z03070).

a congruence on S, then the set $S/\rho := \{[a] | a \in S\}$ ($[a]$ is the ρ-class of S containing a ($a \in S$)) is a semigroup via $[s][a] = [sa]$[2]. Let B be an ideal of semigroup S. The Rees congruence λ_B induced by B is defined as follows:

$$a\lambda_B b \Longleftrightarrow a, b \in B \quad \text{or} \quad a = b.$$

The factor semigroup S/λ_B is called Rees factor semigroup. It is clear that $S/\lambda_B = \{\{x\} | x \in S \backslash B\} \cup B$.

Congruences on ordered semigroups play an important role in studying the structures of ordered semigroups[3−6],[10]. For any congruence σ on an ordered semigroup S, in general, we do not know whether the quotient semigroup S/σ is also an ordered semigroup. Even if S/σ is an ordered semigroup, the order on S/σ is not certainly relative to the order on the original ordered semigroup S. So we need the concept of order congruence and study order congruences on ordered semigroups. Let $(S, \leq)$ be an ordered semigroup. A congruence σ is called an *order congruence* if there exists an order "$\preceq$" on S/σ such that:

(1) $(S/\sigma, \preceq)$ is an ordered semigroup;

(2) The canonical epimorphism $\varphi : S \to S/\sigma, x \mapsto [x]$ is isotone.

2. Main results

Theorem 1 *Let B be an ideal of an ordered semigroup S. Then B is a congruence class of one order congruence on S if and only if B is convex.*

Proof Let B be a congruence class of order congruence ρ, and $a, c \in B$. If $a \leq b \leq c$, by definition of order congruence, then $(a)_\rho \leq (b)_\rho \leq (c)_\rho$. Since $(a)_\rho = B = (c)_\rho$, we have $(a)_\rho = (b)_\rho = (c)_\rho$, that is, $b \in B$. Thus B is convex.

Conversely, let λ_B be the Rees congruence induced by B on S. Then it is clear that B is the congruence class of λ_B. Now we define a relation $\preceq$ on the factor $S/\lambda_B = \{\{x\} | x \in S \backslash B\} \cup B$ as follows:

$$[x] \preceq [y] \Leftrightarrow (x \leq y) \text{ or } (x \leq b, b' \leq y \text{ for some } b, b' \in B).$$

We show that λ_B is an order congruence. First we prove $\preceq$ is an order, that is, $\preceq$ is reflexive, anti-symmetric and transitive.

(1) Since $x \leq x$, we have $[x] \preceq [x]$.

(2) Let $[x] \preceq [y]$ and $[y] \preceq [x]$. Then $x \leq y$ or $x \leq b, b' \leq y$ for some $b, b' \in B$, and $y \leq x$ or $y \leq b_1, b_1' \leq x$ for some $b_1, b_1' \in B$. We consider following four cases:

(α) If $x \leq y$ and $y \leq x$, then $x = y$, therefore $[x] = [y]$.

(β) If $x \leq y$ and $y \leq b_1, b_1' \leq x$ for some $b_1, b_1' \in B$, then $b_1' \leq x \leq y \leq b_1$. Since B is convex and $b_1, b_1' \in B$, we have $x, y \in B$. Hence $[x] = B = [y]$.

(γ) If $x \leq b, b' \leq y$ for some $b, b' \in B$ and $y \leq x$, by similar discussion to (β), we have $[x] = [y]$.

(δ) If $x \leq b, b' \leq y$ for some $b, b' \in B$ and $y \leq b_1, b_1' \leq x$ for some $b_1, b_1' \in B$, then $b_1' \leq x \leq b$ and $b' \leq y \leq b_1$. Since B is convex, we have $x, y \in B$. Thus $[x] = [y]$.

(3) If $[x] \preceq [y]$ and $[y] \preceq [z]$, then $x \leq y$ or $x \leq b, b' \leq y$ for some $b, b' \in B$, and $y \leq z$ or $y \leq b_1, b_1' \leq z$ for some $b_1, b_1' \in B$. We consider following four cases:

(α) If $x \leq y$ and $y \leq z$, then $x \leq z$, therefore $[x] \preceq [z]$.

(β) If $x \leq y$ and $y \leq b_1, b_1' \leq z$ for some $b_1, b_1' \in B$, then $x \leq y \leq b_1$, $b_1' \leq z$. By definition, $[x] \preceq [z]$.

(γ) If $x \leq b, b' \leq y$ for some $b, b' \in B$ and $y \leq z$, by similar discussion to (β), we have $[x] \preceq [z]$.

(δ) If $x \leq b, b' \leq y$ for some $b, b' \in B$ and $y \leq b_1, b_1' \leq z$ for some $b_1, b_1' \in B$, then $x \leq b$, $b_1' \leq z$. Thus $[x] \preceq [z]$.

Now we show that the order is compatible with the binary operation on S. Let $[x] \preceq [y]$ and $[s] \preceq [t]$. Then $x \leq y$ or $x \leq b, b' \leq y$ for some $b, b' \in B$, and $s \leq t$ or $s \leq b_1, b_1' \leq t$ for some $b_1, b_1' \in B$. We consider following four cases:

(α) If $x \leq y$ and $s \leq t$, then $sx \leq ty$, therefore, $[s][x] = [sx] \preceq [ty] = [t][y]$.

(β) If $x \leq y$ and $s \leq b_1, b_1' \leq t$ for some $b_1, b_1' \in B$, then $sx \leq b_1x, b_1'y \leq ty$. Since B is an ideal of S and $b_1, b_1' \in B$, we have $b_1x, b_1'y \in B$. On the other hand, $sx \leq b_1x$, $b_1'y \leq ty$. Therefore, $[s][x] = [sx] \preceq [ty] = [t][y]$.

(γ) If $x \leq b, b' \leq y$ for some $b, b' \in B$ and $s \leq t$, by similar discussion to (β), we have $[s][x] \preceq [t][y]$.

(δ) If $x \leq b, b' \leq y$ for some $b, b' \in B$ and $s \leq b_1, b_1' \leq t$ for some $b_1, b_1' \in B$, then $sx \leq sb$, $tb_1' \leq ty$. Thus $[sx] \preceq [ty]$.

By the definition of $\preceq$, we can get that the canonical epimorphism is isotone.

Proposition 2 *Let S be an ordered semigroup and an ordered relation $\preceq$ on S/λ_B be defined as Theorem 1. Then for any $x, y \in S$*

$$([x], [y]) \in \preceq \iff (x, y) \in \leq \circ \lambda_B \circ \leq .$$

Proof $\Longrightarrow$ If $[x] \preceq [y]$, then $x \leq y$ or $x \leq b, b' \leq y$ for some $b, b' \in B$. If $x \leq y$, then $(x, y) \in \leq \subseteq \leq \circ \lambda_B \circ \leq$. If $x \leq b, b' \leq y$ for some $b, b' \in B$, then $x \leq b \lambda_B b' \leq y$. Thus $(x, y) \in \leq \circ \lambda_B \circ \leq$.

$\Longleftarrow$ Let $(x, y) \in \leq \circ \lambda_B \circ \leq$. Then there exist $z, h \in S$ such that $x \leq z$, $(z, h) \in \lambda_B$, $h \leq y$. Since $(z, h) \in \lambda_B$, we have $z = h \in S \setminus B$ or $z, h \in B$. If $z = h$, then $x \leq z = h \leq y$. If $z, h \in B$, then by $x \leq z, h \leq y$, we have $[x] \preceq [y]$.

Let B be an ordered ideal of an ordered semigroup S. In [7], [8], an ordered relation $\preceq_1$ in S/λ_B is defined as follows:

$$\preceq_1 := \{(B, \{x\}) | x \in S \backslash B\} \cup \{(\{x\}, \{y\}) | x, y \in S \backslash B, x \leq y\} \cup \{(B, B)\}.$$

If $[a] \preceq [b]$ in S/λ_B, then $a \leq b$ or $a \in B$ and $k \leq b$ for some $k \in B$. The latter implies $[a] \preceq_1 [k] \preceq_1 [b]$, that is, $[a] \preceq_1 [b]$. Hence $\preceq \subseteq \preceq_1$. The following example shows that $\preceq \subsetneqq \preceq_1$ in general.

Example We consider the ordered monoid $S = \{1, a, b, c, e, d, e\}$ whose multiplication and order are given below:

	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	a	a	a	a	a
b	b	a	a	a	a	a
c	c	a	a	c	c	e
d	d	a	a	c	d	e
e	e	a	a	c	c	e

$$\begin{aligned}\leq := \{&(1,1),(a,a),(a,c),(a,d),(a,e),(b,b),(b,c),(b,d),\\&(b,e),(c,c),(c,d),(c,e),(d,d),(e,e)\}.\end{aligned}$$

Obviously, $B=\{a\}$ is an ordered ideal of S . Since $a \nleqq b$ and there does not exist $x \in B$ such that $x \leq b$, we have $[a] \npreceq [b]$. But by the definition of $\preceq_1$, we have $[a] \preceq_1 [b]$.

Let S be an ordered semigroup and θ be an equivalence relation on S. A θ-*chain* is a finite sequence of elements $\{a, a_1, a_1^{'}, \ldots, a_i, a_i^{'}, \ldots, a_n, a_n^{'}, b\}$ of S such that $a \leq a_1$, $a_i\theta a_i^{'}$, for $i=1,\ldots,n$, $a_i^{'} \leq a_{i+1}$ for $i=1,\ldots,n-1$ and $a_n^{'} \leq b$. A θ-chain is said to be *close* if $a=b$. Otherwise it is said to be open.

Lemma 3[7] *Let S be an ordered semigroup and θ be a congruence on S. Then θ is an order congruence on S if and only if every closed θ-chain is contained in a single equivalent class of θ. In this case the induced order on S/θ is given by $[a] \leq [b]$ in S/θ if and only if there is a θ-chain from a to b in S.*

In fact, we have more general results (see following Theorem 5). First we recall the concept of order congruence generated by an congruence ρ on an ordered semigroup S.

Definition 4 *Let ρ be a congruence on ordered semigroup S. Order congruence σ is called an order congruence generated by ρ on S, if σ satisfies the following condition:*

(i) $\rho \subseteq \sigma$;

(ii) If there is an order congruence η on S such that $\rho \subseteq \eta$, then $\sigma \subseteq \eta$.

Theorem 5 *Let ρ ba a congruence on ordered semigroup S. Then we have*

(1) If we define a relation $\preceq_\rho$ on S as follows:

$$(a,b \in S)(a,b) \in \preceq_\rho \text{ if and only if there is a } \rho\text{-chain from } a \text{ to } b.$$

Then $\preceq_\rho$ is a quasi-order which is compatible with the binary operation;

(2) R_ρ on S is a relation on S defined as follows

$$(a,b \in S)(a,b) \in R_\rho \Longleftrightarrow (a,b),(b,a) \in \preceq_\rho.$$

Then R_ρ is an order congruence on S, which, in fact, is the order congruence generated by ρ. And S/R_ρ has the induced order relation:

$$[x] \preceq_{R_\rho} [y] \Leftrightarrow x \preceq_\rho y.$$

Proof (1) If $a \leq b$ for $a, b \in S$, then there is a ρ-chain from a to b: $\{a, b, b, b\}$, that is, $a \leq b$ implies $a \preceq_\rho b$. Thus $\leq \subseteq \preceq_\rho$. For any $a \in S$, since $a\rho a$, we have a ρ-chain from a to a. Consequently, $(a, a) \in \preceq_\rho$. Suppose that $(a, b) \in \preceq_\rho$ and $(b, c) \in \preceq_\rho$ for $a, b, c \in S$. Then there exist $a_1,\ a_1^{'},\ \ldots,\ a_i,\ a_i^{'},\ \ldots,\ a_n,\ a_n^{'},\ b_1,\ b_1^{'},\ \ldots,\ b_i,\ b_i^{'},\ \ldots,\ b_m,\ b_m^{'} \in S$ such that $a \leq a_1$, $b \leq b_1$, $a_i \theta a_i^{'}$, $b_j \theta b_j^{'}$, for $i = 1, \ldots, n, j = 1, \ldots, m$, $a_i^{'} \leq a_{i+1}$ $b_j^{'} \leq b_{j+1}$, for $i = 1, \ldots, n-1$, $j = 1, \ldots, m-1$ and $a_n^{'} \leq b$, $b_m^{'} \leq c$. Hence there is a ρ-chain from a to c $\{a, a_1,\ a_1^{'},\ \ldots,\ a_i,\ a_i^{'},\ \ldots,\ a_n,\ a_n^{'}, b,\ b_1,\ b_1^{'},\ \ldots,\ b_i,\ b_i^{'},\ \ldots,\ b_m,\ b_m^{'},\ c\}$. Therefore, $(a, c) \in \preceq_\rho$ and so $\preceq_\rho$ is transitive. Since ρ and $\leq$ are compatible with the binary operation on S, it is easy to see that $\preceq_\rho$ is also compatible with the binary operation on S.

(2) Since ρ is a congruence on S, by definition of R_ρ and (1), we get that R_ρ is also a congruence on S. In order to prove R_ρ is an ordered congruence on S, we only show that every closed R_ρ-chain is contained in a single equivalent class of R_ρ by Lemma 3. For any closed R_ρ-chain $\{a, a_1,\ a_1^{'},\ \ldots,\ a_i,\ a_i^{'},\ \ldots,\ a_n,\ a_n^{'}, a\}$, we have

$$a \leq a_1 R_\rho a_1^{'} \leq \cdots \leq a_i R_\rho a_i^{'} \leq \cdots a_n R_\rho a_n^{'} \leq a.$$

Thus $a \leq a_1 \preceq_\rho a_1^{'} \leq \cdots \leq a_i \preceq_\rho a_i^{'} \leq \cdots a_n \preceq_\rho a_n^{'} \leq a$. Furthermore

$$a \preceq_\rho a_1 \preceq_\rho a_1^{'} \preceq_\rho \cdots \preceq_\rho a_i \preceq_\rho a_i^{'} \preceq_\rho \cdots a_n \preceq_\rho a_n^{'} \preceq_\rho a.$$

Therefore, $(a, a_i), (a_i, a) \in \preceq_\rho$. We have showed that $a_i R_\rho a$. Since $a_i R_\rho a_i^{'}$, we have $a_i^{'} R_\rho a$. Hence the closed R_ρ-chain is contained in a single equivalent class of a_{R_ρ}. Therefore, R_ρ is an order congruence.

Let $(a, b) \in \rho$. Since ρ is a congruence, we have $(b, a) \in \rho$. Consequently, $(a, b), (b, a) \in \preceq_\rho$, thus $(a, b) \in R_\rho$, which shows that $\rho \subseteq R_\rho$.

Suppose that η is an order congruence on S and $\rho \subseteq \eta$. Then $R_\rho \subseteq \eta$. In fact: If $(a, b) \in R_\rho$, then $(a, b), (b, a) \in \preceq_\rho$. By (1), there exist $a_1,\ a_1^{'},\ \ldots,\ a_i,\ a_i^{'},\ \ldots,\ a_n,\ a_n^{'},\ b_1,\ b_1^{'},\ \ldots,\ b_i,\ b_i^{'},\ \ldots,$ $b_m,\ b_m^{'} \in S$ such that

$$a \leq a_1 \rho a_1^{'} \leq \cdots \leq a_i \rho a_i^{'} \leq \cdots a_n \rho a_n^{'} \leq b,$$

$$b \leq b_1 \rho b_1^{'} \leq \cdots \leq b_j \rho b_j^{'} \leq \cdots b_m \rho a_m^{'} \leq a.$$

Hence there exists a closed η-chain $a \leq a_1 \eta a_1^{'} \leq \cdots \leq a_i \eta a_i^{'} \leq \cdots a_n \eta a_n^{'} \leq b \eta b \leq b_1 \eta b_1^{'} \leq \cdots \leq b_j \eta b_j^{'} \leq \cdots b_m \eta a_m^{'} \leq a$. Since η is an order congruence on S, we have that the closed η-chain is contained in a single equivalent class of η. In particular, we have $(a, b) \in \eta$. Therefore R_ρ is the order congruence generated by ρ.

In S/R_ρ, we define a relation as follows:

$$[x] \preceq_{R_\rho} [y] \Leftrightarrow x \preceq_\rho y.$$

By (1) and definition of R_ρ, $\preceq_{R_\rho}$ is a quasi-order which is compatible with the binary operation. Furthermore, if $[x] \preceq_{R_\rho} [y]$ and $[y] \preceq_{R_\rho} [x]$, then $x \preceq_\rho y$ and $y \preceq_\rho x$. Thus $(x, y) \in R_\rho$, that is, $[x]_{R_\rho} = [y]_{R_\rho}$. Furthermore, if $x \leq y$, then $x \preceq_\rho y$. Thus $[x] \preceq_{R_\rho} [y]$.

Remark We will always specify the corresponding order on S/R_ρ that we have in mind.

Let S be an ordered semigroup and B be an ideal of S, and denote by H the Rees congruence λ_B. Then the order congruence R_H generated by H can be characterized as follows:

Theorem 6 *Let S be an ordered semigroup, B an ideal of S, and $H = \lambda_B$. Then for $x, y \in S$, xR_Hy if and only if $x = y$, or there exist $b, b', c, c' \in B$ such that*

$$x \leq b, b' \leq y \quad \text{and} \quad y \leq c, c' \leq x. \tag{*}$$

Therefore,

$$[x] \preceq [y] \textit{ in } S/R_H \Leftrightarrow (x \leq y) \textit{ or } (x \leq b, b' \leq y \textit{ for some } b, b' \in B).$$

Moreover, ($[x] = [y]$ in S/R_H if and only if $x = y$, or $x, y \in B$) $\Leftrightarrow$ B is convex.

Proof For any $x, y \in S$, we define a relation σ on S by: $a\sigma b$ if and only if $x = y$ or a system of inequalities $(*)$ exists. It is easy to check that the relation σ is an equivalence which is compatible with the binary operation.

To see that σ is an order congruence, we assume $x, a_i, a_i' \in S$ for $1 \leq i \leq n$, and suppose

$$x \leq a_1\sigma a_1' \leq a_2\sigma a_2' \leq \cdots \leq a_j\sigma a_j' \leq \cdots \leq a_{n-1}\sigma a_{n-1}' \leq a_n\sigma a_n' \leq x.$$

Then, for each $k \in \{1, \ldots, n\}$ we have a system of inequalities

$$a_k \leq b_k, b_k' \leq a_k' \text{ and } a_k' \leq c_k, c_k' \leq a_k,$$

where $b_k, b_k', c_k, c_k' \in B$. The system below establishes the fact that $x\sigma a_k'$ holds for each $k :$

$$\begin{aligned} &x \leq a_1 \leq b_1, \quad b_k' \leq a_k', \\ &a_k' \leq c_k, \quad b_n' \leq a_n' \leq x. \end{aligned}$$

Because $a_k\sigma a_k'$, we obtain $x\sigma a_k$. Thus, every closed σ-chain is contained in a single equivalence class of σ, so σ is an order congruence on S. Since $H \subseteq \sigma$, $R_H \subseteq \sigma$. On the other hand, if $(x, y) \in \sigma$, then, by the definition of σ, $x = y$, or there exists a system of inequalities $(*)$. Therefore, $x \leq bR_Hb' \leq y$. Thus $x \preceq_{R_H} y$. Similarly, $y \preceq_{R_H} x$. Since R_H is an order congruence, we have $(x, y) \in \lambda(H)$. This means that $\sigma \subseteq R_H$. Thus $\sigma = R_H$.

By Remark, $[x] \preceq [y] \Leftrightarrow x \preceq_{R_H} y \Leftrightarrow x \preceq_H y$. By Proposition 5, $x \preceq_H y$ if and only if there is a H-chain from x to y, that is, there exist $a_1, a_1', \ldots, a_i, a_i', \ldots, a_n, a_n' \in S$ such that $x \leq a_1$, a_iHa_i' for $i = 1, \ldots, n$, $a_i' \leq a_{i+1}$ for $i = 1, \ldots, n-1$, and $a_n' \leq y$. Thus $a_i, a_i' \in B$ for $i = 1, \ldots, n$. Hence

$$x \preceq_H y \Leftrightarrow (x \leq y) \text{ or } (x \leq b, b' \leq y \text{ for some } b, b' \in B).$$

Furthermore, if R_H is the Rees congruence, that is, $[x] = [y]$ in S/R_H if and only if $x = y$, or $x, y \in B$, then by the discussion of Theorem 1, if and only if B is convex.

We note that if B is a convex ideal of an ordered semigroup S, then R_H is the least order congruence containing $B \times B$, and $[x] \preceq [y]$ in $S/R_H \Leftrightarrow (x \leq y)$ or $(x \leq b, b' \leq y$ for some $b, b' \in B)$. The latter is the ordered relation defined in Theorem 1.

References

[1] BLYTH T S, JANOWITZ M F. *Residuation Theory* [M]. Pergamon Press, Oxford-New York-Toronto, Ont., 1972.
[2] HOWIE J M. *Fundamentals of Semigroup Theory* [M]. The Clarendon Press, Oxford University Press, New York, 1995.
[3] KEHAYOPULU N, TSINGELIS M. *On subdirectly irreducible ordered semigroups* [J]. Semigroup Forum, 1995, **50**(2): 161–177.
[4] KEHAYOPULU N, TSINGELIS M. *Pseudo-order in ordered semigroups* [J]. Semigroup Forum, 1995, **50**(3): 389–392.
[5] XIE Xiangyun. *Contributions to theory of congruences on ordered semigroups* [D]. Lanzhou: Lanzhou University, 1995.
[6] XIE Xiangyun. *An Introduction of Partially Ordered Semigroup* [M]. Beijing: Science Press, 2000. (in Chinese)
[7] XIE Xiangyun. *On regular, strongly regular congruences on ordered semigroups* [J]. Semigroup Forum, 2000, **61**(2): 159–178.
[8] XIE Xiangyun. *Regular congruence classes of ordered semigroups* [J]. J. Math. Res. Exposition, 2001, **21**(2): 207–211.
[9] XIE Xiangyun, SHI Xiaoping. *Order-congruences on S-posets* [J]. Commun. Korean Math. Soc., 2005, **20**(1): 1–14.
[10] XIE Xiangyun, WU Mingfen. *On congruences on ordered semigroups* [J]. Math. Japon., 1997, **45**(1): 81–84.