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A Kind of Rectilinear Congruences in the Minkowski 3-Space

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Abstract We consider the rectilinear congruence T generated by the tangents to a one parameter family of geodesics on a space-like surface S_1 in the Minkowski 3-space E_1^3, having S_1 as one of its focal surfaces. We prove that the two families of torsal surfaces of T touch the second focal surface S_2 along the net of orthogonal parametric curves if and only if S_1 is developable. We also obtain the necessary and sufficient condition for the correspondence between the points of S_1 and S_2 at the same rays preserving the net of asymptotic curves. At last, we investigate the orthogonal surface S of T. We proved that the correspondence between S_1 and S_2 preserves the net of asymptotic curves if S is maximal in E_1^3.

Keywords the rectilinear congruence; the focal surfaces; the Minkowski space; the space-like surface; maximal surfaces.

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1. Introduction

In the Euclidean 3-space R^3, the tangents to a one parameter family of curves $\mathcal{C}$ on a regular surface S form a rectilinear congruence, having S as one of its focal surfaces. The two focal surfaces coincide when $\mathcal{C}$ is a family of asymptotic curves.

Let the net of parameter curves of S be the net of lines of curvature $\mathcal{C}_1$ and $\mathcal{C}_2$. Suppose that T_1 and T_2 are congruences generated by the tangents to families of curves $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Abdel-Baky[1,2] proved that the torsal surfaces of T_1 and T_2 touch the focal surfaces along the net of lines of curvature if and only if

$$\bar{q}^2 + \bar{q}_1 = 0, \tag{1.1}$$

$$q^2 - q_2 = 0, \tag{1.2}$$

where q and $\bar{q}$ are the geodesic curvatures of curves in $\mathcal{C}_1$ and $\mathcal{C}_2$, and suffix 1 and 2 imply taking derivative with respect to the arc length parameter of curves in $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. In this case, T_1 and T_2 are the Guichard congruences.

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It is easy to prove that S is developable when both T_1 and T_2 are the Guichard congruences. It follows that one of $\mathcal{C}_1$ and $\mathcal{C}_2$ is a family of straight lines. Thus one of T_1 and T_2 has to be degenerate.

Tsagas[3] investigated the congruences formed by the tangents to a one parametric family of geodesics on S. He obtained the necessary and sufficient conditions for the correspondence between the focal surfaces by focal points at the same rays preserving the area elements. Papantonion et al[4,5] studied the same problems in E_1^3 and generalized the results of Tsagas[3].

In this paper, we study the rectilinear congruence T generated by tangents to a one parametric family of geodesics $\mathcal{G}$ on a space-like surface S in E_1^3.

Firstly we prove that the necessary and sufficient condition for the torsal surfaces of T touching the second focal surface along the net of orthogonal parametric curves is

$$q^2 + q_1 = 0, \tag{1.3}$$

where q is the geodesic curvature of the orthogonal trajectories of $\mathcal{G}$ and the suffix 1 means taking derivative with respect to the arc length parameter of the geodesics in $\mathcal{G}$. It is easy to see that (1.3) has the same form as (1.1). Therefore, we generalize the result of Abdel-Baky[1]. Furthermore, we prove that the Gauss curvature K of S is zero which means that S is developable.

Then we prove that the correspondence between the focal surfaces by focal points at the same rays preserves the net of asymptotic curves if and only if the Gauss curvature K and $\bar{K}$ of the two focal surfaces satisfy the equality $K\bar{K} = -q^4$.

Finally we investigate the surface S which is perpendicular to the rays of T, say the orthogonal surface of T. We prove that if the orthogonal surface S is maximal in E_1^3, then $K\bar{K} = -q^4$, where K and $\bar{K}$ are the Gaussian curvatures of the two focal surfaces. That is to say, for the congruence whose orthogonal surface is maximal, the correspondence between the focal surfaces by focal points at the same rays preserves the net of asymptotic curves.

2. Preliminaries

First we introduce some basic concepts and properties on the rectilinear congruences in E_1^3. More of them can be found in [6].

Definition 2.1 *A rectilinear congruence T in E_1^3 can be defined by*

$$T : \tilde{r}(u,v) = r(u,v) + tR(u,v), \quad (u,v) \in D, \quad t \in (-\infty, +\infty), \tag{2.1}$$

where $S : r = r(u,v)$ is called the reference surface and the unit vector valued function $R = R(u,v)$ gives the directions of the rays of T. If $R = R(u,v)$ is the normal vectors of a surface S' transversal to the rays of T, then T is called a normal congruence, and S' is called the orthogonal surface of T.

Definition 2.2 *The rectilinear congruence T defined by (2.1) is called space-like (resp. time-like) if $R = R(u,v)$ is space-like (resp. time-like) for all $(u,v) \in D$.*

Definition 2.3 *For the rectilinear congruence T defined by (2.1), the first and second Kummer fundamental forms of T are defined by*

$$\mathrm{d}R \cdot \mathrm{d}R = E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2, \tag{2.2}$$

where $E = R_u \cdot R_u$, $F = R_u \cdot R_v$, $G = R_v \cdot R_v$; and

$$\mathrm{d}r \cdot \mathrm{d}R = edu^2 + (f + f')\mathrm{d}u\mathrm{d}v + g\mathrm{d}v^2, \tag{2.3}$$

where $e = r_u \cdot R_u$, $f = r_v \cdot R_u$, $f' = r_u \cdot R_v$, $g = r_v \cdot R_v$.

Lemma 2.1[6] *The necessary and sufficient condition for the space-like or time-like rectilinear congruence in E_1^3 to be normal congruence is $f = f'$.*

Definition 2.4[6] *The surface in E_1^3 is called space-like or time-like if the normal vectors of the surface are time-like or space-like.*

3. Space-like rectilinear congruence in E_1^3

Given a regular space-like surface S in E_1^3 defined by

$$S : r = r(u, v), \quad (u, v) \in D.$$

Denote by g_{ij}, b_{ij} $(i, j = 1, 2)$ the coefficients of the first and second fundamental forms of S, respectively. Let the u-curves be geodesics and the v-curves be their orthogonal trajectories. Let u be the arc length parameter of the u-curves. Then we have

$$g_{11} = 1,\ g_{12} = g_{21} = 0,\ g_{22} = \rho^2\ (\rho > 0).$$

We choose the orthonormal frame field $\{e_1, e_2, e_3\}$ over S so that

$$e_1 = r_u, \quad e_2 = \frac{r_v}{\rho}$$

and e_3 is the unit normal vector to S satisfying $(e_1, e_2, e_3) = 1$ where $(*, *, *)$ denotes the mixed product of three vectors. Then we have

$$e_1^2 = e_2^2 = -e_3^2 = 1,$$

$$e_2 \times e_3 = e_1,\ e_3 \times e_1 = e_2,\ e_2 \times e_1 = e_3,$$

where "$\times$" is the vectorial product of two vectors in E_1^3[6].

Let $\mathrm{d}s = \mathrm{d}u$, $\mathrm{d}\bar{s} = \rho\mathrm{d}v$. Then s and $\bar{s}$ are the arc length parameters of the u-curves and v-curves respectively. The derivative formula of the frame field $\{e_1, e_2, e_3\}$ with respect to s and $\bar{s}$ are

$$\frac{\partial}{\partial s}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & p \\ k & p & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \quad \frac{\partial}{\partial \bar{s}}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & q & p \\ -q & 0 & \bar{k} \\ p & \bar{k} & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{3.1}$$

where

$$k = -b_{11}, \quad p = -\frac{b_{12}}{\rho}, \quad \bar{k} = -\frac{b_{22}}{\rho^2}, \quad q = \frac{\rho_u}{\rho}. \tag{3.2}$$

It is easy to check that q is the geodesic curvature of the v-curves.

We denote the derivatives of function ϕ with respect to s and $\bar{s}$ by $\partial\phi/\partial s = \phi_1$ and $\partial\phi/\partial\bar{s} = \phi_2$ respectively. Then the Gauss-Codazzi equations of S are:

$$K = k\bar{k} - p^2 = q_1 + q^2, \tag{3.3}$$

$$\begin{aligned} -k_2 + p_1 + 2pq = 0, \\ \bar{k}_1 - p_2 + q(\bar{k} - k) = 0, \end{aligned} \tag{3.4}$$

where K is the Gaussian curvature of S.

The collection of tangents of S along the u-curves forms a rectilinear congruence T having S as one of its focal surfaces. The parametric equation of T is

$$T : Y(u, v, t) = r(u, v) + te_1(u, v), \quad (u, v) \in D, \quad t \in (-\infty, +\infty). \tag{3.5}$$

The coefficients of the first and second Kummer fundamental forms of T are:

$$E = e_{1u}^2 = -k^2, \quad F = e_{1u} \cdot e_{1v} = -k\rho p, \quad G = e_{1v}^2 = \rho^2 \left(q^2 - p^2\right), \tag{3.6}$$

$$e = e_{1u} \cdot r_u = 0, \quad f = e_{1u} \cdot r_v = 0, \quad f' = e_{1v} \cdot r_u = 0, \quad g = e_{1v} \cdot r_v = \rho^2 q. \tag{3.7}$$

It is easy to see that $p \equiv 0$ implies that the net of parametric curves is the net of curvature lines. $k \equiv 0$ implies that all of the u-curves are straight lines and $q \equiv 0$ implies that S is developable and the second Kummer fundamental form of T is zero.

From now on, we assume $p \neq 0$, $k \neq 0$ and $q \neq 0$.

Theorem 3.1 *Let T be the rectilinear congruence generated by the tangents to a one parametric family of geodesics (non straight lines) on a space-like surface S_1 in E_1^3. Then the distance between the foci on the same rays is the geodesic curvature radius of the orthogonal trajectories of the family of geodesics. The second focal surface S_2 of T is time-like and the normal vectors of S_2 are parallel to the tangent vectors of the orthogonal trajectories.*

Proof Denote the parametric equation of the focal surface by

$$Z = r(u, v) + t(u, v)e_1(u, v).$$

Then it follows that $t(u, v)$ is the root of the quadratic equation

$$(Y_u, Y_v, Y_t) = 0, \tag{3.8}$$

where

$$Y_u = r_u + te_{1u} = e_1 + tke_3, \quad Y_v = r_v + te_{1v} = \rho[e_2 + t(qe_2 + pe_3)], \quad Y_t = e_1. \tag{3.9}$$

From (3.8) and (3.9), we have $t_1(u, v) = 0$ and $t_2(u, v) = -1/q$. It follows that the distance between the loci on the same rays is $d = |t_2(u, v) - t_1(u, v)| = 1/|q|$. So we proved the first claim.

The parametric equation of the second focal surface S_2 of T is

$$S_2 : Z = r(u, v) - \frac{1}{q}\, e_1(u, v). \tag{3.10}$$

Direct computation yields

$$Z_1 = \frac{\partial Z}{\partial s} = e_1 + \frac{q_1}{q^2}e_1 - \frac{1}{q}(ke_3) = \frac{q^2+q_1}{q^2}e_1 - \frac{k}{q}e_3, \tag{3.11}$$

$$Z_2 = \frac{\partial Z}{\partial \bar{s}} = e_2 + \frac{q_2}{q^2}e_1 - \frac{1}{q}(qe_2 + pe_3) = \frac{q_2}{q^2}e_1 - \frac{p}{q}e_3. \tag{3.12}$$

The unit normal vector of S_2 is

$$n = \varepsilon\frac{Z_1 \times Z_2}{\|Z_1 \times Z_2\|} = \varepsilon e_2 \quad (\varepsilon = \pm 1). \tag{3.13}$$

Thus $n^2 = 1$ which implies that S_2 is time-like and the normal vectors of S_2 are the tangent vectors of the orthogonal trajectories. The second claim is proved. □

Theorem 3.2 *Let T be the rectilinear congruence given in Theorem 3.1. If the second focal surface is non-degenerate, then the necessary and sufficient condition that the two torsal surfaces of T touch the second focal surface along the net of orthogonal parametric curves is the reference surface S is developable. In this case, the second focal surface is not developable.*

Proof Let $\{A, B, C\}$ and $\{L, M, N\}$ be the coefficients of the first and second fundamental forms of the second focal surface S_2. It follows from (3.11)–(3.13) that

$$A = \frac{(q_1+q^2)^2}{q^4} - \frac{k^2}{q^2}, \quad B = \rho\left\{\left(\frac{q_1+q^2}{q^4}\right)q_2 - \frac{kp}{q^2}\right\}, \quad C = \rho^2\left(\frac{q_2^2}{q^4} - \frac{p^2}{q^2}\right), \tag{3.14}$$

$$L = -\varepsilon\frac{kp}{q}, \qquad M = -\varepsilon\rho\frac{p^2}{q}, \quad N = \varepsilon\rho^2\left(\frac{q_2 - \bar{k}p}{q}\right). \tag{3.15}$$

Let $\mathrm{d}u : \mathrm{d}v$, $\delta u : \delta v$ be the directions of two torsal surfaces of T. Then it follows from the differential equation of torsal surfaces of T

$$\begin{vmatrix} E\mathrm{d}u + F\mathrm{d}v & F\mathrm{d}u + G\mathrm{d}v \\ e\mathrm{d}u + f\mathrm{d}v & f'\mathrm{d}u + g\mathrm{d}v \end{vmatrix} = 0$$

that

$$\mathrm{d}u : \mathrm{d}v = -\rho p : k, \quad \delta u : \delta v = 1 : 0. \tag{3.16}$$

If the two torsal surfaces of T touch the second focal surface S_2 along the net of orthogonal parametric curves, then (3.16) must fulfil the equation

$$A\mathrm{d}u\delta u + B(\mathrm{d}u\delta v + \mathrm{d}v\delta u) + C\mathrm{d}v\delta v = 0. \tag{3.17}$$

From (3.14), (3.16) and (3.17), we immediately derive that

$$(q_1 + q^2)[(q_1 + q^2)p - q_2 k] = 0, \tag{3.18}$$

that is

$$q_1 + q^2 = 0, \tag{3.19}$$

or

$$(q_1 + q^2)p - q_2 k = 0. \tag{3.20}$$

It follows from (3.11) and (3.12) that

$$Z_1 \times Z_2 = \frac{(q_1 + q^2)p - kq_2}{p^3}\ e_2. \tag{3.21}$$

Thus (3.20) implies that the second focal surface S_2 is degenerate, which contradicts the assumption of Theorem 3.2. Therefore (3.19) holds .

(3.19) together with (3.3) yields $K = 0$. That is to say, the reference surface S is developable. From(3.14) and (3.15), we have that the Gauss curvature of the second focal surface S_2 is

$$\bar{K} = \frac{LN - M^2}{AC - B^2} = -\frac{q^4(k\bar{k}p^2 - kpq_2 - p^4)}{[(q_1 + q^2)p - kq_2]^2}.$$

From (3.3) we obtain

$$\bar{K} = \frac{pq^4}{kq_2 - Kp}. \tag{3.22}$$

(3.19) together with (3.22) implies that

$$\bar{K} = \frac{pq^4}{kq_2} \neq 0,$$

since $p \neq 0$ and $q \neq 0$. So S_2 is not developable. Hence Theorem 3.2 is proved. □

One can establish a one to one correspondence ξ between S_1 and S_2 by focal points belonging to the same rays. Now we look for the necessary and sufficient condition for the correspondence ξ preserving the net of asymptotic curves. We shall prove the following

Theorem 3.3 *Let T be the rectilinear congruence given in Theorem 3.1. Then the correspondence ξ between the non-degenerate focal surfaces of T preserves the net of asymptotic curves if and only if that the Gaussian curvatures K and $\bar{K}$ of the focal surfaces satisfy*

$$K\bar{K} = -q^4,$$

where q is the geodesic curvature of the orthogonal trajectories to the family of geodesics given in Theorem 3.1.

Proof The correspondence ξ preserves the net of asymptotic curves if and only if $\{b_{11}, b_{12}, b_{22}\}$ and $\{L, M, N\}$ satisfy the condition:

$$\frac{b_{11}}{L} = \frac{b_{12}}{M} = \frac{b_{22}}{N}. \tag{3.23}$$

It follows from (3.2) and (3.15) that (3.23) is equivalent to

$$q_2 = 0. \tag{3.24}$$

From (3.22) and (3.24) we have

$$K\bar{K} = -q^4.$$

Conversely, suppose that $K\bar{K} = -q^4$. Then it follows from (3.22) that

$$\frac{pq^4}{kq_2 - Kp} = -\frac{q^4}{K},$$

which is equivalent to

$$kq_2 = 0.$$

So (3.23) holds because $k \neq 0$. Hence Theorem 3.3 is proved. $\square$

If (3.19) and (3.24) are both satisfied, then $Z_1 \times Z_2 = 0$. It follows that the focal surface S_2 is degenerate. Hence we have the following

Corollary 3.1 *Let S_1 be a space-like surface in E_1^3. Let T be the rectilinear congruence generated by the tangents to a one parametric family of geodesics. If the two torsal surfaces of T touch the second focal surface S_2 along the net of orthogonal parametric curves and the correspondence ξ between S_1 and S_2 preserves the net of asymptotic curves, then S_2 is degenerate.*

From (3.7) we have $f = f' = 0$. It follows that T is a normal congruence. We investigate the surfaces that are perpendicular to the rays of T. In [6], the authors proved that the parameter equations of these surfaces can be expressed as

$$\bar{S} : \bar{Z}(u, v) = r(u, v) + t(u, v)e_1(u, v), \tag{3.25}$$

where

$$t = -\int r_u \cdot e_1 \mathrm{d}u + r_v \cdot e_1 \mathrm{d}v + c, \quad c \in (-\infty, +\infty).$$

All of them are time-like due to $e_1^2 = 1$. As

$$r_u \cdot e_1 = e_1 \cdot e_1 = 1, \quad r_v \cdot e_1 = \rho e_2 \cdot e_1 = 0,$$

we obtain

$$t = c - u.$$

Therefore, (3.25) becomes

$$\bar{S} : \bar{Z}(u, v) = r(u, v) + (c - u)e_1(u, v). \tag{3.26}$$

Direct computation yields

$$\bar{Z}_u = (c - u)k\ e_3, \quad \bar{Z}_v = \rho\{1 + (c - u)q\}e_2 + \rho(c - u)p\ e_3. \tag{3.27}$$

It is noted that $\bar{S}$ is not regular along the curve $u = c$. So we assume $u \neq c$. Let $\{A_1, B_1, C_1\}$ and $\{L_1, M_1, N_1\}$ be the coefficients of the first and second fundamental forms of $\bar{S}$. It follows from (3.26) and (3.1) that

$$A_1 = -t^2k^2, \quad B_1 = -\rho t^2 kp, \quad C_1 = \rho^2(1 + tq)^2 - \rho^2t^2p^2, \tag{3.28}$$

$$L_1 = tk^2, \quad M_1 = \rho tkp, \quad N_1 = \rho^2 tp^2 - \rho^2 q(1 + tq), \quad t = c - u. \tag{3.29}$$

The Gaussian and mean curvatures K_1 and H_1 of $\bar{S}$ are

$$K_1 = \frac{q}{(u - c)\{1 + (c - u)q\}} = q\left\{\frac{1}{u - c} + \frac{q}{1 + (c - u)q}\right\}, \tag{3.30}$$

$$H_1 = \frac{1 + 2q(c - u)}{2(u - c)\{1 + (c - u)q\}} = \frac{q}{2}\left\{\frac{1}{u - c} - \frac{q}{1 + (c - u)q}\right\}. \tag{3.31}$$

From $q \neq 0$ and (3.30) we can see that $\bar{S}$ is not developable.

Theorem 3.4 *If the orthogonal surfaces of the rectilinear congruence T are maximal in E_1^3, then the correspondence between the focal surfaces of T preserves the net of asymptotic curves.*

Proof Assume that $\bar{S}$ is maximal in E_1^3. Then $H_1 = 0$. From (3.31) we have

$$q = \frac{1}{2(u-c)},$$

which implies that

$$q_2 = 0.$$

Hence Theorem 3.3 follows from (3.23) and (3.24). □

Remark 3.1 From (3.2) we have $q = \rho_u/\rho$. It follows that $q_2 = 0$ if and only if $\rho_{uv} = 0$ which implies that $\rho = \alpha(u) + \beta(v)$.

Remark 3.2 From (3.30) and (3.30), we can see that

$$q^2 = \frac{K_1^2 - 4H_1^2}{4K_1}.$$

It follows that $q_2 = 0$ if and only if

$$\frac{\partial}{\partial v}\left(\frac{K_1^2 - 4H_1^2}{4K_1}\right) = 0,$$

which implies that the orthogonal surfaces $\bar{S}$ of T are the Weingarten surfaces.

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