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Some Matrix Versions and Generalizations for Inequalities of Hua-Wang Type

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Abstract This paper is concerned with inequalities of Hua-Wang type. Some generalizations for inequalities of Hua-Wang type are obtained by the method of theory of Matrix. Moreover, some new Hua-Wang type inequalities are established.

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1. Introduction

The following inequality due to Hua[1] is important to number theory:

$$\alpha\sum_{i=1}^{n}x_i^2+(\delta-\sum_{i=1}^{n}x_i)^2\geq\frac{\alpha}{n+\alpha}\delta^2, \tag{1}$$

where $\alpha>0,\delta>0,x_i\geq 0,i=1,2,\ldots,n$ and $\sum_{i=1}^{n}x_i\leq\delta$. In 1992, inequality (1) was generalized by Wang[2] by making use of the majorization and dynamic programming. If $\alpha>0,\delta>0$ and $r>1$, then the inequality

$$\alpha^{r-1}\sum_{i=1}^{n}x_i^r+(\delta-\sum_{i=1}^{n}x_i)^r\geq(\frac{\alpha}{n+\alpha})^{r-1}\delta^r \tag{2}$$

holds for all non-negative numbers $x_i,i=1,\ldots,n$ with $\sum_{i=1}^{n}x_i\leq\delta$. The sign of inequality in (2) is reserved for $0<r<1$.

Inequalities (1) and (2) have been extensively studied by many authors in recent years. Some new generalizations and proofs are obtained[3−10].

In this paper, our main purpose is to extend inequalities (1) and (2) by using the theory of matrix.

First we collect some lemmas which will be useful later.

Lemma 1[11] *Let f be a convex function on I ($I\subseteq R$) and A_j, $j=1,2,\ldots,k$ be Hermitian*

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matrices with eigenvalues in I; $x_j \in C^n$, $j = 1, 2, \ldots, k$ with $\sum_{j=1}^{k}\langle x_j, x_j\rangle = 1$, Then

$$f(\sum_{j=1}^{k}\langle A_j x_j, x_j\rangle) \leq \sum_{j=1}^{k}\langle f(A_j)x_j, x_j\rangle.$$

Let $\langle x, y\rangle = x^*y$, where x, $y \in C^n$, $x^* = \bar{x}^T$. Set $k = 1$. It is easy to obtain the following lemma by Lemma 1.

Lemma 2 *Suppose that A is an $n \times n$ Hermitian matrix, $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \lambda_n(A)$ are eigenvalues of A. Let f be a convex function on $[\lambda_n(A), \lambda_1(A)]$. If $x \in C^n$ and $x^*x = 1$, then*

$$f(x^*Ax) \leq x^*f(A)x; \tag{3}$$

If f is a concave function on$[\lambda_n(A), \lambda_1(A)]$, then the inequality of (3) in reverse order holds.

It is not difficult to verify those lemmas below by making use of the property of the convex function, power mean and the maximum principle.

Lemma 3 *Let $p > 0$, $q > 0$, $rs \neq 0$, $0 < u < \infty$, $0 < v < q$. If $\frac{r}{s} > 1$ or $\frac{r}{s} < 0$. Then the following inequality holds:*

$$\frac{p^{r-1}u^r + (q-v)^r}{[p^{s-1}u^s + (q-v)^s]^{\frac{r}{s}}} \geq (\frac{p+1}{p})^{\frac{s-r}{s}}; \tag{4}$$

If $0 < \frac{r}{s} < 1$, then the inequality of (4) in reverse order holds.

Lemma 4 *Given $p > 0$, $0 < v < q$. Let $f(v) = p^{r-1}v^r + (q-v)^r$. If $r > 1$ or $r < 0$, then we have*

$$f(v) \geq \frac{p^{r-1}q^r}{(1+p)^{r-1}}; \tag{5}$$

If $0 < r < 1$, then the inequality of (5) in reverse order holds.

Lemma 5 *Assume that $p > 0$, $q > 1$, $0 < v < q$. Let $f(v) = v + \frac{1}{r}(q-v)^r$. If $r > 1$, then the following inequality*

$$f(v) \geq \frac{1}{r} + q - 1 \tag{6}$$

holds. If $0 < r < 1$, then the inequality of (6) in reverse order holds.

2. Main results

In this section, we give our main results. The first result can be stated as follows:

Theorem 1 *Assume that $p > 0$, $q > 0$. Let A be an $n \times n$ positive definite Hermitian matrix and $x \in C^n$ with $x^*x = 1$, $x^*Ax < q$. For some $r \in R$, set $F(r) = p^{r-1}x^*A^rx + (q - x^*Ax)^r$. Then we have the following inequalities:*

(i) If $r > 1$ or $r < 0$, then

$$F(r) \geq \frac{p^{r-1}q^r}{(1+p)^{r-1}}; \tag{7}$$

The inequality of (7) in reverse order holds for $0 < r < 1$.

(ii) If $\frac{r}{s} > 1$ or $\frac{r}{s} < 0$, then

$$F(r) \geq (\frac{p+1}{p})^{\frac{s-r}{s}}[F(s)]^{\frac{r}{s}}; \tag{8}$$

The inequality of (8) in reverse order holds for $0 < \frac{r}{s} < 1$.

Proof Case (i) If $r > 1$ or $r < 0$, then $f(x) = x^r$ is a convex function on $(0, \infty)$. By Lemma 2, we get $x^*A^rx \geq (x^*Ax)^r$. Combining with Lemma 4, we have

$$F(r) \geq p^{r-1}(x^*Ax)^r + (q - x^*Ax)^r \geq \frac{p^{r-1}q^r}{(1+p)^{r-1}}. \tag{9}$$

Similarly, we can prove the inequality of (7) in reverse order holds for $0 < r < 1$.

Case (ii) If $\frac{r}{s} > 1$ or $\frac{r}{s} < 0$, then $f(x) = x^{\frac{r}{s}}$ is a convex function on $(0, \infty)$. Also, we have $x^*A^rx = x^*(A^s)^{\frac{r}{s}}x \geq (x^*A^sx)^{\frac{r}{s}}$, by Lemma 2. Set $u = (x^*A^sx)^{\frac{1}{s}}$ and $v = x^*Ax$. Together with Lemma 3, we get

$$F(r) \geq p^{r-1}u^r + (q - v)^r \geq (\frac{p+1}{p})^{\frac{s-r}{s}}[F(s)]^{\frac{r}{s}}. \tag{10}$$

Similarly, we can prove the inequality of (8) in reverse order holds for $0 < \frac{r}{s} < 1$. The proof is completed. □

Choosing suitable values of A, x, p and q, we can get some relative conclusions of [1–3] and derive some general results below. The following corollaries are consequences of Theorem 1.

Corollary 1 *Assume that $k_i \geq 0$, $x_i \geq 0$, $i = 1, 2, \dots, n$ and $\sum_{i=1}^n k_i = 1, 0 < \sum_{i=1}^n k_ix_i < q$. If $r > 1$ or $r < 0$, then*

$$p^{r-1}\sum_{i=1}^n k_ix_i^r + (q - \sum_{i=1}^n k_ix_i)^r \geq \frac{p^{r-1}q^r}{(1+p)^{r-1}}; \tag{11}$$

The inequality of (11) in reverse order holds for $0 < r < 1$.

Proof Let $A = \operatorname{diag}\{x_1, x_2, \dots, x_n\}$, $x = \{\sqrt{k_1}, \sqrt{k_2}, \dots, \sqrt{k_n}\}^{\mathrm{T}}$. The conclusion follows from (7). The proof is completed. □

Set $np = \alpha$, $nq = \delta$, and $k_i = \frac{1}{n}, i = 1, 2, \dots, n$. Combining with Corollary 1, we have the following corollary:

Corollary 2[2] *Suppose that $\alpha > 0$, $\delta > 0$, $x_i > 0$ and $\sum_{i=1}^n x_i < \delta$. If $r > 1$ or $r < 0$, then*

$$\alpha^{r-1}\sum_{i=1}^n x_i^r + (\delta - \sum_{i=1}^n x_i)^r \geq (\frac{\alpha}{n+\alpha})^{r-1}\delta^r; \tag{12}$$

The inequality of (12) in reverse order holds for $0 < r < 1$.

Corollary 3 *Set $G(r) = \alpha^{r-1}\sum_{i=1}^n x_i^r + (\delta - \sum_{i=1}^n x_i)^r$, where $\alpha > 0$, $\delta > 0$, $x_i > 0$, and $\sum_{i=1}^n x_i < \delta$. If $\frac{r}{s} > 1$ or $\frac{r}{s} < 0$, then*

$$G(r) \geq (\frac{n+\alpha}{\alpha})^{\frac{s-r}{s}}[G(s)]^{\frac{r}{s}}; \tag{13}$$

The inequality of (13) in reverse order holds for $0 < \frac{r}{s} < 1$.

Proof Let $A=\mathrm{diag}\{x_1,x_2,\ldots,x_n\}$, $x=(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}},\ldots,\frac{1}{\sqrt{n}})^{\mathrm{T}}$ and $np=\alpha$, $nq=\delta$. Thus we have the conclusion from (8). □

Taking $s=r-1$ or $s=r+1$, respectively, in the Corollary 3 when $r>\bar{N}+1(\bar{N}\in N)$ or $r<-\bar{N}$, then we can get chains of inequalities in [3]. Now, by means of the theory of matrix, we give some generalizations of (1) and (2).

Theorem 2 *Assume that A is an $n\times n$ positive definite Hermitian matrix. Let $p>0,q>0$, and r,s,t_1,t_2 be constants satisfying $rt_2=st_1$, and $x\in C^n$ with $x^*x=1$, $x^*A^{t_2}x<q$. Then we have the following results:*

(i) If $\frac{t_1}{t_2}>1,r>1$ or $\frac{t_1}{t_2}<0$, $r<0$, then

$$p^{r-1}(x^*A^{t_1}x)^s+(q-x^*A^{t_2}x)^r\geq\frac{p^{r-1}q^r}{(1+p)^{r-1}}.\tag{14}$$

(ii) If $0<\frac{t_1}{t_2}<1$ and $0<r<1$, then

$$p^{r-1}(x^*A^{t_1}x)^s+(q-x^*A^{t_2}x)^r\leq\frac{p^{r-1}q^r}{(1+p)^{r-1}}.\tag{15}$$

Proof Case (i) Since $rt_2=st_1$, it is obvious to see that $s>0$ if $\frac{t_1}{t_2}>1,r>1$ or $\frac{t_1}{t_2}<0,r<0$. At the same time, we can find that $f(x)=x^{\frac{t_1}{t_2}}$ is a convex function on $(0,\infty)$. Thus

$$x^*A^{t_1}x=x^*(A^{t_2})^{\frac{t_1}{t_2}}x\geq(x^*A^{t_2}x)^{\frac{t_1}{t_2}}.\tag{16}$$

In view of (16) and Lemma 4, we get

$$p^{r-1}(x^*A^{t_1}x)^s+(q-x^*A^{t_2}x)^s\geq p^{r-1}(x^*A^{t_2}x)^r+(q-x^*A^{t_2}x)^r\geq\frac{p^{r-1}q^r}{(1+p)^{r-1}}.\tag{17}$$

Similarly, we can prove (ii). The proof is completed. □

Set $s=t_2=1,t_1=r$, it is easy to see that (7) follows from Theorem 2. Therefore, Theorem 2 is a generalization of (1) and (2). By choosing a suitable parameter, we can obtain some interesting inequalities.

Corollary 4 *Suppose that $\alpha>0$, $\delta>0$, $x_i>0$ and $\sum_{i=1}^n x_i^{\frac{1}{r}}<\delta$. If $r>1$, then*

$$\alpha^{r-1}(\sum_{i=1}^n x_i^r)^{\frac{1}{r}}+(\delta-\sum_{i=1}^n x_i^{\frac{1}{r}})^r\geq(\frac{\alpha}{n^{\frac{r+1}{r}}+\alpha})^{r-1}\delta^r;\tag{18}$$

The inequality of (18) in reverse order holds for $0<r<1$.

Proof Set $s=t_2=\frac{1}{r}$, $t_1=r$, $A=\mathrm{diag}\{x_1,x_2,\ldots,x_n\}$, $x=(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}},\ldots,\frac{1}{\sqrt{n}})^{\mathrm{T}}$ and $n^{\frac{r+1}{r}}p=\alpha$, $nq=\delta$. The conclusion follows from Theorem 2.

Now, we can obtain the following refinement of Hua-Wang type inequality.

Theorem 3 *Assume that A is an $n\times n$ positive definite Hermitian matrix. Let $q>1$ and $x\in C^n$ with $x^*x=1$, $x^*Ax<q$. If $r>1$, then*

$$(x^*A^rx)^{\frac{1}{r}}+\frac{1}{r}(q-x^*Ax)^r\geq q+\frac{1}{r}-1;\tag{19}$$

If $0<r<1$, then the inequality of (19) in reverse order holds.

Proof From Lemma 2, we know that $x^*A^rx \geq (x^*Ax)^r$ for $r > 1$. Combining with Lemma 5, we have

$$(x^*A^rx)^{\frac{1}{r}} + \frac{1}{r}(q - x^*Ax)^r \geq x^*Ax + \frac{1}{r}(q - x^*Ax)^r \geq q + \frac{1}{r} - 1.$$

Using the similar methods, we can show that the inequality of (19) in reverse order holds for $0 < r < 1$. The proof is completed. □

In particular, let $A = \mathrm{diag}\{x_1, x_2, \ldots, x_n\}$, $x = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})^{\mathrm{T}}$ and $\delta = nq$, we can get the following corollary as a consequence of Theorem 3.

Corollary 5 *Suppose that* $x_i > 0$, $\delta > n$ *and* $\sum_{i=1}^n x_i < \delta$. *If* $r > 1$, *then*

$$n^{1-\frac{1}{r}}(\sum_{i=1}^n x_i^r)^{\frac{1}{r}} + \frac{1}{r}(\delta - \sum_{i=1}^n x_i)^r \geq n^{r-1}\delta + \frac{n}{r} - n; \tag{20}$$

The inequality of (20) in reverse order holds for $0 < r < 1$.

Remark 1 It is easy to show that Lemma 1 holds when A is a Hermitian operator of Hilbert space by using the spectral decomposition of the linear operator. Therefore, conclusions of this paper still hold for positive definite operators of Hilbert spaces. Suppose that $H = L^2[a, b]$, we can obtain some integrals version of Hua-Wang type inequalities by choosing suitable operators.

Remark 2 Under the condition that the equality holds in Jensen inequlity, we can get a new condition that the equality holds in this paper. Details are omitted.

Remark 3 The condition in Hua-Wang inequality is

$$x_i \geq 0, \quad \sum_{i=1}^n x_i \leq \delta. \tag{21}$$

However, this text conclusion statement always remove the sign of equality. It is not difficult to see that the sign of equality can be easily added to the condition of inequality of the related conclusion by utilizing the limit method.

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