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Monotone CQ Algorithm of Fixed Points for Weak Relatively Nonexpansive Mappings and Applications

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Abstract Matsushita, Takahashi[4] proved a strong convergence theorem for relatively nonexpansive mappings in a Banach space by using the hybrid method (CQ method) in mathematical programming. The purpose of this paper is to modify the hybrid method of Matsushita, Takahashi by monotone CQ method, and to prove strong convergence theorems for weak relatively nonexpansive mappings and maximal monotone operators in Banach spaces. The convergence rate of monotone CQ method is faster than the hybrid method of Matsushita, Takahashi. In addition, the Cauchy sequence method is used in this paper without using the Kadec-Klee property. The results of this paper modify and improve the results of Matsushita, Takahashi and some others.

Keywords weak relatively nonexpansive mapping; generalized projection; asymptotic fixed point; monotone CQ method; maximal monotone operator.

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1. Introduction

In recent years, the CQ iteration methods for approximating fixed points of nonlinear mappings have been introduced and studied by various authors[1−4].

In 2003, Nakajo and Takahashi[1] proposed the following modification of Mann iteration method for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.1}$$

where C is a closed convex subset of H and P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one,

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then the sequence $\{x_n\}$ generated by (1.1) converges strongly to $P_{F(T)}(x_0)$, where $F(T)$ denotes the fixed points set of T.

In 2006, Kim and Xu[2] proposed the following modification of the Mann iteration method for asymptotically nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.2}$$

where C is bounded closed convex subset and

$$\theta_n = (1-\alpha_n)(k_n^2 - 1)(\mathrm{diam} C)^2 \to 0 \quad \text{as} \quad n \to \infty.$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.2) converges strongly to $P_{F(T)}(x_0)$.

They also proposed the following modification of the Mann iteration method for asymptotically nonexpansive semigroup $\Im$ in a Hilbert space H:

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) \dfrac{1}{t_n} \displaystyle\int_0^{t_n} T(s) x_n \mathrm{d}s, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \overline{\theta}_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.3}$$

where C is bounded closed convex subset and

$$\overline{\theta}_n = (1-\alpha_n)[(\frac{1}{t_n}\int_0^{t_n} L(u)\mathrm{d}u)^2 - 1](\mathrm{diam} C)^2 \to 0 \quad \text{as} \quad n \to \infty.$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.3) converges strongly to $P_{F(\Im)}(x_0)$, where $F(\Im)$ denotes the common fixed points set of $\Im$.

In 2006, Martinez-Yanes and Xu[3] proposed the following modification of the Ishikawa iteration method for nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) T z_n, \\ z_n = \beta_n x_n + (1-\beta_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \\ \qquad (1-\alpha_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, z \rangle)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.4}$$

where C is a closed convex subset of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one and $\beta_n \to 0$, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to

$P_{F(T)}(x_0)$.

Martinez-Yanes and Xu[3] proposed also the following modification of the Halpern iteration method for nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily,} \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \\ \qquad \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.5}$$

where C is a closed convex subset of H. They proved that if the sequence $\alpha_n \to 0$, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{F(T)}(x_0)$.

In 2005, Matsushita and Takahashi[4] proposed the following hybrid iteration method with generalized projection for relatively nonexpansive mapping T in a Banach space E:

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\} \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.6}$$

They proved the following convergence theorem.

Theorem MT *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by (1.6), where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}(\cdot)$ is the generalized projection from C onto $F(T)$.*

The purpose of this paper is to modify the hybrid method of Matsushita, Takahashi by monotone CQ method, and to prove strong convergence theorems for relatively nonexpansive mappings and maximal monotone operators in Banach spaces. The convergence rate of monotone CQ method is faster than the hybrid method of Matsushita, Takahashi. In addition, the Cauchy sequence method is used in this paper instead of using the Kadec-Klee property. The results of this paper modify and improve the results of Matsushita, Takahashi and some others.

2. Preliminaries

Let E be a Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber[5] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined as in [5, 6] by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for} \quad x, y \in E. \tag{2.1}$$

Observe that, in a Hilbert space H, (2.1) reduces to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$.

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x). \tag{2.2}$$

Existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J[5−7]. In Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2 \quad \text{for all} \quad x, y \in E. \tag{2.3}$$

Remark If E is a reflexsive strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definitions of j, we have $Jx = Jy$. That is, $x = y$; see [8, 9] for more details.

Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point of p in C is said to be an asymptotic fixed point of T[10] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(Tx_n - x_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$ and relatively nonexpansive[10−12] if $\widehat{F}(T) = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Now, we present the definition of weak relatively nonexpansive mappings.

Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point of p in C is said to be a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that the strong $\lim_{n\to\infty}(Tx_n - x_n) = 0$. The set of strong asymptotic fixed points of T will be denoted by $\overline{F}(T)$. A mapping T from C into itself is called weak relatively nonexpansive if $\overline{F}(T) \subset F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

It is obvious, relatively nonexpansive mapping is weak relatively nonexpansive mapping.

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. A Banach space is said to have the Kadec-Klee property if a sequence $\{x_n\} \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex. Then E has the Kadec-Klee property; see [8, 9] for more details.

We need the following Lemmas for the proof of our main results.

Lemma 2.1[7] *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2.2[5] *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0 \ \text{ for } \ y \in C.$$

Lemma 2.3[5] *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_c x) + \phi(\Pi_c x, x) \leq \phi(y, x) \ \text{ for all } \ y \in C.$$

Lemma 2.4 *Let E be a strictly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a weak relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Proof We first show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence of $F(T)$ such that $x_n \to q \in C$. From the definition of T we have

$$\phi(x_n, Tq) \leq \phi(x_n, q)$$

for each $n \geq 1$. This implies

$$\phi(q, Tq) = \lim_{n\to\infty} \phi(x_n, Tq) \leq \lim_{n\to\infty} \phi(x_n, q) = \phi(q, q) = 0.$$

Therefore, we obtain $q = Tq$, so that $q \in F(T)$. Next, we show that $F(T)$ is convex. For $x, y \in F(T)$ and $t \in (0, 1)$, put $z = tx + (1-t)y$. It suffices to show $Tz = z$. In fact, we have

$$\begin{aligned}
\phi(z, Tz) &= \|z\|^2 - 2\langle z, JTz\rangle + \|Tz\|^2 \\
&= \|z\|^2 - 2\langle tx + (1-t)y, JTz\rangle + \|Tz\|^2 \\
&= \|z\|^2 - 2t\langle x, JTz\rangle - 2(1-t)\langle y, JTz\rangle + \|Tz\|^2 \\
&\leq \|z\|^2 + t\phi(x, z) + (1-t)\phi(y, z) - t\|x\|^2 - (1-t)\|y\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \|z\|^2 - 2\langle tx + (1-t)y, Jz\rangle + \|z\|^2 \\
&= \|z\|^2 - 2\langle z, Jz\rangle + \|z\|^2 \\
&= \phi(z, z) = 0.
\end{aligned}
$$

This implies $z = Tz$. This completes the proof. □

3. Main results

Now, we can prove a strong convergence theorem for weak relatively nonexpansive mappings in a Banach space by using the monotone CQ method.

Theorem 3.1 *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a weak relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$
\begin{cases}
x_0 \in C, \quad \text{chosen arbitrarily}, \\
y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\
C_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \phi(z, y_n) \le \phi(z, x_n)\}, \\
C_0 = \{z \in C : \phi(z, y_0) \le \phi(z, x_0)\}, \\
Q_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\
Q_0 = C, \\
x_{n+1} = \Pi_{C_n \cap Q_n}(x_0),
\end{cases}
\tag{3.1}
$$

where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Proof We first show that C_n and Q_n are closed and convex for each $n \ge 0$. From the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \ge 0$. We show that C_n is convex. Since $\phi(z, y_n) \le \phi(z, x_n)$ is equivalent to

$$
2\langle z, Jx_n - Jy_n\rangle + \|y_n\|^2 - \|x_n\|^2 \le 0,
$$

it follows that Q_n is convex.

Next, we show that $F(T) \subset C_n \bigcap Q_n$ for each $n \ge 0$. For any $p \in F(T)$ and $n \ge 0$,

$$
\begin{aligned}
\phi(p, y_n) &= \phi(p, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n)) \\
&= \|p\|^2 - 2\langle p,\ \alpha_n Jx_n + (1-\alpha_n)JTx_n\rangle + \|\alpha_n Jx_n + (1-\alpha_n)JTx_n)\|^2 \\
&\le \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2(1-\alpha_n)\langle p, JTx_n\rangle + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Tx_n\|^2 \\
&= \alpha_n(\|p\|^2 - 2\langle p, Jx_n\rangle + \|x_n\|^2) + (1-\alpha_n)(\|p\|^2 - 2\langle p, JTx_n\rangle + \|Tx_n\|^2) \\
&= \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, Tx_n) \\
&\le \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, x_n) \\
&= \phi(p, x_n),
\end{aligned}
$$

we have $p \in F(T)$. Therefore, we obtain $F(T) \subset C_n$ for each $n \ge 0$.

Next, we show that $F(T) \subset Q_n$ for all $n \geq 0$, we prove this by induction. For $n = 0$, we have $F(T) \subset C = Q_0$. Assume that $F(T) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Lemma 2.2 we have

$$\langle x_{n+1} - z, Jx_0 - Jx_{n+1} \rangle \geq 0, \quad \forall z \in C_n \cap Q_n.$$

As $F(T) \subset C_n \cap Q_n$ by the induction assumptions, the last inequality holds, in particular, for all $z \in F(T)$. This together with the definition of Q_{n+1} implies that $F(T) \subset Q_{n+1}$.

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$ and $C_n \bigcap Q_n \subset C_{n-1} \bigcap Q_{n-1}$ for all $n \geq 1$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \tag{3.2}$$

for all $n \geq 0$. Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. In addition, it follows from definition of Q_n and Lemma 2.2 that $x_n = \Pi_{Q_n} x_0$. Therefore, by Lemma 2.3 we have

$$\phi(x_n, x_0) = \phi(\Pi_{Q_n} x_0, x_0) \leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0),$$

for each $p \in F(T) \subset Q_n$ for all $n \geq 0$. Therefore, $\phi(x_n, x_0)$ is bounded. This together with (3.2) implies that the limit of $\{\phi(x_n, x_0)\}$ exists. Put

$$\lim_{n \to \infty} \phi(x_n, x_0) = d. \tag{3.3}$$

From Lemma 2.3, we have, for any positive integer m, that

$$\begin{aligned} \phi(x_{n+m}, x_n) &= \phi(x_{n+m}, \Pi_{C_n} x_0) \leq \phi(x_{n+m}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_{n+m}, x_0) - \phi(x_n, x_0), \end{aligned} \tag{3.4}$$

for all $n \geq 0$.

We claim that $\{x_n\}$ is a Cauchy sequence, if not, there exists a positive real number $\varepsilon_0 > 0$ and the subsequence $\{n_k\}, \{m_k\} \subset \{n\}$ such that $\|x_{n_k+m_k} - x_{n_k}\| \geq \varepsilon_0$.

On the other hand, from (3.3) and (3.4) we have

$$\begin{aligned} \phi(x_{n_k+m_k}, x_{n_k}) &\leq \phi(x_{n_k+m_k}, x_0) - \phi(x_{n_k}, x_0) \\ &\leq |\phi(x_{n_k+m_k}, x_0) - d| + |d - \phi(x_{n_k}, x_0)| \to 0, \;\; k \to \infty. \end{aligned}$$

Because from (2.3) we know that the $\phi(x_n, x_0)$ is bounded implies the $\{x_n\}$ is also bounded, by using Lemma 2.1, we obtain

$$\lim_{k \to \infty} \|x_{n_k+m_k} - x_{n_k}\| = 0.$$

This is a contradiction, so that $\{x_n\}$ is a Cauchy sequence. Therefore, there exists a point $p \in C$ such that $\{x_n\}$ converges strongly to p. Hence we have

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.5}$$

In addition, from (3.3) and (3.4) we have $\lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0$. This together with the fact $x_{n+1} \in C_n$ implies that

$$\lim_{n \to \infty} \phi(x_{n+1}, y_n) = 0.$$

By using again Lemma 2.1, we have

$$\lim_{n \to \infty} \|x_{n+1} - y_n\| = 0. \tag{3.6}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{3.7}$$

On the other hand, we have, for each $n \geq 0$,

$$\begin{aligned}\|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_n Jx_n + (1-\alpha_n)JTx_n)\| \\ &= \|\alpha_n(Jx_{n+1} - Jx_n) + (1-\alpha_n)(Jx_{n+1} - JTx_n)\| \\ &= \|(1-\alpha_n)(Jx_{n+1} - JTx_n) - \alpha_n(Jx_n - Jx_{n+1})\| \\ &\geq (1-\alpha_n)\|Jx_{n+1} - JTx_n\| - \alpha_n\|Jx_n - Jx_{n+1}\|\end{aligned}$$

and hence

$$\begin{aligned}\|Jx_{n+1} - JTx_n\| &\leq \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \alpha_n\|Jx_n - Jx_{n+1}\| \\ &\leq \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \|Jx_n - Jx_{n+1}\|).\end{aligned}$$

From (3.7) and $\lim_{n\to\infty} \alpha_n < 1$, we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - JTx_n\| = 0.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - Tx_n\| = \lim_{n\to\infty} \|J^{-1}Jx_{n+1} - J^{-1}JTx_n\| = 0.$$

Therefore, from

$$\|x_n - Tx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\|,$$

we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. This together with the fact $\{x_n\}$ converges strongly to $p \in C$ and the definition of weak relatively nonexpansive mappings implies that $p \in F(T)$.

Finally, we prove that $p = \Pi_{F(T)}x_0$. From Lemma 2.3, we have

$$\phi(p, \Pi_{F(T)}x_0) + \phi(\Pi_{F(T)}x_0, x_0) \leq \phi(p, x_0). \tag{3.8}$$

On the other hand, since $x_{n+1} = \Pi_{C_n \bigcap Q_n}(x_0)$ and $C_n \bigcap Q_n \supset F(T)$, for all n. Also from Lemma 2.3, we have

$$\phi(\Pi_{F(T)}x_0, x_{n+1}) + \phi(x_{n+1}, x_0) \leq \phi(\Pi_{F(T)}x_0, x_0). \tag{3.9}$$

By the definition of $\phi(x, y)$, we know that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_0) = \phi(p, x_0). \tag{3.10}$$

Combining (3.8), (3.9) and (3.10), we know that $\phi(p, x_0) = \phi(\Pi_{F(T)}x_0, x_0)$. Therefore, it follows from the uniqueness of $\Pi_{F(T)}x_0$ that $p = \Pi_{F(T)}x_0$. This completes the proof. □

4. Applications

In a similar fashion, we can modify iteration methods (1.1)–(1.5) by monotone CQ methods. So we can obtain some strong convergence theorems, respectively, we omit here.

Now, we apply Theorem 3.1 to prove a strong convergence theorem concerning maximal monotone operators in a Banach space E.

Let A be a multi-valued operator from E to E^* with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \{z \in E : z \in D(A)\}$. An operator A is said to be monotone if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$$

for each $x_1, x_2 \in D(A)$ and $y_1 \in Ax_1, y_2 \in Ax_2$. A monotone operator A is said to be maximal if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. We know that if A is a maximal monotone operator, then $A^{-1}0$ is closed and convex. The following result is also well-known.

Theorem 4.1[13] *Let E be a reflexive, strictly convex and smooth Banach space and let A be a monotone operator from E to E^*. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Let E be a reflexive, strictly convex and smooth Banach space, and let A be a maximal monotone operator from E to E^*. Using Theorem 4.1 and strict convexity of E, we obtain that for every $r > 0$ and $x \in E$, there exists a unique x_r such that

$$Jx \in Jx_r + rAx_r.$$

Then we can define a single valued mapping $J_r : E \to D(A)$ by $J_r = (J + rA)^{-1}J$ and such a J_r is called the resolvent of A. We know that $A^{-1} = F(J_r)$ for all $r > 0$, see [9, 14] for more details. Using Theorem 3.1, we can consider the problem of strong convergence concerning maximal monotone operators in a Banach space. Such a problem has been also studied in [1], [7], [15]–[18].

Theorem 4.2 *Let E be a uniformly convex and uniformly smooth Banach space, let A be a maximal monotone operator from E to E^*, let J_r be a resolvent of A, where $r > 0$ and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$\begin{cases} x_0 \in E, \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JJ_r x_n), \\ C_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ C_0 = \{z \in E : \phi(z, y_0) \leq \phi(z, x_0)\}, \\ Q_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ Q_0 = E, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases}$$

where J is the duality mapping on E. If $A^{-1}0$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0} x_0$, where $\Pi_{A^{-1}0}$ is the generalized projection from E onto $A^{-1}0$.

Proof We first show that $\widehat{F}(J_r) \subset A^{-1}0$. Let $p \in \widehat{F}(J_r)$. Then there exists $\{z_n\} \subset E$ such that $z_n \rightharpoonup p$ and $\lim_{n\to\infty} \|z_n - J_r z_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\frac{1}{r}(Jz_n - JJ_r z_n) \to 0.$$

It follows from

$$\frac{1}{r}(Jz_n - JJ_rz_n) \in AJ_rz_n$$

and the monotonicity of A that

$$\langle w - J_rz_n, w^* - \frac{1}{r}(Jz_n - JJ_rz_n)\rangle \geq 0$$

for all $w \in D(A)$ and $w^* \in Aw$. Letting $n \to \infty$, we have $\langle w - p, w^*\rangle \geq 0$ for all $w \in D(A)$ and $w^* \in Aw$. Therefore, from the maximality of A, we obtain $p \in A^{-1}0$. On the other hand, we know that $F(J_r) = A^{-1}0$ and $F(J_r) \subset \widehat{F}(J_r)$, therefore, $A^{-1}0 = F(J_r) = \widehat{F}(J_r)$. Next we show that J_r is a relatively nonexpansive mapping with respect to $A^{-1}0$. Let $w \in E$ and $p \in A^{-1}0$. From the monotonicity of A, we have

$$\begin{aligned}
\phi(p, J_rw) &= \|p\|^2 - 2\langle p, JJ_rw\rangle + \|J_rw\|^2\\
&= \|p\|^2 + 2\langle p, Jw - JJ_rw - Jw\rangle + \|J_rw\|^2\\
&= \|p\|^2 + 2\langle p, Jw - JJ_rw\rangle - 2\langle p, Jw\rangle + \|J_rw\|^2\\
&= \|p\|^2 - 2\langle J_rw - p - J_rw, Jw - JJ_rw - Jw\rangle - 2\langle p, Jw\rangle + \|J_rw\|^2\\
&= \|p\|^2 - 2\langle J_rw - p, Jw - JJ_rw - Jw\rangle +\\
&\quad 2\langle J_rw, Jw - JJ_rw\rangle - 2\langle p, Jw\rangle + \|J_rw\|^2\\
&\leq \|p\|^2 + 2\langle J_rw, Jw - JJ_rw\rangle - 2\langle p, Jw\rangle + \|J_rw\|^2\\
&= \|p\|^2 - 2\langle p, Jw\rangle + \|w\|^2 - \|J_rw\|^2 + 2\langle J_rw, Jw\rangle - \|w\|^2\\
&= \phi(p, w) - \phi(J_rw, w)\\
&\leq \phi(p, w).
\end{aligned}$$

This implies that J_r is a relatively nonexpansive mapping. Using Theorem 3.1, we can conclude that $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}x_0$. This completes the proof. □

Remark In the monotone CQ iteration methods, because $\{C_n \bigcap Q_n\}$ is monotone sequence of sets, that is, $C_n \bigcap Q_n \subset C_{n-1} \bigcap Q_{n-1}$ for all $n \geq 1$, the convergence rate of monotone CQ iteration method is faster than the hybrid(CQ) method of Matsushita, Takahashi and others. In addition, by using the monotone CQ iteration method, we can obtain the strong convergence theorem for weak relatively nonexpansive mappings.

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