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Quasi-Hereditary Orderings of A_n-Type Algebras with Two Generators

ZHANG Yue Hui, WU Liu San, GAO Chun Yan
(Institute of Mathematics, Tianjin University of Technology and Education, Tianjin 300222, China)
(E-mail: zhangyh@tute.edu.cn)

Abstract This short note is devoted to an approach of the quasi-hereditary orderings of A_n-type algebras with exactly two generators. A necessary and sufficient condition for a quasi-hereditary ordering is obtained. Moreover, the numbers of quasi-hereditary orderings of such algebras are explicitly given.

Keywords A_n-type algebras; quasi-hereditary algebras; quasi-hereditary orderings.

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1. Introduction

Since the famous paper[1] by Cline, Parshall and Scott was published in 1988, quasi-hereditary theory has received broad development. A remarkable feature of a quasi-hereditary algebra is that its properties depend heavily on the selected ordering on simple modules. So an important and interesting problem of quasi-hereditary theory is how to count the different quasi-hereditary orderings for a quasi-hereditary algebra in general. A classical example is that Dlab and Ringel proved that all orderings of hereditary algebras (and only hereditary algebras) are quasi-hereditary[2]. Henceforth, the first author and Li proved that the different quasi-hereditary orderings of a tree-type quasi-hereditary algebra are $\frac{2}{3}n!$ at most in [3]. However, it seems to be very difficult to give a method for the computation of quasi-hereditary orderings for all quasi-hereditary algebras. This short note is devoted to an approach of the quasi-hereditary orderings of A_n-type algebras with exactly two generators. We first obtain a necessary and sufficient condition for an ordering to be quasi-hereditary, then give explicitly the number of quasi-hereditary orderings of all A_n-type algebras with exactly two generators by combinatoric technique. We hope that the method given in this paper could produce a new way to this question.

2. Preliminaries

Throughout, algebras are all finite dimensional algebra (associative, having a unit) over an algebraically closed field k. Modules are finitely generated (=finite dimensional) right modules.

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For an algebra A, $\mod A$ stands for the category of all A-modules. The composition of mappings are from left to right, that is, fg means first f, then g. For $M \in \text{mod}A$, $\text{Grad}(M)$ denotes the good radical of M and $\text{pd}.M$ the projective dimension of M. We use $\text{gld}.A$ to denote the global dimension of A.

In order to save space, we use freely the standard terminologies and basic properties of quasi-hereditary algebras, see also [2] and [4].

For an algebra A, denote by $\mathcal{O}(A)$ the collection of all orderings on simple modules. Let Λ be the weight poset of A (That is, Λ is a partially ordered set in bijective correspondence with the set of iso-class of simple modules over A). For each $\lambda \in \Lambda$, denote by $E(\lambda)$ (or more precisely, $E(A, \lambda)$) the corresponding simple module, $P(\lambda)$ a projective cover of $E(\lambda)$; Denote by $\Delta(\lambda)$ the maximal factor module of $P(\lambda)$ with all simple composition factors of the form $E(\mu)$ with $\mu \le \lambda$. Denote by Δ the full subcategory of $\mod A$ consisting of all modules $\Delta(\lambda)$, $\lambda \in \Lambda$. The modules in Δ are called to be Weyl modules (or standard modules). Denote by $\mathcal{F}(\Delta)$ the collection of all A-modules having Δ-filtrations, namely, M satisfies

$$0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M$$

such that M_{i-1}/M_i is isomorphic to some $\Delta(\lambda) \in \Delta,\ 1 \le i \le t$.

Definition 1 *The algebra A is said to be quasi-hereditary with respect to the weight poset Λ if for each $\lambda \in \Lambda$ we have*

1) $\text{End}_A(\Delta(\lambda)) \simeq k$;

2) $P(\lambda) \in \mathcal{F}(\Delta)$.

No matter the feature of a quasi-hereditary algebra is closely related with the poset Λ or the selected ordering on its simple modules from the above definition. It is well-known that any quasi-hereditary ordering of an algebra is equivalent to some total one (i.e., Weyl modules coincide under two orderings). We take this fact for a convention in the present paper. So all of the elements in $\mathcal{O}(A)$ are total orderings. Let $\{E(i) | 1 \le i \le n\}$ be the set of iso-classes of the simple A-modules. Denote by $\mathcal{O}(E(i))$ or $\mathcal{O}(i)$ the subset of $\mathcal{O}(A)$ consisting of those elements with the largest element $E(i)$. Notice that two different orderings may generate the same quasi-hereditary algebra–such two orderings are called to be equivalent. In literature, some authors identify such two ordering. However, we do not make this hypothesis in this paper. We always suppose that "Two orderings are the same if and only if they are same as totally ordered sets". Therefore, $|O(A)| = n!$, $|\mathcal{O}(i)| = (n-1)!$. Denote by the number of the quasi-hereditary ordering of A by $q(A)$, and the number of the nonquasi-hereditary orderings by $n(A)$. Apparently, $n(A) = n! - q(A)$. By [1], A is hereditary if and only if $q(A) = |\mathcal{O}(A)| = n!$. For nonhereditary algebra of A, we always have $q(A) < n!$.

3. Quasi-hereditary orderings of A_n-type algebras with two generators

Later on, $\Lambda = \{1 < 2 < \cdots < n\}$, and $I(\Lambda)$ is its incidence algebra, namely, $I(\Lambda)$ is the hereditary algebra with ordinary quiver Q the Hasse diagram of Λ, where the arrows of Q are given by $\alpha_i : i+1 \longrightarrow i,\ 1 \leq i \leq n-1$.

Definition 3.1 *Let $\Lambda = \{1 < 2 < \cdots < n\}$, $I(\Lambda)$ be as above. Suppose $1 \leq i < j \leq n-2$. Then the (i,j)-th algebra $A^{(i,j)}$ is defined by the following bounden quiver*

$$A^{(i,j)} = I(\Lambda)/(I(i) + I(j)),$$

where $I(i)$ respectively $I(j)$ is a (two-sided) ideal of $I(\Lambda)$ generated by $\alpha_i\alpha_{i+1}$ respectively $\alpha_j\alpha_{j+1}$.

Therefore, $A^{(i,j)},\ 1 \leq i < j \leq n-2$, is the following Nakayama algebra:

$$Q : 1 \to 2 \to \cdots \to i \xrightarrow{\alpha_i} i+1 \xrightarrow{\alpha_{i+1}} i+2 \to \cdots \to j \xrightarrow{\alpha_j} j+1 \xrightarrow{\alpha_{j+1}} j+2 \to \cdots \to n-1 \to n$$

with defining relations $\alpha_i\alpha_{i+1} = 0$ and $\alpha_j\alpha_{j+1} = 0$.

Proposition 3.1 *Let $1 \leq i < j \leq n-2$. Then $\dim_k A^{(i,j)} = [(n-i)^2 + (j-i)^2 + i^2 + n + 2j]/2$.*

Proof Because $A^{(i,j)}$ is a Nakayama algebra, its indecomposable modules can be determined uniquely by the dimension vectors. So the indecomposable projective modules $P(r),\ 1 \leq r \leq n$ can be denoted by the dimension vectors. According to the definition of $I(i)$ and $I(j)$, we have (From left to right: top to socle):

$$P(r) = \begin{cases} E(r),\ E(r+1), \ldots, E(i),\ E(i+1), & \text{if } 1 \leq r \leq i, \\ E(r),\ E(r+1), \ldots, E(j),\ E(j+1), & \text{if } i+1 \leq r \leq j, \\ E(r),\ E(r+1), \ldots, E(n-1),\ E(n), & \text{if } j+1 \leq r \leq n. \end{cases} \tag{1}$$

So we can compute the dimension of indecomposable projective modules $P(r),\ 1 \leq r \leq n$, namely

$$\dim_k P(r) = \begin{cases} i(i+3)/2, & \text{if } 1 \leq r \leq i, \\ (j-i)(j-i+3)/2, & \text{if } i+1 \leq r \leq j, \\ (n-j)(n-j+1)/2, & \text{if } j+1 \leq r \leq n. \end{cases} \tag{2}$$

So

$$\dim_k A^{(i,j)} = \sum_{r=1}^{n} \dim_k P(r) = \frac{1}{2}[(n-j)^2 + (j-i)^2 + i^2 + 2j + n].$$

Proposition 3.2 *Let $1 \leq i < j \leq n-2$. Then*

1) If $i+1 < j$, then $A^{(i,j)}$ is a tilted algebra with global dimension 2;

2) If $i+1 = j$, then $A^{(i,j)}$ is an algebra with global dimension 3.

Proof 1) Let $i+1 < j$. According to formula (1), we have the following exact sequence in mod $A^{(i,j)}$:

$$0 \longrightarrow P(r+1) \longrightarrow P(r) \longrightarrow E(r) \longrightarrow 0, \quad \text{if } \ r \neq i, j \tag{3}$$

and

$$0 \longrightarrow P(r+2) \longrightarrow P(r+1) \longrightarrow P(r) \longrightarrow E(r) \longrightarrow 0, \quad \text{if } \ r = i \text{ or } r = j. \tag{4}$$

So

$$\text{pd}.E(r) = \begin{cases} 1, & \text{if } r \neq i, j, \\ 2, & \text{if } r = i, j. \end{cases}$$

According to the famous result by Goodearl[5], $\mathrm{gld}.A = \sup\{\mathrm{pd}.M | M \text{ is simple-module}\}$, we then have $\mathrm{gld}.A = 2$.

In order to prove $A^{(i,j)}$ is a tilted algebra, consider A as the following path-algebra A_n:

$$A_n : 1 \to 2 \to \cdots \to r \to \cdots \to n-1 \to n.$$

Denote by e_r the primitive idempotent of A corresponding to the vertex r of A_n, and let $P_A(r) = e_rA$ be the corresponding indecomposable projective module of A. Put

$$T(r) = \begin{cases} P_A(r), & \text{if } r \neq i, j, \\ P_A(r)/P_A(r+2), & \text{if } r = i \text{ or } r = j. \end{cases}$$

Suppose $T = \oplus_{r=1}^{n} T(r)$. Since A is hereditary, $\mathrm{pd}.T \leq 1$ (In fact $\mathrm{pd}.T = 1$). Furthermore, it is easy to prove that $\mathrm{Ext}_A^1(T, T) = 0$. So T_A is a tilting module. All of the $T(r)$'s are Schurian-modules (that is, $\mathrm{End}_A(T(r)) \cong k$). If $r < s$, then $\mathrm{Hom}_A(T(r), T(s)) = 0$; If $r > s$, we have

$$\mathrm{Hom}_A(T(r), T(s)) \cong \begin{cases} k, & \text{if } 1 \leq s < r \leq i+1, \\ 0, & \text{if } 1 \leq s < i+1 < r \leq j+1, \\ k, & \text{if } i+1 \leq s < r \leq j+1, \\ 0, & \text{if } i+1 \leq s < j+1 < r \leq n, \\ k, & \text{if } j+1 \leq s < r \leq n. \end{cases}$$

Therefore, $\mathrm{End}_A T = \mathrm{Hom}_A(T, T) \cong A^{(i,j)}$, so $A^{(i,j)}$ is a tilted algebra[6].

2) Let $i + 1 = j$. Then the defining relations of $A^{(i,j)} = A^{(i,i+1)}$ are $\alpha_i\alpha_{i+1} = 0$ and $\alpha_{i+1}\alpha_{i+2} = 0$. We have the following exact sequence in $\mod A^{(i,j)}$ from (1):

$$0 \longrightarrow P(r+1) \longrightarrow P(r) \longrightarrow E(r) \longrightarrow 0, \quad \text{if } \ r \neq i, i+1 \tag{5}$$

and

$$0 \longrightarrow P(i+3) \longrightarrow P(i+2) \longrightarrow P(i+1) \longrightarrow P(i) \longrightarrow E(i) \longrightarrow 0 \tag{6}$$

and

$$0 \longrightarrow P(i+3) \longrightarrow P(i+2) \longrightarrow P(i+1) \longrightarrow E(i+1) \longrightarrow 0. \tag{7}$$

Therefore

$$\mathrm{pd}.E(r) = \begin{cases} 1, & \text{if } \ r \neq i, i+1, \\ 3, & \text{if } \ r = i, \\ 2, & \text{if } \ r = i+1. \end{cases}$$

We have $\mathrm{gld}.A^{(i,j)} = 3$. Since the global dimension of a classical tilted algebra ≤ 2[7], $A^{(i,j)}$ is tilted (in fact,$A^{(i,j)}$ is even not a generalized tilted algebra).

Lemma 3.3 *An ordering "$\leq$" of $A^{(i,j)}$ is quasi-hereditary if and only if one of the following conditions holds.*

1) $i + 1 < i$ *and* $j + 1 < j$*;*

2) $i + 1 < i$ *and* $j < j + 1 < j + 2$*;*

3) $j + 1 < j$ *and* $i < i + 1 < i + 2$*;*

4) $i < i+1 < i+2$ *and* $j < j+1 < j+2$.

Proof Necessity. Suppose $(A^{(i,j)}, \leq)$ is a quasi-hereditary algebra. Then $P(i) \in \mathcal{F}(\Delta)$. Since the dimension vector of $P(i)$ is $[i, i+1]$, we know that, $P(i) \in \Delta$, that is, $P(i) = \Delta(i)$, or $P(i)$ has a $\mathcal{F}(\Delta)$-filtration of length 2. Notice that $\mathrm{Hom}_{A^{(i,j)}}(P(r), P(i)) \cong (\delta_{ir} + \delta_{i+1,r})k$, where δ_{xy} is the Kronecker symbol, so the former implies that $i < i+1$, while the latter requires that $i < i+1$ and $\Delta(i+1)$ is a simple-module. So we must have $i+1 < i+2$. Similar argument works for the case $P(j) \in \mathcal{F}(\Delta)$. So we also have that $j+1 < j$ or $j < j+1 < j+2$.

Sufficiency. We prove only the sufficiency of condition 1), the proof of other conditions are similar. For any $r < s$, there is $\mathrm{Hom}_{A^{(i,j)}}(P(r), P(s)) = 0$, so whether $P(r)$ in Δ or not only relates to $P(s)(s > r)$. Because the quotient algebra $A^{(i,j)}/(\sum_{r=1}^{j} e_r) \cong kA_{n-j}$ of $A^{i,j}$ is hereditary, $P(r) \in \mathcal{F}(\Delta),\ r \geq j+1$ for all orderings "$\leq$" of simple modules. So we only need to discuss those $P(r)$ with $r \leq j$.

Now suppose condition 1) holds, that is, $i+1 < i$ and $j+1 < j$. Notice that for all $r > j$, we have $\mathrm{Hom}_{A^{(i,j)}}(P(r), P(j)) = \delta_{j+1,j}k$. But $j+1 < j$, so $\mathrm{Grad}(P(j)) = 0$ and $P(j) \in \Delta$. Thus, $P(r) \in \Delta$ for all $i < r < j$. According to the same reason, if $i+1 < i$, then $\mathrm{Grad}(P(i)) = 0$ and $P(i) \in \Delta$. Whence, $P(r) \in \Delta$ for all $1 \leq r < i$. Therefore $(A^{(i,j)}, \leq)$ is quasi-hereditary. The proof is completed.

Now, we can prove our main result:

Theorem 3.4 *Let* $1 \leq i < j \leq n-2$. *Then*

1) If $i+1 = j$, *then* $\mathcal{O}(A^{(i,j)}) = \frac{n!}{3}$;

2) If $i+2 = j$, *then* $\mathcal{O}(A^{(i,j)}) = \frac{7n!}{15}$;

3) If $i+2 < j$, *then* $\mathcal{O}(A^{(i,j)}) = \frac{4n!}{9}$.

Proof By Theorem 3.3, we may obtain $\mathcal{O}(A^{(i,j)})$ by computing $\mathcal{O}(i+1 < i \text{ and } j+1 < j)$ and other three cases.

1) Suppose $i+1 = j$. Then

(i) $\mathcal{O}(i+1 < i \text{ and } j+1 < j) = \mathcal{O}(i+2 < i+1 < i) = \frac{n!}{3!} = \frac{n!}{6}$;

(ii) $\mathcal{O}(i+1 < i \text{ and } j < j+1 < j+2) = \mathcal{O}(i+1 < i \text{ and } i+1 < i+2 < i+3) = \frac{n!}{4!} \times 3 = \frac{n!}{8}$;

(iii) $\mathcal{O}(j+1 < j \text{ and } i < i+1 < i+2) = \mathcal{O}(i+2 < i+1 \text{ and } i < i+1 < i+2) = 0$;

(iv) $\mathcal{O}(i < i+1 < i+2 \text{ and } j < j+1 < j+2) = \mathcal{O}(i < i+1 < i+2 < i+3) = \frac{n!}{4!} = \frac{n!}{24}$.

Therefore, $\mathcal{O}(A^{(i,i+1)}) = \frac{n!}{6} + \frac{n!}{8} + \frac{n!}{24} = \frac{n!}{3}$.

2) Suppose $i+2 = j$. Then

(i) $\mathcal{O}(i+1 < i \text{ and } j+1 < j) = \mathcal{O}(i+1 < i \text{ and } i+3 < i+2) = \frac{n!}{2! \times 2!} = \frac{n!}{4}$;

(ii) $\mathcal{O}(i+1 < i \text{ and } j < j+1 < j+2) = \mathcal{O}(i+1 < i \text{ and } i+2 < i+3 < i+4) = \frac{n!}{2! \times 3!} = \frac{n!}{12}$;

(iii) $\mathcal{O}(j+1 < j \text{ and } i < i+1 < i+2) = \mathcal{O}(i+3 < i+2 \text{ and } i < i+1 < i+2) = \frac{n!}{4!} \times 3 = \frac{n!}{8}$;

(iv) $\mathcal{O}(i < i+1 < i+2 \text{ and } j < j+1 < j+2) = \mathcal{O}(i < i+1 < i+2 < i+3 < i+4) = \frac{n!}{5!} = \frac{n!}{120}$.

Therefore, $\mathcal{O}(A^{(i,i+2)}) = \frac{n!}{4} + \frac{n!}{12} + \frac{n!}{8} + \frac{n!}{120} = \frac{7n!}{15}$.

3) Suppose $i+2 < j$. Then

(i) $\mathcal{O}(i+1 < i \text{ and } j+1 < j) = \frac{n!}{2! \times 2!} = \frac{n!}{4}$;

(ii) $\mathcal{O}(i+1<i \text{ and } j<j+1<j+2)=\frac{n!}{2!\times 3!}=\frac{n!}{12}$;

(iii) $\mathcal{O}(j+1<j \text{ and } i<i+1<i+2)=\frac{n!}{4!}\times 3=\frac{n!}{12}$;

(iv) $\mathcal{O}(i<i+1<i+2 \text{ and } j<j+1<j+2)=\frac{n!}{3!\times 3!}=\frac{n!}{36}$.

Therefore, $\mathcal{O}(A^{(i,j)})=\frac{n!}{4}+\frac{n!}{12}+\frac{n!}{8}+\frac{n!}{120}=\frac{4n!}{9}$.

Remark If $i+2<j$, there is an interesting relationship between the numbers of quasi-hereditary orderings of A_n-type algebras with two generators and one generator:

$$\frac{\mathcal{O}(A^{(i,j)})}{n!}=(\frac{\mathcal{O}(A^{(i)})}{n!})^2.$$

We guess that the above formula is true in general.

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