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On the Minimum Real Roots of the Adjoint Polynomials of Graphs

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Abstract In this paper, we are concerned with the minimum real root of the adjoint polynomial of the connected graph G with cut-vertex u, in which $G-u$ contains paths, circles or D_n components. Here D_n is the graph obtained from K_3 and path P_{n-2} by identifying a vertex of K_3 with an end-vertex of P_{n-2}. Some relevant ordering relations are obtained. This extends several previous results on the minimum roots of the adjoint polynomials of graphs.

Keywords chromatic polynomial; adjoint polynomial; roots.

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1. Introduction

All the graphs considered here are finite, undirected and simple. Undefined notation and terminology will refer to those in [1]. For a graph G, let $\overline{G}, V(G)$ and $E(G)$, respectively, be the complement, vertex set and edge set of G. For a vertex v of G, we denote by $N_G(v)$ the set of vertices of G which are adjacent to v. Let P_n and $C_n (n \geq 4)$ denote the path and cycle with order n, resp. $D_n (n \geq 4)$ denotes the graph obtained from K_3 and P_{n-2} by identifying a vertex of K_3 with an end-vertex of P_{n-2}, and $F_n (n \geq 6)$ denotes the graph obtained from K_3 and D_{n-2} by identifying a vertex of K_3 with the vertex of degree 1 of D_{n-2}.

The adjoint polynomial was introduced for solving the chromaticity problem of the complements of graphs. For details, one can refer to [2, 6]. Roots and properties of the polynomials related to the chromatic polynomials of graphs have been studied for several years. For example, Brenti, Royle and Wagner studied the roots and log-concavity of the coefficients of the σ-polynomials of graphs [3, 4] (In fact, the adjoint polynomial of the graph G can be considered as the σ-polynomials of the graph $\overline{G}$ in [3]). The ordering relations of the minimum real roots of the adjoint polynomials of graphs can be applied to sort out graphs that are not adjointly equivalent. Recently, by comparing the minimum roots of adjoint polynomials of graphs, many new classes of chromatically unique (chromatically equivalent) graphs have been obtained[10−12].

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But comparing the minimum roots of adjoint polynomials of graphs is not easy. In this paper, we are concerned with the minimum real root of the adjoint polynomial of the connected graph G with cut-vertex u, in which $G-u$ contains paths, circles or D_n components. Some relevant new ordering relations are obtained. This extends several previous results on the minimum roots of the adjoint polynomials of graphs in [10–13].

2. Prelimininaries

In this section, we first introduce some basic definitions and results.

A partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$, where k is a positive integer, is called a k-independent partition of a graph G if each A_i is a nonempty independent set of G. Let $\alpha(G,i)$ denote the number of i-independent partitions of G. Then

$$P(G,\lambda) = \Sigma_{i=1}^{|V(G)|}\alpha(G,i)(\lambda)_i,$$

is called the chromatic polynomial of G, where $(\lambda)_i = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-i+1)$ for each $i \geq 1$[5].

In [2, 6], $N(G,i)$ denotes the number of i-ideal subgraphs of G, that is, the number of ways of partitioning G into i cliques, then $N(G,i) = \alpha(\overline{G},i)$.

Definition 2.1[2,6] *Let G be a graph with p vertices and*

$$P(\overline{G},\lambda) = \Sigma_{i=1}^{p}\alpha(\overline{G},i)(\lambda)_i.$$

Then the polynomial

$$h(G,x) = \Sigma_{i=1}^{p}N(G,i)x^i$$

is called the adjoint polynomial of G.

Let $\beta(h(G,x))$ denote the minimum real root of $h(G,x)$. For brevity we shall write $h(G)$ instead of $h(G,x)$, and $\beta(G)$ instead of $\beta(h(G,x))$. There is at least one real root in $h(G,x)$. Since $h(G,0)=0$ this is in fact obvious, and $\beta(G) \leq 0$ follows.

Let H be a subgraph of G. The graph $G-H$ is obtained from G by deleting the vertices of H and all edges incident to these vertices.

Lemma 2.1[7] *Let $e \in E(G)$ with u and v as end points, writing $H = G * e$ for the graph obtained from G by omitting u, v and adding a new vertex x to G, such that $E(H) = \{xy \mid y \in N_G(u) \cap N_G(v)\} \cup E(G-\{u,v\})$. Then*

$$h(G,x) = h(G-e,x) + h(G*e,x).$$

Corollary 2.1[7] *If vertices u and v are adjacent and edge uv does not belong to any triangle in G. Then*

$$h(G,x) = h(G-uv,x) + xh(G-\{u,v\},x).$$

For convention, let $h(P_0) = 1, h(P_1) = x, h(D_0) = -x, h(D_1) = 2x+1, h(D_2) = h(P_2) = x^2+x, h(D_3) = h(C_3) = h(K_3) = x^3+3x^2+x$.

Lemma 2.2[8] *Let $1 \le r_1 \le r_2, r_1 < s_1 \le s_2$ and $r_1 + r_2 = s_1 + s_2$, where r_1, r_2, s_1, s_2 are the positive integers. Then*

$$h(P_{r_1})h(P_{r_2}) - h(P_{s_1})h(P_{s_2}) = (-1)^{r_1}x^{r_1+1}h(P_{s_1-r_1-1})h(P_{s_2-r_1-1}).$$

Lemma 2.3[13] *Let $4 \le r_1 \le r_2, r_1 < s_1 \le s_2$ and $r_1 + r_2 = s_1 + s_2$, where r_1, r_2, s_1, s_2 are the positive integers. Then*

$$h(D_{r_1})h(D_{r_2}) - h(D_{s_1})h(D_{s_2}) = (-1)^{r_1-1}x^{r_1-3}[x^5 + h(P_4)h(P_2)]h(P_{s_1-r_1-1})h(P_{s_2-r_1-1}).$$

Lemma 2.4 *Let $r_1 + r_2 = s_1 + s_2$, $r_1 \ne s_1$ and $r_2, s_2 \ge 4$, where r_1, r_2, s_1, s_2 are positive integers. Suppose that $\min\{r_1, r_2\} \le \min\{s_1, s_2\}$.*

(i) If $\min\{r_1, r_2\} = r_1$, then

$$h(P_{r_1})h(D_{r_2}) - h(P_{s_1})h(D_{s_2}) = (-1)^{r_1}x^{r_1+1}h(P_{s_1-r_1-1})h(D_{s_2-r_1-1}),$$

where $h(D_{-1}) = x^{-1}(x+1)^2$.

(ii) If $\min\{r_1, r_2\} = r_2$, then

$$h(P_{r_1})h(D_{r_2}) - h(P_{s_1})h(D_{s_2}) = (-1)^{r_2-1}x^{r_2-1}h(P_{s_2-r_2-1})[h(D_{s_1-r_2+3}) - xh(P_{s_1-r_2+1})].$$

Proof By Lemma 2.1 we have

$$h(P_n) = x[h(P_{n-1}) + h(P_{n-2})] \tag{1}$$

and

$$h(D_n) = h(P_n) + h(P_2)h(P_{n-3}). \tag{2}$$

Thus,

$$h(P_{r_1})h(D_{r_2}) - h(P_{s_1})h(D_{s_2}) = h(P_{r_1})h(P_{r_2}) - h(P_{s_1})h(P_{s_2}) + h(P_2)[h(P_{r_1})h(P_{r_2-3}) - h(P_{s_1})h(P_{s_2-3})].$$

(i) If $\min\{r_1, r_2\} = r_1$, note that $s_2 - r_1 = r_2 - s_1$, then by (1), Lemmas 2.1 and 2.2 it is seen that the first assertion holds for $s_2 - r_1 = 0, 1, 2, 3$ or $s_2 - r_1 \ge 4$.

(ii) If $\min\{r_1, r_2\} = r_2$. It is easy to check the second assertion by (1), Lemmas 2.1 and 2.2. The proof is completed. □

Lemma 2.5[12] *Let G be a connected graph and H a proper subgraph of G. Then $\beta(G) < \beta(H)$.*

Let $f(x)$ be a polynomial in x, denote by $\partial(f(x))$ the degree of $f(x)$.

Lemma 2.6[13] *Let $f_i(x)$ be the real coefficient polynomials in the form $f_i(x) = \sum_{j=1}^{n_i} a_{ij}x^j$ for $i = 1, 2$ such that $a_{in_i} > 0$, where $n_i = \partial(f(x_i))$. Suppose that $\beta_1 \ne \beta_2$. If (1) $f_3(x) = f_2(x) + f_1(x)$ and $n_2 - n_1 \equiv 0 \,(\mathrm{mod}2)$ or (2) $f_3(x) = f_2(x) - f_1(x)$ and $n_2 - n_1 \equiv 1 \,(\mathrm{mod}2)$, then there exists at least one real root β_3 such that $\beta_3 > \min\{\beta_1, \beta_2\}$, where β_i denotes the minimum root of $f_i(x)$ for $i = 1, 2, 3$.*

Lemma 2.7[9] *Let G be a graph and $u \in V(G)$. Then $h(G) = x\sum_{u \in V(K_j), j \ge 1} h(G - K_j)$, where the summation is over all the complete subgraphs of G which contain u.*

Let K_{i+1} be the complete subgraph of G. Suppose that $V(K_{i+1}) = V(K_i) \cup \{w_{i+1}\}$. Let $G - K_i + e_{i+1}$ denote the graph obtained from $G - K_i$ by adding a pendant edge e_{i+1} to the vertex w_{i+1}. Since $h(G - K_i + e_i) = xh(G - K_i) + xh(G - K_{i+1})$ by Corollary 2.1. Denote by K_n^u the complete graph of order n which contains the vertex u. Then Lemma 2.7 implies the following corollary 2.2.

Corollary 2.2 *Let $\mathcal{K} = \{K_i^u | i \geq 2, K_i^u \subset G\}$. For each $K_i^u \in \mathcal{K}$, if $i \equiv 1\,(\mathrm{mod}2)$, then set $V(K_i^u) = V(K_{i-1}^u) \cup \{w_i\}$ such that $u \neq w_i$, where $G - K_{i-1}^u + e_i$ denotes the graph obtained from $G - K_{i-1}^u$ by adding a pendant edge e_i to the vertex w_i. So we have*

$$\begin{aligned} & h(G) - xh(G-u) \\ & = x \sum_{K_i^u \in \mathcal{K}, i \equiv 0(\mathrm{mod}2)} h(G - K_i^u) + \sum_{K_i^u \in \mathcal{K}, i \equiv 1(\mathrm{mod}2)} [h(G - K_{i-1}^u + e_{i-1}) - xh(G - K_{i-1}^u)]. \end{aligned}$$

Lemma 2.8[14] *Let u be a cut vertex of graph G. If $G_1 - u$ and $G_2 - u$ are two components of $G - u$ such that $G_1 \cap G_2 = \{u\}$. Then*

$$h(G, x) = h(G_1 - u, x)h(G_2, x) + h(G_1, x)h(G_2 - u, x) - xh(G_1 - u, x)h(G_2 - u, x).$$

3. Main results and proofs

For a connected graph G of order n, pick a vertex $u \in V(G)$. Let s and t be the positive integers.

Theorem 3.1 *Let $H_m(G, P_{s+1}, P_{t+1})$ be the graph with order m obtained from G by identifying u with an end-vertex of P_{s+1} (resp. P_{t+1}), where $m = n + s + t$, $n \geq 2$ and $1 \leq s \leq t$. Then*

$$\beta(H_m(G, P_{s+1}, P_{t+1})) < \beta(H_m(G, P_s, P_{t+2})).$$

Proof By Lemma 2.8 we have

$$H_m(G, P_{s+1}, P_{t+1}) = h(G - u)h(P_{s+t+2}) + [h(G) - xh(G - u)]h(P_s)h(P_t).$$

Since $n \geq 2$ and $1 \leq s \leq t$. Then, by Lemma 2.2, we have

$$H_m(G, P_{s+1}, P_{t+1}) - H_m(G, P_s, P_{t+2}) = (-1)^s x^s h(P_{t-s})[h(G) - xh(G - u)].$$

Considering the parity of s and by Lemmas 2.5, 2.6 and Corollary 2.2, we know that the assertion holds. The proof is completed. □

Theorem 3.2 *Let $H_m(G, C_{s+3}, C_{t+3})$ be the graph with order m obtained from G by identifying u with a vertex of C_{s+3} (resp. C_{t+3}), where $m = n + s + t + 4$, $n \geq 1$ and $2 \leq s \leq t$. Then*

$$\beta(H_m(G, C_{s+3}, C_{t+3})) < \beta(H_m(G, C_{s+2}, C_{t+4})).$$

Proof By Lemma 2.1 we have

$$h(C_n) = h(P_n) + xh(P_{n-2}). \tag{3}$$

Also, by Lemma 2.8

$$H_m(G, C_{s+3}, C_{t+3}) = h(G)h(P_{s+2})h(P_{t+2}) + 2xh(G-u)[h(P_{s+1})h(P_{t+2}) + h(P_{t+1})h(P_{s+2})].$$

Since $2 \le s \le t$. Then, by Lemma 2.2, it is easy to see that

$$\begin{aligned} &H_m(G, C_{s+3}, C_{t+3}) - H_m(G, C_{s+2}, C_{t+4}) \\ &\quad = (-1)^{s+1}x^{s+2}h(P_{t-s})\{xh(G-u) - [h(G) - xh(G-u)]\}, \end{aligned}$$

and the rest is as in Theorem 3.1, so the proof is completed. □

Theorem 3.3 *Let $H_m(G, D_{s+3}, D_{t+3})$ be the graph with order m obtained from G by identifying u with the vertex of degree 1 of D_{s+3} (resp. D_{t+3}), where $m = n+s+t+4$, $n \ge 2$ and $2 \le s \le t$. Then*

$$\beta(H_m(G, D_{s+3}, D_{t+3})) > \beta(H_m(G, D_{s+2}, D_{t+4})).$$

Proof By Lemma 2.8 we have

$$H_m(G, D_{s+3}, D_{t+3}) = h(G-u)h(F_{s+t+5}) + [h(G) - xh(G-u)]h(D_{t+2})h(D_{s+2}).$$

Note that $2 \le s \le t$. Then, by Lemma 2.3, we easily know that

$$\begin{aligned} &H_m(G, D_{s+3}, D_{t+3}) - H_m(G, D_{s+2}, D_{t+4}) \\ &\quad = (-1)^{s+1}x^{s-2}h(P_{t-s})[x^5 + h(P_4)h(P_2)][h(G) - xh(G-u)]. \end{aligned}$$

Since $\beta(F_6) < \beta(x^5 + h(P_4)h(P_2))$([13]). As in Theorem 3.1, also by Lemmas 2.5, 2.6 and Corollary 2.2, we know that the assertion holds.

Theorem 3.4 *Let $H_m(G, P_{s+1}, D_{t+3})$ be the graph with order m obtained from G by identifying u with an end vertex of P_{s+1} and the vertex of degree 1 of D_{t+3}, respectively, where $m = n+s+t+2$, $n \ge 2$, $s \ge 1$ and $t \ge 1$. Then*

$$\beta(H_m(G, P_{s+1}, D_{t+3})) < \beta(H_m(G, P_s, D_{t+4})).$$

Proof By Lemma 2.8 we have

$$H_m(G, P_{s+1}, D_{t+3}) = h(G-u)h(D_{s+t+3}) + [h(G) - xh(G-u)]h(D_{t+2})h(P_s).$$

Note that $s \ge 1$ and $t \ge 1$. Then, by Lemma 2.4

$$\begin{aligned} &H_m(G, P_{s+1}, D_{t+3}) - H_m(G, P_s, D_{t+4}) \\ &= \begin{cases} (-1)^{t+1}x^{t+1}[h(D_{s-t}) - xh(P_{s-t-2})] \cdot [h(G) - xh(G-u)], & \text{if } s \ge t+3; \\ (-1)^s x^s h(D_{t-s+2})[h(G) - xh(G-u)], & \text{Otherwise.} \end{cases} \end{aligned}$$

Considering the parity of s(or t), as in Theorem 3.1 we easily know that the assertion holds.

Theorem 3.5 *Let $H_m(G, P_{s+1}, C_{t+3})$ be the graph with order m obtained from G by identifying u with an end vertex of P_{s+1} and a vertex of C_{t+3}, respectively, where $m = n+s+t+2$, $n \ge 2$, $s \ge 1$ and $t \ge 1$. If $s \le t+1$, then $\beta(H_m(G, P_{s+1}, C_{t+3})) < \beta(H_m(G, P_s, C_{t+4}))$; Otherwise, $\beta(H_m(G, P_{s+1}, C_{t+3})) > \beta(H_m(G, P_s, C_{t+4}))$.*

Proof As in Theorem 3.1, also by Lemmas 2.1 and 2.8 we have

$$H_m(G, P_{s+1}, C_{t+3}) = h(G)h(P_s)h(P_{t+2}) + xh(G-u)[h(P_{s-1})h(P_{t+2}) + 2h(P_{t+1})h(P_s)].$$

Note that $s \geq 1$ and $t \geq 1$. Then, by Lemma 2.2, we distinguish with the following cases:

Case 1 If $s \leq t+1$, then

$$\begin{aligned}&H_m(G, P_{s+1}, C_{t+3}) - H_m(G, P_s, C_{t+4})\\ &\quad= (-1)^s x^s \{h(P_{t-s+2})[h(G) - xh(G-u)] + xh(G-u)h(P_{t-s+1})\}.\end{aligned}$$

Case 2 If $s = t+2$, then

$$H_m(G, P_{s+1}, C_{t+3}) - H_m(G, P_s, C_{t+4}) = (-1)^s x^s [h(G) - xh(G-u)].$$

Case 3 If $s = t+3$, then

$$H_m(G, P_{s+1}, C_{t+3}) - H_m(G, P_s, C_{t+4}) = (-1)^t x^t h(G-u).$$

Case 4 If $s \geq t+4$, then

$$\begin{aligned}&H_m(G, P_{s+1}, C_{t+3}) - H_m(G, P_s, C_{t+4})\\ &\quad= (-1)^t x^{t+3} \{h(P_{s-t-4})[h(G) - xh(G-u)] - h(G-u)h(P_{s-t-3})\}.\end{aligned}$$

By considering the parity of s(or t) in the cases 1-4, as in Theorem 3.1 we easily know that the assertion holds.

In the proof of Theorem 3.5, let $n = 1$. Then we have the following immediate corollary 3.1.

Corollary 3.1 *Let $H_m(P_{s+1}, C_{t+3})$ be the graph obtained from P_{s+1} and C_{t+3} by identifying an end-vertex of P_{s+1} with a vertex of C_{t+3}, where $m = s+t+3$, $s \geq 1$ and $t \geq 1$.*

(i) If $s \leq t+1$, then $\beta(H_m(P_{s+1}, C_{t+3})) < \beta(H_m(P_s, C_{t+4}))$;

(ii) If $s = t+2$, then $\beta(H_m(P_{s+1}, C_{t+3})) = \beta(H_m(P_s, C_{t+4}))$;

(iii) If $s \geq t+3$, then $\beta(H_m(P_{s+1}, C_{t+3})) > \beta(H_m(P_s, C_{t+4}))$.

Theorem 3.6 *Let $H_m(G, C_{s+3}, D_{t+3})$ be the graph with order m obtained from G by identifying u with a vertex of C_{s+3} and the vertex of degree 1 of D_{t+3}, respectively, where $m = n+s+t+4$, $n \geq 1$, $s \geq 2$ and $t \geq 1$. Then*

$$\beta(H_m(G, C_{s+3}, D_{t+3})) < \beta(H_m(G, C_{s+2}, D_{t+4})).$$

Proof By Lemma 2.1 we have

$$h(D_n) = x[h(D_{n-1}) + h(D_{n-2})].$$

As in Theorem 3.1, also by Lemma 2.8 we have

$$H_m(G, C_{s+3}, D_{t+3}) = h(G)h(P_{s+2})h(D_{t+2}) + xh(G-u)[2h(P_{s+1})h(D_{t+2}) + h(D_{t+1})h(P_{s+2})].$$

By Lemma 2.4, we distinguish with the following five cases:

Case 1 If $t \geq s+2$, then

$$H_m(G, C_{s+3}, D_{t+3}) - H_m(G, C_{s+2}, D_{t+4})$$

$$= (-1)^{s+1}x^{s+2}\{h(G-u)h(D_{t-s+1}) - h(D_{t-s})[h(G) - xh(G-u)]\}.$$

Case 2 If $t = s+1$, then

$$\begin{aligned} &H_m(G, C_{s+3}, D_{t+3}) - H_m(G, C_{s+2}, D_{t+4}) \\ &= (-1)^{s+1}x^{s+1}\{[h(P_2) + x^2] \cdot [h(G) - xh(G-u)] - xh(G-u)h(P_2)\}. \end{aligned}$$

Case 3 If $t = s$, then

$$\begin{aligned} &H_m(G, C_{s+3}, D_{t+3}) - H_m(G, C_{s+2}, D_{t+4}) \\ &= (-1)^{s+1}x^{s+1}\{x^2[h(G) - xh(G-u)] + h(G-u)[h(P_2) + x^2]\}. \end{aligned}$$

Case 4 If $t = s-1$, then

$$\begin{aligned} &H_m(G, C_{s+3}, D_{t+3}) - H_m(G, C_{s+2}, D_{t+4}) \\ &= (-1)^{t+1}x^{t}\{[h(G) - xh(G-u)]h^2(P_2) + x^4h(G-u)]\}. \end{aligned}$$

Case 5 If $t \le s-2$, then

$$\begin{aligned} &H_m(G, C_{s+3}, D_{t+3}) - H_m(G, C_{s+2}, D_{t+4}) \\ &= (-1)^{t+1}x^{t+1}\{[h(G) - xh(G-u)] \cdot [h(D_{s-t+2}) - xh(P_{s-t})] + \\ &\quad xh(G-u)[h(D_{s-t+1}) - xh(P_{s-t-1})]\}. \end{aligned}$$

And the rest is as in Theorem 3.5. So the proof is completed. □

References

[1] BONDY J A, MURTY U S R. *Graph Theory with Application* [M]. North-Holland, Amsterdam, 1976.

[2] LIU Ruying. *A new method for finding chromatic polynomial of graphs and its applications* [J]. Kexue Tongbao, 1987, **32**: 1508–1509.

[3] BRENTI F. *Expansions of chromatic polynomials and log-concavity* [J]. Trans. Amer. Math. Soc., 1992, **332**(2): 729–756.

[4] BRENTI F, ROYLE G F, WAGNER D G. *Location of zeros of chromatic and related polynomials of graphs* [J]. Canad. J. Math., 1994, **46**(1): 55–80.

[5] READ R C. *An introduction to chromatic polynomials* [J]. Combinatorial Theory, 1968, **4**: 52–71.

[6] LIU Ruying, ZHAO Lianchang. *A new method for proving chromatic uniqueness of graphs* [J]. Discrete Math., 1997, **171**(1-3): 169–177.

[7] LIU Ruying. *Adjoint polynomials of graphs* [J]. J. Qinghai Normal Univ., 1990, **3**: 1–8.

[8] LIU Ruying. *Chromatic uniqueness of complementary graphs of a kind of tree* [J]. Math. Appl., 1996, **9**(suppl): 170–173.

[9] LIU Ruying. *Some results on the adjoint polynomials of graphs* [J]. J. Qinghai Normal Univ., 1992, **1**: 1–6.

[10] WANG Shouzhong, LIU Ruying. *Chromatic uniqueness of complementary graphs of cycle and D_n* [J]. J. Math. Res. Exposition, 1998, **18**(2): 296.

[11] ZHAO Haixing, HUO Bofeng, LIU Ruying. *Chromaticity of the complements of paths* [J]. J. Math. Study, 2000, **33**(4): 345–353.

[12] ZHAO Haixing, LI Xueliang, ZHANG Shenggui. et al. *On the minimum real roots of the σ-polynomials and chromatic uniqueness of graphs* [J]. Discrete Math., 2004, **281**(1-3): 277–294.

[13] REN Haizhen, Liu Ruying. *On extremes of the minimum real roots of the adjoint polynomials of graphs with $R(G) \ge 1$* [J]. Math. Res. Exposition, 2006, **26**(4): 819–824.

[14] MA Haicheng. *Some new methods for constructing chromatically equivalent graphs* [J]. Appl. Math. J. Chinese Univ. Ser. A, 2004, **19**(2): 135–140. (in Chinese)