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Strong Convergence Theorems for Asymptotically Strictly Pseudocontractive Maps in Hilbert Spaces

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Abstract The Mann iterations have no strong convergence even for nonexpansive mappings in Hilbert spaces. The aim of this paper is to propose a modification of the Mann iterations for strictly asymptotically pseudocontractive maps in Hilbert spaces to have strong convergence. Our results extend those of Kim, Xu[4], Nakajo, Takahashi[3] and many others.

Keywords nonexpansive mappings; strictly pseudocontractive mappings; Hilbert spaces; projection.

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1. Introduction and preliminaries

Mann's iteration process[1] is often used to approximate a fixed point of a nonexpansive mapping. Mann's iteration process is defined as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.1}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval [0, 1].

If T is a nonexpansive mapping with a fixed point and if the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n)=\infty$, then the sequence $\{x_n\}$ generated by Mann's algorithm (1.1) converges weakly to a fixed point of T (This is also valid in a uniformly convex Banach space with a Fréchet differentiable norm[2]).

Attempts to modify the Mann iteration method (1.1) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi[3] proposed the following modification of the Mann iteration (1.1) for a single nonexpansive mapping T in a Hilbert space:

Theorem 1.1 *Let C be a closed convex subset of a Hilbert space H and let $T: C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in [0, 1] such*

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that $\alpha_n \le 1-\delta$ *for some* $\delta \in (0,1]$. *Define a sequence* $\{x_n\}_{n=0}^{\infty}$ *in* C *by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.2}$$

Then $\{x_n\}$ *converges in norm to* $P_{F(T)}x_0$.

Recently, Kim and Xu[4] adapted the iteration (1.2) in Hilbert spaces. They extended the recent one of Nakajo and Takahashi[3] from nonexpansive mappings to asymptotically nonexpansive mappings. More precisely, they gave the following result.

Theorem 1.2[4] *Let* C *be a nonempty bounded closed convex subset of a Hilbert space* H *and let* $T: C \to C$ *be an asymptotically nonexpansive mapping with a sequence* $\{k_n\}$ *such that* $k_n \to 1$ *as* $n \to \infty$. *Assume that* $\{\alpha_n\}_{n=0}^{\infty}$ *is a sequence in [0,1] such that* $\limsup_{n\to\infty} \alpha_n < 1$. *Define a sequence* $\{x_n\}$ *in* C *by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.3}$$

where

$$\theta_n = (1-\alpha_n)(k_n^2 - 1)(\mathrm{diam} C)^2 \to 0, \quad \text{as } n \to \infty.$$

Then $\{x_n\}$ *defined by (1.3) converges strongly to* $P_{F(T)}x_0$.

On the other hand, Halpern iterations process[5] which is defined as

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n, \quad n \ge 0, \tag{1.4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in the interval [0,1], is also usually used to approximate a fixed point of nonexpansive mappings. The iteration process (1.4) has been proved to be strongly convergent in both Hilbert spaces[5−7] and uniformly smooth Banach spaces[8−10] provided that the sequence $\{\alpha_n\}$ satisfies the conditions (C_1): $\alpha_n \to 0$; (C_2): $\sum_{n=0}^{\infty} \alpha_n = \infty$ and (C_3): either $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}|$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$. It is well known that the iterative process (1.4) is widely believed to have slow convergence because of the restriction of condition (C_2). Moreover, Halpern[5] proved that condition (C_1) and (C_2) are indeed necessary in the sense that if process (1.4) is strongly convergent for all closed convex subsets C of a Hilbert space H and all nonexpansive mappings T on C, then the sequence $\{\alpha_n\}$ must satisfy conditions (C_1) and (C_2). Thus to improve the rate of convergence of process (1.4), one cannot rely only on the process itself. In [11], Martinez-Yanes and Xu proved the following theorem:

Theorem 1.3 *Let* H *be a real Hilbert space,* C *a closed convex subset of* H *and* $T: C \to C$ *a nonexpansive mapping such that* $F(T) \neq \emptyset$. *Assume that* $\alpha_n \subset (0,1)$ *is chosen such that*

$\lim_{n\to\infty} \alpha_n = 0$. *Then the sequence* $\{x_n\}_{n=0}^{\infty}$ *generated by*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases}$$

converges strongly to $P_{F(T)}x_0$.

In [12], Qin and Su extended the results of Mantinez-Yanes and Xu[11] from Hilbert spaces to Banach spaces by using generalized projection operators. Recently, Kim and Xu[13] introduced another modification of Mann's iteration method which is a convex combination of a fixed point in subset C of a Banach space E and the Mann's iteration method (1.1) to get a strong convergence theorem for nonexpansive mappings. More precisely, they proved the following theorem:

Theorem 1.4 *Let C be a closed convex subset of a uniformly smooth Banach space X and let $T : C \to C$ be a nonexpansive mapping such that the set of fixed points $F(T) \neq \emptyset$. Given a point $u \in C$ and given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$, the following conditions are satisfied:*

(i) $\alpha_n \to 0$, $\beta_n \to 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$. *Define a sequence $\{x_n\}$ in C by*

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ x_{n+1} = \beta_n u + (1-\beta_n)y_n. \end{cases}$$

Then $\{x_n\}$ strongly converges to a fixed point of T.

The purpose of this paper is to combine Nakajo and Takahashi[3] with Kim and Xu[13]'s idea to modify Mann iterative process (1.1) for asymptotically k-strictly pseudocontractive mappings and k-strictly pseudocontractive mappings, respectively to have strong convergence theorems in Hilbert spaces without any compactness on T. Our results improve and extend the recent ones announced by Nakajo and Takahashi[3], Kim and Xu[13] and some others.

Let K be a nonempty subset of a Hilbert space H. Recall that a mapping $T : K \to K$ is said to be asymptotically k-strictly pseudocontractive (The class of asymptotically k-strictly pseudocontractive maps was first introduced in Hilbert spaces by Liu[14].) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\|^2 \le k_n^2 \|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2, \tag{1.5}$$

for some $k \in [0, 1)$, for all x, $y \in K$ and $n \in \mathbb{N}$.

Note that the class of asymptotically k-strictly pseudocontractive mappings strictly includes the class of asymptotically nonexpansive mappings[15] which are mappings T on K such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad \text{for all} \quad x, y \in K,$$

where the sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$. That is, T is asymptotically nonexpansive if and only if T is asymptotically 0-strictly pseudocontractive.

Recall that a mapping $T : K \to K$ is said to be asymptotically demicontractive (The class of asymptotically demicontractive maps was first introduced in Hilbert spaces by Liu[14].) if the set of fixed point of T, that is, $F(T)$, is nonempty and if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - p\|^2 \le k_n^2 \|x-p\|^2 + k\|x - T^n x\|^2, \tag{1.6}$$

for some $k \in [0,1)$, $\forall p \in F(T)$, for all $x \in K$ and $n \in \mathbb{N}$.

Recall that a mapping $T : K \to K$ is said to be strictly pseudocontractive if there exists a constant $0 \le k < 1$ such that

$$\|Tx - Ty\|^2 \le \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2, \tag{1.7}$$

for all x, $y \in K$. (If (1.7) holds, we also say that T is a k-strictly pseudocontractive map.)

Note that the class of k-strictly pseudocontractive maps strictly includes the class of nonexpansive mappings which are mappings T on K such that

$$\|Tx - Ty\| \le \|x-y\|, \quad \text{for all} \quad x, y \in K.$$

That is, T is nonexpansive if and only if T is 0-strictly pseudocontractive.

In order to prove our main results, we need the following Lemmas.

Lemma 1.1 *Let H be a real Hilbert space. There hold the following identities:*

(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$, $\forall x, y \in H$.

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$, $\forall t \in [0,1]$, $\forall x, y \in H$.

Lemma 1.2[16] *Let H be a real Hilbert space. Let K be a nonempty closed convex subset of E and $T : K \to K$ a asymptotically k-strictly pseudocontractive mapping with a nonempty fixed point set. Then $(I - T)$ is demiclosed at zero.*

Lemma 1.3 *Let H be a real Hilbert space, K a nonempty subset of E and $T : K \to K$ a asymptotically k-strictly pseudocontractive mapping. Then T is uniformly L-Lipschitzian.*

Proof It follows from the definition of asymptotically k-strictly pseudocontractive mappings that

$$\begin{aligned}\|T^n x - T^n y\|^2 \le& k_n^2\|x-y\|^2 + k\|(x - T^n x) - (y - T^n x)\|^2\\ \le& (k_n\|x-y\| + \sqrt{k}\|(x - T^n x) - (y - T^n x)\|)^2.\end{aligned}$$

That is,

$$\begin{aligned}\|T^n x - T^n y\| \le& k_n\|x-y\| + \sqrt{k}\|(x - T^n x) - (y - T^n x)\|\\ \le& k_n\|x-y\| + \sqrt{k}\|x-y\| + \sqrt{k}\|T^n x - T^n y\|,\end{aligned}$$

which yields that

$$\|T^n x - T^n y\| \le \frac{k_n + \sqrt{k}}{1-\sqrt{k}}\|x-y\|.$$

Since $\{k_n\}$ is bounded, we have $k_n \le M$ for all $n \ge 0$ and for some $M > 0$. Therefore, we obtain

$$\|T^n x - T^n y\| \le L\|x - y\|,$$

where $L = \frac{M+\sqrt{k}}{1-\sqrt{k}}$. This completes the proof. □

Lemma 1.4[16] *Let H be a real Hilbert space, K a nonempty subset of H and $T : K \to K$ a asymptotically k-strictly pseudocontractive mapping. Then the fixed points set $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well defined.*

2. Main results

Theorem 2.1 *Let C be a closed bounded convex subset of a Hilbert space H and let $T : C \to C$ be a asymptotically k-strictly pseudocontractive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ z_n = \beta_n x_n + (1-\beta_n)T^n x_n, \\ y_n = \alpha_n x_0 + (1-\alpha_n) z_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (k_n^2 - 1)(1-\alpha_n)M + \\ \qquad \alpha_n(\|x_0\|^2 - \|x_n\|^2 + 2\langle x_n - x_0, z\rangle) + \\ \qquad (k-\beta_n)(1-\beta_n)(1-\alpha_n)\|T^n x_n - x_n\|^2 - \alpha_n(1-\alpha_n)\|z_n - x_0\|^2\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where M is a constant such that $M \ge \|x_n - p\|^2$ for any $p \in F(T)$. Assume that the control sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are chosen such that $\lim_{n\to\infty} \alpha_n = 0$ and $\beta_n \in [0, a]$ for some $a \in [0, 1)$. Then $\{x_n\}$ converges in norm to $P_{F(T)} x_0$.

Proof We first show that C_n and Q_n are closed and convex for each $n \ge 0$. From the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \ge 0$. We prove that C_n is convex. Since

$$\begin{aligned} \|y_n - z\|^2 \le & \|x_n - z\|^2 + (k_n^2 - 1)(1-\alpha_n)M + \alpha_n(\|x_0\|^2 - \|x_n\|^2 + 2\langle x_n - x_0, z\rangle) + \\ & (k-\beta_n)(1-\beta_n)(1-\alpha_n)\|T^n x_n - x_n\|^2 - \alpha_n(1-\alpha_n)\|z_n - x_0\|^2 \end{aligned} \tag{2.1}$$

is equivalent to

$$\begin{aligned} & \langle 2(1-\alpha_n)x_n - 2y_n - 2\alpha_n x_0, z\rangle \\ & \le \|x_n\|^2 - \|y_n\|^2 + (k_n^2 - 1)(1-\alpha_n)M + \alpha_n(\|x_0\|^2 - \|x_n\|^2) + \\ & \quad (k-\beta_n)(1-\beta_n)(1-\alpha_n)\|Tx_n - x_n\|^2 - \alpha_n(1-\alpha_n)\|z_n - x_0\|^2. \end{aligned} \tag{2.2}$$

So, C_n is convex. Next, we show that $F(T) \subset C_n$ for all n. Indeed, we have, for all $p \in F(T)$

$$\begin{aligned} \|y_n - p\|^2 &= \|\alpha_n(x_0 - p) + (1-\alpha_n)(z_n - p)\|^2 \\ &= \alpha_n\|x_0 - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 - \alpha_n(1-\alpha_n)\|z_n - x_0\|^2 \\ &= \alpha_n\|x_0 - p\|^2 + (1-\alpha_n)\|\beta_n(x_n - p) + (1-\beta_n)(T^n x_n - p)\|^2 - \end{aligned}$$

$$\begin{aligned}
&\alpha_n(1-\alpha_n)\|z_n-x_0\|^2\\
\le{}& \alpha_n\|x_0-p\|^2+(1-\alpha_n)\beta_n\|x_n-p\|^2+\\
&(1-\alpha_n)(1-\beta_n)\|T^nx_n-p\|^2-\beta_n(1-\beta_n)(1-\alpha_n)\|T^nx_n-x_n\|^2-\\
&\alpha_n(1-\alpha_n)\|z_n-x_0\|^2\\
\le{}& \|x_n-p\|^2+(1-\alpha_n)(k_n^2-1)M+\alpha_n(\|x_0\|^2-\|x_n\|^2+2\langle x_n-x_0,p\rangle)+\\
&(1-\alpha_n)(1-\beta_n)(k-\beta_n)\|T^nx_n-x_n\|^2-\alpha_n(1-\alpha_n)\|z_n-x_0\|^2.
\end{aligned}$$

So $p\in C_n$ for all n. Next we show that

$$F(T)\subset Q_n,\quad \text{for all}\quad n\ge 0. \tag{2.3}$$

We prove this by induction. For $n=0$, we have $F(T)\subset C=Q_0$. Assume that $F(T)\subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n\cap Q_n$, by Lemma 1.2 we have $\langle x_0-x_{n+1},x_{n+1}-z\rangle\ge 0$, $\forall\, z\in C_n\cap Q_n$. As $F(T)\subset C_n\cap Q_n$ by the induction assumptions, the last inequality holds, in particular, for all $z\in F(T)$. This together with the definition of Q_{n+1} implies that $F(T)\subset Q_{n+1}$. Hence (2.3) holds for all $n\ge 0$. In order to prove $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, from the definition of Q_n we have $x_n=P_{Q_n}x_0$ which together with the fact that $x_{n+1}\in C_n\cap Q_n\subset Q_n$ implies that $\|x_0-x_n\|\le\|x_0-x_{n+1}\|$. This shows that the sequence $\|x_n-x_0\|$ is nondecreasing. Since C is bounded, we obtain that $\lim_{n\to\infty}\|x_n-x_0\|$ exists. Noticing again that $x_n=P_{Q_n}x_0$ and $x_{n+1}\in Q_n$, which give that $\langle x_{n+1}-x_n,x_n-x_0\rangle\ge 0$, we have

$$\begin{aligned}
\|x_{n+1}-x_n\|^2&=\|(x_{n+1}-x_0)-(x_n-x_0)\|^2\\
&=\|x_{n+1}-x_0\|^2-\|x_n-x_0\|^2-2\langle x_{n+1}-x_n,x_n-x_0\rangle\\
&\le\|x_{n+1}-x_0\|^2-\|x_n-x_0\|^2.
\end{aligned}$$

It follows that

$$\|x_n-x_{n+1}\|\to 0,\quad \text{as}\quad n\to\infty. \tag{2.4}$$

On the other hand, it follows from $x_{n+1}\in C_n$ that

$$\begin{aligned}
\|y_n-x_{n+1}\|^2\le{}&\|x_n-x_{n+1}\|^2+(1-\alpha_n)(k_n^2-1)M+\\
&\alpha_n(\|x_0\|^2-\|x_n\|^2+2\langle x_n-x_0,x_{n+1}\rangle)+\\
&(1-\alpha_n)(1-\beta_n)(k-\beta_n)\|T^nx_n-x_n\|^2-\alpha_n(1-\alpha_n)\|z_n-x_0\|^2.
\end{aligned}\tag{2.5}$$

Observing that

$$y=\alpha_nx_0+(1-\alpha_n)z_n, \tag{2.6}$$

we have

$$\|y_n-x_{n+1}\|^2=\alpha_n\|x_0-x_n\|^2+(1-\alpha_n)\|z_n-x_{n+1}\|^2-\alpha_n(1-\alpha_n)\|z_n-x_0\|^2. \tag{2.7}$$

Combining (2.5) and (2.7), we obtain

$$\begin{aligned}
(1-\alpha_n)\|z_n-x_{n+1}\|^2\le{}&(1-\alpha_n)\|x_n-x_{n+1}\|^2+(1-\alpha_n)(k_n^2-1)M+\\
&(k-\beta_n)(1-\beta_n)(1-\alpha_n)\|T^nx_n-x_n\|^2.
\end{aligned}$$

Since $\alpha_n < 1$, we obtain

$$\|z_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + (k_n^2 - 1)M + (k - \beta_n)(1-\beta_n)\|T^n x_n - x_n\|^2. \qquad (2.8)$$

Similarly, observing that $z_n = \beta_n x_n + (1-\beta_n)T^n x_n$, we have that

$$\|z_n - x_{n+1}\|^2 = \beta_n\|x_n - x_{n+1}\|^2 + (1-\beta_n)\|T^n x_n - x_{n+1}\|^2 - \beta_n(1-\beta_n)\|T^n x_n - x_n\|^2. \quad (2.9)$$

Combining (2.8) and (2.9), we have

$$\begin{aligned}&\beta_n\|x_n - x_{n+1}\|^2 + (1-\beta_n)\|T^n x_n - x_{n+1}\|^2 - \beta_n(1-\beta_n)\|T^n x_n - x_n\|^2 \\ &\leq \|x_n - x_{n+1}\|^2 + (k_n^2 - 1)M + (k-\beta_n)(1-\beta_n)\|T^n x_n - x_n\|^2.\end{aligned}$$

Since $\beta_n \in [0, a]$ for some $a \in [0, 1)$ and $\lim_{n\to\infty} k_n = 1$, we obtain

$$\|Tx_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + k\|T^n x_n - x_n\|^2 + \frac{(k_n^2 - 1)M}{1-\beta_n}. \qquad (2.10)$$

On the other hand, we have

$$\begin{aligned}\|T^n x_n - x_{n+1}\|^2 =& \|T^n x_n - x_n + x_n - x_{n+1}\|^2 \\ =& \|T^n x_n - x_n\|^2 + \|x_n - x_{n+1}\|^2 + 2\langle T^n x_n - x_n, x_n - x_{n+1}\rangle. \qquad (2.11)\end{aligned}$$

Substituting (2.11) into (2.10), we obtain

$$(1-k)\|T^n x_n - x_n\|^2 \leq \frac{(k_n^2 - 1)M}{1-\beta_n} + 2\|T^n x_n - x_n\|\|x_n - x_{n+1}\|.$$

It follows from (2.4) and $k < 1$ that $\lim_{n\to\infty}\|T^n x_n - x_n\| = 0$. Observe that

$$\begin{aligned}\|Tx_n - x_n\| \leq& \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \\ &\|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|.\end{aligned}$$

Since T is uniformly L-Lipschitzian, we obtain $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Assume that $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \tilde{x}$. by Lemma 1.3 we have $\tilde{x} \in F(T)$. Next we show that $\tilde{x} = P_{F(T)}x_0$ and convergence is strong. Put $\bar{x} = P_{F(T)}x_0$ and consider the sequence $\{x_0 - x_{n_i}\}$. Then we have $x_0 - x_{n_i} \rightharpoonup x_0 - \tilde{x}$. By the weak lower semicontinuity of the norm and by the fact that $\|x_0 - x_{n+1}\| \leq \|x_0 - \bar{x}\|$ for all $n \geq 0$, which is implied by the fact that $x_{n+1} = P_{C_n\cap Q_n}x_0$, we have

$$\|x_0 - \bar{x}\| \leq \|x_0 - \tilde{x}\| \leq \liminf_{i\to\infty}\|x_0 - x_{n_i}\| \leq \limsup_{i\to\infty}\|x_0 - x_{n_i}\| \leq \|x_0 - \bar{x}\|,$$

which gives that $\|x_0 - \bar{x}\| = \|x_0 - \tilde{x}\|$ and $\|x_0 - x_{n_i}\| \to \|x_0 - \bar{x}\|$. It follows that $x_0 - x_{n_i} \to x_0 - \bar{x}$. Hence, we have $x_{n_i} \to \bar{x}$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we conclude that $x_n \to \bar{x}$. The proof is completed. □

Theorem 2.2 *Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be a k-strictly pseudocontractive maps and assume that the fixed point set $F(T)$ of T is nonempty.*

Define a sequence $\{x_n\}_{n=0}^{\infty}$ *in* C *by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ z_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ y_n = \alpha_n x_0 + (1-\alpha_n)z_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \alpha_n(\|x_0\|^2 - \|x_n\|^2 + 2\langle x_n - x_0, z\rangle) + \\ \qquad (k-\beta_n)(1-\beta_n)(1-\alpha_n)\|Tx_n - x_n\|^2 - \alpha_n(1-\alpha_n)\|z_n - x_0\|^2\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases}$$

Assume that the control sequences $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *are chosen such that* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\beta_n < 1$. *Then* $\{x_n\}$ *converges in norm to* $P_{F(T)}x_0$.

Proof Taking the sequence $\{k_n\} = 1$ and from the proof of Theorem 2.1, we can get the desired conclusion easily.

As corollaries of Theorems 2.1 and 2.2, we have the following results.

Corollary 2.3[11] *Let* H *be a real Hilbert space,* C *a closed convex subset of* H *and* $T : C \to C$ *a nonexpansive mapping such that* $F(T) \ne \emptyset$. *Assume that* $\{\alpha_n\} \subset (0,1)$ *is chosen such that* $\lim_{n\to\infty}\alpha_n = 0$. *Then the sequence* $\{x_n\}_{n=0}^{\infty}$ *generated by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases}$$

converges strongly to $P_{F(T)}x_0$.

Proof Note that the class of k-strictly pseudocontractive maps strictly includes the class of nonexpansive mappings. That is, T is a nonexpansive mapping if and only if T is a 0-strictly pseudocontractive mapping. By using Theorem 2.2, we can obtain the desired conclusion immediately. This completes the proof.

Corollary 2.4 *Let* H *be a real Hilbert space,* C *a bounded closed convex subset of* H *and* $T : C \to C$ *an asymptotically nonexpansive mapping. Assume that* $\{\alpha_n\} \subset (0,1)$ *is chosen such that* $\lim_{n\to\infty}\alpha_n = 0$. *The sequence* $\{x_n\}_{n=0}^{\infty}$ *generated by*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (k_n^2 - 1)(1-\alpha_n)M \\ \quad + \alpha_n(\|x_0\|^2 - \|x_n\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where M *is a constant such that* $M \ge \|x_n - p\|^2$ *for any* $p \in F(T)$, *converges strongly to* $P_{F(T)}x_0$.

Proof Note that the class of asymptotically k-strictly pseudocontractive maps strictly includes the class of asymptotically nonexpansive mappings. That is, T is an asymptotically nonexpansive mapping if and only if T is a asymptotically 0-strictly pseudocontractive mapping. By using Theorem 2.1, we can obtain the desired conclusion easily. This completes the proof. □

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