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The Separation and N-Compactness of Induced $R(L)$-Fuzzy Topological Spaces

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Abstract In this paper, we prove that (L^X, δ) is T_0, T_1, T_2, regular (T_3), normal (T_4) and completely regular spaces if and only if $(R(L)^X, \omega(\delta))$ is T_0, T_1, T_2, regular (T_3), normal (T_4) and completely regular spaces, respectively, and (L^X, δ) is N-compact if and only if $(R(L)^X, \omega(\delta))$ is N-compact.

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1. Introduction

The induced fuzzy topological space plays an important role in fuzzy topological spaces. For a topological space $(X, \mathcal{T})$, the all of L-valued lower semicontinuous mappings form LF-topology on L^X, $(L^X, \omega_L(\mathcal{T}))$ is called the induced fuzzy topological space[2] of the topological space $(X, \mathcal{T})$. The notion of induced fuzzy topological spaces was extended to the case of $R(L)$-fuzzy topological spaces[4] by using the $R(l)$-valued lower semicontinuous mappings. In this way, to every L-fuzzy topological space (L^X, δ) one can assign a unique induced $R(L)$-fuzzy topological space $(R(L)^X, \omega(\delta))$. As Lowen[6] proposed that a property P in fuzzy topology is called "a good extension" of a property P' in general topology if $(X, \mathcal{T})$ has P' if and only if $(L^X, \omega_L(\mathcal{T}))$ has P. For induced $R(L)$-fuzzy topological space, an interesting question is what properties of $(R(L)^X, \omega(\delta))$ is "a good extension". In this paper, we discuss the separation and N-compactness of induced $R(L)$-fuzzy topological spaces.

Throughout this paper L denotes a fuzzy lattice, a completely distributive lattice with an order-reversing involution, and $M(L)$ denotes the set of all molecule in L. We refer to [2, 3, 4] for some notions and symbols.

2. The separation of induced $R(L)$-fuzzy topological space

Let R be real line. Define a mapping $\lambda : R \to L$ satisfying $\lambda(s) \geq \lambda(t)$ when $s \leq t$ for each

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$s, t \in R$. We denote all of such mapping by Σ, and for each $\lambda \in \Sigma, t \in R$ let

$$\lambda(t+) = \vee\{\lambda(s) | s > t\}, \quad \lambda(t-) = \wedge\{\lambda(s) | s < t\}.$$

For each $\lambda,\ \mu \in \Sigma$, define $\lambda \sim \mu$ if and only if $\lambda(t+) = \mu(t+)$ and $\lambda(t-) = \mu(t-)$ for every $t \in R$. Obviously, $\sim$ is an equivalence relation. Let $R(L) = \Sigma/\sim$. For every $\lambda \in R(L),\ t \in R$ define

$$L_t([\lambda]) = \lambda(t-)', \quad R_t([\lambda]) = \lambda(t+).$$

An induced $R(L)$-fuzzy topological space of (L^X, δ) is a pair $(R(L)^X, \omega(\delta))$, where $\omega(\delta) = \{\mu \in R(L)^X \mid \sigma_t(\mu) \in \delta,\ t \in R\}$, $\sigma_t(\mu) = R_t \circ \mu = \mu(t+)$, $\omega_t(\mu) = L_t \circ \mu = \mu(t-)$.

Define a mapping $* : L^X \to R(L)^X$ by letting $\gamma^*(x)(t+) = \gamma(x)$ for each $\gamma \in L^X, x \in X, t \in R$. Moreover, for each $t \in R$, $\alpha \in L$, let

$$\lambda_{\alpha,t}(s+) = \begin{cases} \alpha, & s < t, \\ 0, & s \geq t. \end{cases}$$

Theorem 2.1 *Let $\lambda \in R(L)$. Then λ is a molecule of $R(L)$ if and only if there exists a molecule $\alpha \in L$ and $t \in R$ such that $\lambda = \lambda_{\alpha,t}$.*

Proof Let $\alpha \in M(L)$, $t \in R$, and $\mu, \nu \in R(L)$. Suppose that $\lambda_{\alpha,t} \leq \mu \vee \nu$, we have $\alpha = \lambda_{\alpha,t}(t-) \leq \mu(t-) \vee \nu(t-)$ by the definition of $\lambda_{\alpha,t}$. Since α is a molecule, we have $\alpha \leq \mu(t-)$ or $\alpha \leq \nu(t-)$. Without loss of generality we assume $\alpha \leq \mu(t-)$. Note that μ is decreasing, we have $\lambda_{\alpha,t}(s-) \leq \mu(s-)$ for any $s \in R$. In the same way we can prove $\lambda_{\alpha,t}(s+) \leq \mu(s+)$ for any $s \in R$. Therefore $\lambda_{\alpha,t} \leq \mu$, which implies $\lambda_{\alpha,t}$ is a molecule of $R(L)$.

Conversely, assume that λ is a molecule of $R(L)$. Without loss of generality, we assume that λ is left continuous and $\lambda \neq 0$. If there exist $t_1,\ t_2 \in R$ with $t_1 < t_2$ and $\alpha,\ \beta \in L - \{0, 1\}$ with $\alpha > \beta$ such that $\lambda(t_1) = \alpha$ and $\lambda(t_2) = \beta$, we shall show that it is impossible.

(1) If there exists a discontinuous point $t_0 \in [t_1, t_2)$, then $\lambda(t_0) = \lambda(t_0-) > \lambda(t_0+)$. Let

$$\mu(t) = \begin{cases} \lambda(t), & t \leq t_0, \\ 0, & t > t_0 \end{cases}$$

and

$$\nu(t) = \begin{cases} \lambda(t_0+), & t \leq t_0, \\ \lambda(t), & t > t_0. \end{cases}$$

Obviously, $\lambda = \mu \vee \nu$ and $\lambda \neq \mu$, $\lambda \neq \nu$, which contradicts with the fact that λ is a molecule of $R(L)$.

(2) If λ is continuous at each point t with $t \in [t_1, t_2)$, then $\lambda(t_2-) = \lambda(t_2) = \beta < \alpha = \lambda(t_1+)$. Thus for any $\gamma \in (\beta, \alpha)$, there exists $t_3 \in (t_1, t_2)$ such that $\lambda(t_3) = r$. Let

$$\mu(t) = \begin{cases} \lambda(t), & t \leq t_3, \\ 0, & t > t_3 \end{cases}$$

and

$$\nu(t) = \begin{cases} \gamma, & t \leq t_3, \\ \lambda(t), & t > t_3. \end{cases}$$

Then we have $\lambda = \mu \vee \nu$ and $\lambda \neq \mu$, $\lambda \neq \nu$, which is a contradiction. This implies that there exist $\alpha \in L$ and $t \in R$ such that $\lambda = \lambda_{\alpha,t}$. It is easy to show that α is a molecule in L when $\lambda_{\alpha,t}$ is molecule in $R(L)$.

Lemma 2.2 *(1) Let $\mu \in R(L)^X$ and $\alpha \in M(L)$. Then $\tau_\alpha(\omega_t(\mu)) = \tau_{\lambda_{\alpha,t}}(\mu)$. Where $\tau_r(A) = \{x \mid A(x) \geq r\}$.*

(2) Let (L^X, δ) be a L-fts, $A \in L^X$, $\alpha \in M(L)$ and $t \in R$. Then $\iota_{\alpha'}(A) = \iota_{\lambda'_{\alpha,t}}(A^*)$. Where $\iota_\alpha(A) = \{x \mid A(x) \not\leq \alpha\}$.

Proof (1) For any $y \in \tau_\alpha(\omega_t(\mu))$, we have $\omega_t(\mu)(y) = \mu(y)(t-) \geq \alpha = \lambda_{\alpha,t}(t-)$. Thus $\mu(y) \geq \lambda_{\alpha,t}$, which implies $y \in \tau_{\lambda_{\alpha,t}}(\mu)$.

Conversely, for any $y \in \tau_{\lambda_{\alpha,t}}(\mu)$, we have $\mu(y) \geq \lambda_{\alpha,t}$. Then $\mu(y)(t-) \geq \lambda_{\alpha,t}(t-) = \alpha$, which implies $y \in \tau_\alpha(\omega_t(\mu))$.

(2) The proof is analogous to (1). □

Lemma 2.3[2] *Let $(L^X, \omega_L(\mathcal{T}))$ be the induced fuzzy topological space of $(X, \mathcal{T})$. Then $A \in \omega_L(\mathcal{T})$ if and only if $\xi_p(A) = \{x \in X \mid A(x) \leq p\} \in \mathcal{T}'$ for each prime element $p \in L$.*

Lemma 2.4 *Let $(L^X, \omega_L(\mathcal{T}))$ be an induced fuzzy topological space of $(X, \mathcal{T})$. Then $(R(L)^X, \omega(\omega_L(\mathcal{T})))$ is also an induced fuzzy topological space of $(X, \mathcal{T})$.*

Proof Assume that $\mu' \in \omega(\omega_L(\mathcal{T}))$. Then $\omega_t(\mu) \in \omega_L(\mathcal{T})'$ for any $t \in R$. By Lemma 2.3, $\tau_\alpha(\omega_t(\mu)) = \tau_{\lambda_{\alpha,t}}(\mu) \in \mathcal{T}'$ for each $\lambda_{\alpha,t} \in M(R(L)^X)$. Thus $\mu \in \omega_{R(L)}(\mathcal{T})'$, i.e, $\mu' \in \omega_{R(L)}(\mathcal{T})$.

Conversely, suppose that $\mu' \in \omega_{R(L)}(\mathcal{T})$. By Lemma 2.3, $\tau_\alpha(\omega_t(\mu)) = \tau_{\lambda_{\alpha,t}}(\mu) \in \mathcal{T}'$ for each $\alpha \in M(L)$. Thus $\omega_t(\mu) \in \omega_L(\mathcal{T})'$, that is, $\mu \in \omega(\omega_L(\mathcal{T}))'$.

Lemma 2.5 *Let $((R(L)^X, \omega(\delta))$ be an induced fuzzy topological space of $(X, \mathcal{T})$. Then (L^X, δ) is also an induced fuzzy topological space of $(X, \mathcal{T})$.*

Proof Assume that $\mu \in \delta$. Then $\mu^* \in \omega(\delta) = \omega_{R(L)}(\mathcal{T})$. By Lemmas 2.2 and 2.3, we have $\iota_{\alpha'}(\mu) = \iota_{\lambda'_{\alpha,t}}(\mu^*)$ for each $t \in R$ and $\alpha \in M(L)$, and $\iota_{\lambda'_{\alpha,t}}(\mu^*) = \{x \in X \mid \mu'(x) \not\leq \lambda'_{\alpha,t}\} \in \mathcal{T}$, that is, $\iota_{\alpha'}(\mu) \in \mathcal{T}$. Thus $\delta \subseteq \omega_L(\mathcal{T})$.

Conversely, suppose that $\mu \in \omega_L(\mathcal{T})$. Then $\mu^* \in \omega(\omega_L(\mathcal{T})) = \omega_{R(L)}(\mathcal{T}) = \omega(\delta)$. Thus $\delta \supseteq \omega_L(\mathcal{T})$ for each $t \in R$.

Summarizing Lemmas 2.4 and 2.5 we have

Theorem 2.6 *Let (L^X, δ) be an L-fts. Then (L^X, δ) is an induced fuzzy topological space of $(X, \mathcal{T})$ if and only if $(R(L)^X, \omega(\delta))$ is an induced fuzzy topological space of $(X, \mathcal{T})$.*

Next, we consider the separation of (L^X, δ) and $(R(L)^X, \omega(\delta))$, and refer to [2] for some relative notions and results.

Theorem 2.7 *Let (L^X, δ) be an L-fts. Then (L^X, δ) is a Hausdorff space if and only if $(R(L)^X, \omega(\delta))$ is a Hausdorff space.*

Proof Assume (L^X, δ) is a Hausdorff space. For each $x_{\lambda_{\alpha,t}},\ y_{\lambda_{\beta,s}} \in M(R(L)^X)$ with $x \neq y$, we have $x_\alpha,\ y_\beta \in M(L^X)$. Since (L^X, δ) is a Hausdorff space, there exist $P \in \eta^-(x_\alpha)$ and $Q \in \eta^-(y_\beta)$ such that $P \vee Q = 1$. By the definition $*$ and Theorem 2.1, we have $\lambda_{\alpha,t} \not\leq P^*(x)$, $\lambda_{\beta,s} \not\leq Q^*(y)$ and $P^*(x) \vee Q^*(x) = 1$ for each $x \in X$. Therefore, $(R(L)^X, \omega(\delta)$ is a Hausdorff space.

Conversely, suppose that $(R(L)^X, \omega(\delta))$ is a Hausdorff space. For each $x_\alpha,\ y_\beta \in M(L^X)$ with $x \neq y$, we have $x_{\lambda_{\alpha,t}},\ y_{\lambda_{\beta,t}} \in M(R(L)^X)$ for each $t \in R$. Since $(R(L)^X, \omega(\delta))$ is a Hausdorff space, there exist $P \in \eta^-(x_{\lambda_{\alpha,t}})$ and $Q \in \eta^-(y_{\lambda_{\beta,t}})$ such that $P \vee Q = 1$, that is, $\lambda_{\alpha,t} \not\leq P(x)$, $\lambda_{\beta,t} \not\leq Q(y)$ and $P(x) \vee Q(x) = 1$ for each $x \in X$. Thus there exist $r \leq t$, such that $\alpha = \lambda_{\alpha,t}(r-) \not\leq P(x)(r-) = \omega_r(P)(x)$ and $\beta = \lambda_{\beta,t}(r-) \not\leq Q(y)(r-) = \omega_r(Q)(y)$, which implies $\omega_r(P) \in \eta^-(x_\alpha)$, $\omega_r(Q) \in \eta^-(y_\beta)$ and $\omega_r(P) \vee \omega_r(Q) = \omega_r(P \vee Q) = 1$. Hence (L^X, δ) is a Hausdorff space.

Theorem 2.8 *Let (L^X, δ) be an L-fts. Then*

(1) (L^X, δ) is a T_0-space if and only if $(R(L)^X, \omega(\delta))$ is a T_0-space.

(2) (L^X, δ) is a T_1-space if and only if $(R(L)^X, \omega(\delta))$ is a T_1-space.

Theorem 2.9 *Let (L^X, δ) be an L-fts. Then*

(1) (L^X, δ) is a regular (T_3-) space if and only if $(R(L)^X, \omega(\delta))$ is a regular (T_3-) space.

(2) (L^X, δ) is a normal (T_4-) space if and only if $(R(L)^X, \omega(\delta))$ is a normal (T_4-) space.

Proof (1) Assume that (L^X, δ) is a regular space. For any $x_{\lambda_{\alpha,t}} \in M(R(L)^X)$, μ is a quasi-general closed set of $(R(L)^X, \omega(\delta))$ and $x \notin \mathrm{supp}\mu$, there exists $s \in R$ such that $\omega_s(\mu)$ is a quasi-general closed set of (L^X, δ) and $\mathrm{supp}\mu = \mathrm{supp}\omega_s(\mu)$. Obviously, $x_\alpha \in M(L^X)$, $\omega_s(\mu) \in \delta'$. Since (L^X, δ) is a regular space, there exist $P \in \eta^-(x_\alpha)$ and $Q \in \eta^-(\omega_s(\mu))$ such that $P \vee Q = 1$. Thus $\alpha \not\leq P(x)$, $\omega_s(\mu)(y) = \mu(y)(s-) \not\leq Q(y)$ for any $y \in \mathrm{supp}\omega_s(\mu)$, which implies $\lambda_{\alpha,t} \not\leq P^*(x)$ and $\mu(y) \not\leq Q^*(y)$. By Theorem 2.1[4], we have $P^* \in \eta^-(x_{\lambda_{\alpha,t}})$, $Q^* \in \eta^-(\mu)$ and $P^* \vee Q^* = 1$. Therefore, $(R(L)^X, \omega(\delta))$ is a regular space.

Conversely, Assume that $(R(L)^X, \omega(\delta))$ is a regular space. For any $x_\alpha \in M(L^X)$, μ is a quasi-general closed set of (L^X, δ) and $x \notin \mathrm{supp}\mu$, we have $x_{\lambda_{\alpha,t}} \in M(R(L)^*)$ for any $t \in R$, μ^* is a quasi-general closed set of $(R(L)^X, \omega(\delta))$ and $\mathrm{supp}\mu = \mathrm{supp}\mu^*$. Since $(R(L)^X, \omega(\delta))$ is a regular space, there exist $P \in \eta^-(x_{\lambda_{\alpha,t}})$ and $Q \in \eta^-(\mu^*)$ such that $P \vee Q = 1$, which implies $\lambda_{\alpha,t} \not\leq P(x)$, $\mu^*(y) \not\leq Q(y)$ for any $y \in \mathrm{supp}\mu^*$. Thus $\alpha = \lambda_{\alpha,t}(s-) \not\leq P(x)(s-) = \omega_s(P)(x)$ for some $s \leq t$, and $\mu(y) = \mu^*(y)(s-) \not\leq Q(y)(s-) = \omega_s(Q)(y)$ for any $y \in \mathrm{supp}\mu$. By Theorem 2.1[4], we have $\omega_s(P) \in \eta^-(x_\alpha)$, $\omega_s(Q) \in \eta^-(\mu)$ and $\omega_s(P) \vee \omega_s(Q) = \omega_s(P \vee Q) = 1$. Therefore, (L^X, δ) is a regular space.

(2) The proof is similar to (1).

For a mapping $f : X \to Y$, we use $\bar{f} : L^X \to L^Y$ to denote the L-valued Zadeh function induced by f, and use $\tilde{f} : R(L)^X \to R(L)^Y$ to denote the $R(L)$-valued Zadeh function induced by f.

Lemma 2.10[4] *Let (L^X, δ) be an L-fts and $f : X \to Y$ be a mapping. Then $\sigma_t(\tilde{f}^{-1}(\mu)) =$*

$\bar{f}^{-1}(\sigma_t(\mu))$ for each $\mu \in R(L)^Y$ and $t \in R$.

Here, I denotes the unit interval $[0,1]$, ε denotes usual topology on I, $(L^I, \omega_L(\varepsilon))$ and $(R(L)^I, \omega_{R(L)}(\varepsilon))$ are both induced fuzzy topological spaces of (I, ε). For a mapping $f : X \to [0,1]$, we use $\bar{f}_1 : L^X \to L^{[0,1]}$ to denote the L-valued Zadeh function induced by f, and use $\bar{f}_2 : R(L)^X \to R(L)^{[0,1]}$ to denote the $R(L)$-valued Zadeh function induced by f.

Lemma 2.11 *Let (L^X, δ) be an L-fts and $f : X \to [0,1]$ be a mapping. Then $\bar{f}_1$ is continuous if and only if $\bar{f}_2$ is continuous.*

Proof Assume that $\bar{f}_1$ is continuous. For each $\mu \in \omega_{R(L)}(\varepsilon)$, since $\omega(\omega_L(\varepsilon)) = \omega_{R(L)}(\varepsilon)$ (Lemma 2.4), we have $\sigma_t(\mu) \in \omega_L(\varepsilon)$ for each $t \in R$. By Lemma 2.10 and $\bar{f}_1$ is continuous, we have $\sigma_t(\bar{f}_2^{-1}(\mu)) = \bar{f}_1^{-1}(\sigma_t(\mu)) \in \delta$, i.e, $\bar{f}_2^{-1}(\mu) \in \omega(\delta)$. Thus $\bar{f}_2$ is continuous.

Conversely, assume that $\bar{f}_2$ is continuous. Let $\mu \in \omega_L(\varepsilon)$, then $\mu^* \in \omega(\omega_L(\varepsilon)) = \omega_{R(L)}(\varepsilon)$ (Lemma 2.4). By Lemma 2.10, we have $\sigma_t(\bar{f}_2^{-1}(\mu^*)) = \bar{f}_1^{-1}(\sigma_t(\mu^*)) = \bar{f}_1^{-1}(\mu)$ for each $t \in R$, thus $\bar{f}_1^{-1}(\mu) = \sigma_t(\bar{f}_2^{-1}(\mu^*)) \in \delta$. Therefore, $\bar{f}_1$ is continuous.

Theorem 2.12 *Let (L^X, δ) be an L-fts. Then (L^X, δ) is a completely regular topological space if and only if $(R(L)^X, \omega(\delta))$ is a completely regular topological space.*

Proof Assume that (L^X, δ) is a completely regular topological space. For each nonzero quasi-general closed set $A \in R(L)^X$ and LF point $x_\lambda \in R(L)^X$ with $x \notin \mathrm{supp}A$, there exists $t \in R$ such that $\omega_t(A) \in L^X$ is a nonzero quasi-general closed set, $\omega_t(x_\lambda) \in L^X$ and $\mathrm{supp}\omega_t(A) = \mathrm{supp}A$. Since (L^X, δ) is a completely regular topological space, there exists a continuous L-valued Zadeh function $\bar{f}_1 : L^X \to L^{[0,1]}$ induced by f, such that $\omega_t(x_\lambda) \le \bar{f}^{-1}(0_1)$, $\omega_t(A) \le \bar{f}^{-1}(1_1)$. For $R(L)$-valued Zadeh function $\bar{f}_2 : R(L)^X \to R(L)^{[0,1]}$ induced by f, obviously $x_\lambda \le \bar{f}_2{}^{-1}(0_1)$, $A \le \bar{f}_2{}^{-1}(1_1)$. By Lemma 2.10, $\bar{f}_2$ is continuous, thus $(R(L)^X, \omega(\delta))$ is a completely regular topological space.

Conversely, for each nonzero quasi-general closed set $A \in L^X$ and LF point $x_\lambda \in L^X$ with $x \notin \mathrm{supp}A$. Obviously, A^* and x_λ^* are nonzero quasi-general closed set and LF point of $(R(L)^X, \omega(\delta))$ respectively, and $\mathrm{supp}A = \mathrm{supp}A^*$. Since $(R(L)^X, \omega(\delta))$ is a completely regular topological space, there exists a continuous $R(L)$-valued Zadeh function $\bar{f}_2 : R(L)^X \to R(L)^{[0,1]}$ such that $x_\lambda^* \le \bar{f}^{-1}(0_1)$, $A^* \le \bar{f}^{-1}(1_1)$. By the definition $\bar{f}_1$ and Lemma 2.10, we have $\bar{f}_1$ is continuous and $x_\lambda \le \bar{f}^{-1}(0_1)$, $A \le \bar{f}^{-1}(1_1)$. Thus (L^X, δ) is a completely regular topological space.

3. The N-compactness of induced $R(L)$-fuzzy topological space

The notion of N-compactness was first introduced by Wang[7], Zhao[8] and Peng[9] generalized the notion to general L-fts. For each $a \in L$, using $\beta(a)$ denotes the greatest minimal set of a, using $\beta^*(a)$ denotes the standard minimal set of a, that is $\beta^*(a) = \beta(a) \cap M(L) = \cup\{\pi(x) \mid x \in \beta(a)\}$, where $\pi(x) = \{y \in M(L) \mid y \le x\}$.

Lemma 3.1 *Let $\alpha,\ \beta \in M(L)$ and $s,\ t \in R$. Then $\lambda_{\alpha,t} \in \beta^*(\lambda_{\beta,s})$ if and only if $\alpha \in \beta^*(\beta)$ and $t \le s$.*

Proof Let $\lambda_{\alpha,t} \in \beta^*(\lambda_{\beta,s})$. Then $\lambda_{\alpha,t} \leq \lambda_{\beta,s}$ and $t \leq s$. For each $r \leq \min\{s,t\}$, we have $\alpha = \lambda_{\alpha,t}(r-) \leq \lambda_{\beta,s}(r-) = \beta$. Thus $\alpha \in \beta^*(\beta)$

Conversely, let $\alpha \in \beta^*(\beta)$ and $t \leq s$, then $\lambda_{\alpha,t} \leq \lambda_{\beta,s}$. Since $\lambda_{\alpha,t}$ is molecule of $R(L)$, thus $\lambda_{\alpha,t} \in \beta^*(\lambda_{\beta,s})$.

Theorem 3.2 *Let (L^X, δ) be an L-fts and $A \in L^X$. Then A is an N-compact set if only if A^* is an N-compact set.*

Proof Suppose that A is an N-compact set. Let Ψ is $\lambda_{\alpha,t}$-remote neighborhood family of A^*. Then we have $x_{\lambda_{\alpha,t}} \not\leq \wedge\Psi$ for each $x_{\lambda_{\alpha,t}} \in A^*$, which implies that $\lambda_{\alpha,t} \not\leq \wedge\Psi(x) = \wedge\{\varphi(x) \mid \varphi \in \Psi\}$. Hence there exists s with $s < t$ such that $\alpha = \lambda_{\alpha,t}(s-) \not\leq \wedge\{\varphi(x)(s-) \mid \varphi \in \Psi\}$, i.e., $x_\alpha \not\leq \wedge\{\omega_s(\varphi) \mid \varphi \in \Psi\}$. Let $\Phi = \{\omega_s(\varphi) \mid \varphi \in \Psi\}$. Then Φ is an α-remote neighborhood family of A. Since A is an N-compact set, there exist a finite subfamily $\Phi_0 \subseteq \Phi$ and $\beta \in \beta^*(\alpha)$ such that $x_\beta \not\leq \wedge\Phi_0$, i.e., Φ_0 is β^- remote neighborhood family of A. Let $\Psi_0 = \{\varphi \mid \omega_s(\varphi) \in \Phi_0\}$. Then Ψ_0 is a finite subfamily of Ψ and $x_\beta \not\leq \{\omega_s(\varphi) \mid \varphi \in \Psi_0\}$, which implies that $\beta \not\leq \{\omega_s(\varphi)(x) \mid \varphi \in \Psi_0\} = \wedge\{\varphi(x)(s-) \in \Psi_0\}$. Taking $r \in (s,t)$, we have $\lambda_{\beta,r} \not\leq \wedge\Psi_0(x)$, which is equivalent to $x_{\lambda_{\beta,r}} \not\leq \wedge\Psi_0$. Thus Ψ_0 is $\lambda_{\beta,r}$-remote neighborhood family of A^*. By Lemma 3.1, $\lambda_{\beta,r} \in \beta^*(\lambda_{\alpha,t})$, which implies that Ψ_0 is a $\lambda^-_{\beta,t}$ remote neighborhood family of A^*. Thus A^* is an N-compact set.

Conversely, suppose $A^* \in R(L)^X$ is an N-compact set. Let Φ be an α-remote neighborhood family of A, where $\alpha \in M(L)$. Then for each $x_\alpha \in A$, we have $x_\alpha \not\leq \wedge\Phi$, and $\alpha \not\leq \wedge\Phi(x) = \wedge\{\phi(x) \mid \phi \in \Phi\} = \wedge\{\phi^*(x)(t-) \mid \phi \in \Phi\}$ for each $t \in R$. Hence $\lambda_{\alpha,t} \not\leq \wedge\{\phi^*(x) \mid \phi \in \Phi\}$ and $x_{\lambda_{\alpha,t}} \not\leq \wedge\{\phi^* \mid \phi \in \Phi\}$, where $\lambda_{\alpha,t}$ is a molecule of $R(L)$ by Theorem 2.1. Let $\Phi^* = \{\phi^* \mid \phi \in \Phi\}$. Then Φ^* is an $\lambda_{\alpha,t}$-remote neighborhood family of A^*. Since A^* is an N-compact set, there exist $\lambda_{\beta,r} \in \beta^*(\lambda_{\alpha,t})$ and a finite subfamily Φ_0^* of Φ^* such that Φ_0^* is an $\lambda_{\beta,r}$-remote neighborhood family of A^*, which implies that $x_{\lambda_{\beta,r}} \not\leq \wedge\Phi_0^*$ for each molecule $x_{\lambda_{\beta,r}} \in A^*$. Let $\Phi_0 = \{\phi \mid \phi^* \in \Phi_0^*\}$. Then Φ_0 is a finite subfamily of Φ. Thus, for each $s \in R$ with $s < r$, we have

$$\beta = \lambda_{\beta,r}(s-) \not\leq \wedge\{\phi^*(x)(s-) \mid \phi \in \Phi_0\} = \wedge\{\phi(x) \mid \phi \in \Phi_0\},$$

which implies $x_\beta \not\leq \{\phi \mid \phi \in \Phi_0\}$. By Lemma 3.1 we have $\beta \in \beta^*(\alpha)$. Hence Φ_0 is an α^- remote neighborhood family of A. Therefore, A is an N-compact set.

Corollary 3.3 *(L^X, δ) is N-compact if and only if $(R(L)^X, \omega(\delta))$ is N-compact.*

Corollary 3.4 *If (L^X, δ) is an induced fuzzy topological space of $(X, \mathcal{T})$, then $(R(L)^X, \omega(\delta))$ is N-compact if and only if $(X, \mathcal{T})$ is compact.*

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