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Asymptotic Behavior of Asymptotically Nonexpansive Type Mappings in Banach Space

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Abstract Let X be a uniformly convex Banach space X such that its dual X^* has the KK property. Let C be a nonempty bounded closed convex subset of X and G be a directed system. Let $\Im = \{T_t : t \in G\}$ be a family of asymptotically nonexpansive type mappings on C. In this paper, we investigate the asymptotic behavior of $\{T_t x_0 : t \in G\}$ and give its weak convergence theorem.

Keywords asymptotically nonexpansive type mappings; Kadec-Klee property; directed system; asymptotic behavior.

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1. Introduction

Let C be a nonempty bounded closed convex subset of Banach space X. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings from C into itself. Recall that $\{T_n\}_{n=1}^{\infty}$ is said to be asymptotically nonexpansive type, if $\|T_n x - T_n y\| \le \|x - y\| + r_n(x)$ for all x, y in C with $r_n(x) \ge 0$ and $\lim_{n\to+\infty} r_n(x) = 0$. And $\{T_n\}_{n=1}^{\infty}$ is said to be asymptotically nonexpansive, if $\|T_n x - T_n y\| \le K_n\|x - y\|$ for all x, y in C with $\lim_{n\to+\infty} K_n = 1$.

Bose[1], Feathers and Dotson[2] gave the weak convergence theorem of asymptotically nonexpansive mappings in a uniformly convex Banach space with weak continuous duality mapping by using Opial's Lemma[3]. Using Bruck's Lemma[4], Passty[5] extended to the results of [1, 2] to a uniformly convex Banach space with a Fréchet differentiable norm. Recently, Huang and Li[6] extended the results of Passty[5] to a uniformly convex Banach space with its dual having the KK property. However, Bruck's Lemma does not extend beyond Lipschitzian Mappings, new techniques are needed for this more general case. Li[7] first gave the convergence theorem of $\Im = \{T_t : t \in G\}$ of asymptotically nonexpansive type (Non-Lipschitzian) mappings in a uniformly convex Banach space with a Fréchet differentiable norm, where G is a directed system. The objective of this paper is to generalize the weak convergence theorem in [7] to the case that

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the dual space X^* has KK property. We would like to remark that the condition that X^* has the KK property is strictly weaker than the condition that X has a Fréchet differentiable norm. Our results are generalizations of the main results in [5,6,7].

2. Preliminaries

Throughout this paper, let C be a nonempty bounded closed convex subset of uniformly convex Banach space X. Let X^* be the dual of X. Then the value of $x^* \in X^*$ at $x \in X$ will be denoted by $\langle x, x^* \rangle$ and we associate the set

$$J(x) = \{x^* \in X^* : \ \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

Using the Hahn-Banach theorem, it is immediately clear that $J(x) \neq \emptyset$ for any $x \in X$. Then the multi-valued operator $J : X \mapsto X^*$ is called the normalized duality mapping of X. We need the following lemma which plays a crucial role in the proof of our main theorem.

Lemma 2.1[8] *Let X be a Banach space and J be the normalized duality mapping. Then for given $x, y \in X$ and $j(x+y) \in J(x+y)$, we have*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y) \rangle.$$

Recall that X has a Fréchet differentiable norm if for each $x \neq 0$,

$$\lim_{t \to 0} (\|x + ty\| - \|x\|)/t$$

exists uniformly in $y \in B_r$, where $B_r = \{z \in X : \|z\| \leq r\}$, $r > 0$. We say that X has the Kadec-Klee property (KK property, for short) if for every sequence $\{x_n\}_{n \in N}$ in X, whenever $\omega - \lim_{n \to \infty} x_n = x$ with $\lim_{n \to \infty} \|x_n\| = \|x\|$, it follows that $\lim_{n \to \infty} x_n = x$.

It is well known that if X is a reflexive Banach space with a Fréchet differentiable norm, then X^* has KK property, while the converse implication fails[9].

Example 2.1[9] Let us take $X_1 = L^p[0,1]$, $1 < p < +\infty$, $p \neq 2$, and $X_2 = R^2$ with the norm defined by $\|x\| = \sqrt{|x_1|^2 + |x_2|^2}$ $(x = (x_1, x_2) \in R^2)$. The Cartersian product of X_1 and X_2 furnished with the l^2-norm is a uniformly convex Banach space. Its norm is not Fréchet differentiable, but its dual X^* does have KK property.

Let $(G, \leq)$ be a directed system. We extend the definition of [1] to a family of mappings which are not necessarily semigroups.

Definition 2.1[7] *Let $\Im = \{T_t : t \in G\}$ be a family self-mappings of C. $\Im$ is said to be asymptotically nonexpansive type if for each $x \in C$, there exists a function $R_{(\cdot)}(x) : G \longmapsto [0, +\infty)$ with $\lim_{t \in G} R_t(x) = 0$ such that*

$$\|T_t x - T_t y\| \leq \|x - y\| + R_t(x)$$

for all $y \in C$ and $t \in G$, where $\lim_{t \in G} R_t(x)$ denotes the limit of the net $R_{(\cdot)}(x)$ on the directed system G.

Let $L(\Im)$ denote the set of all asymptotically fixed points of $\Im = \{T_t : t \in G\}$, i.e., $L(\Im) =$

$\{x \in C : \lim_{t\in G} T_t x = x\}$. It is easily seen that if $\Im$ is a semigroup and for each $t \in G$, T_t is continuous, then $L(\Im)$ is exactly the set of all fixed points of $\Im$. Let $\omega_\omega(x)$ denote the set of all weak limit points of subnet of $\{T_t x : t \in G\}$, i.e., $\omega_\omega(x) = \{y \in C$: there exists a subnet t_α of G such that $T_{t_\alpha} x \rightharpoonup y\}$, where $\rightharpoonup$ denotes weak convergence.

3. Main results

In order to prove the main theorem, we proceed with proving several lemmas.

Lemma 3.1 *If X is a reflexive space, then the following are equivalent:*

(a) X has the KK property;

(b) If $\{x_\alpha\} \subset X$, $x_\alpha \rightharpoonup x$ and $\|x_\alpha\| \to \|x\|$, then $x_\alpha \to x$, where $\alpha \in I$ and I is a directed system.

Proof It suffices to prove $(a) \Rightarrow (b)$. Let us assume that this is not the case. Then there exists $\varepsilon_0 > 0$ such that for all $\alpha \in I$, there exists $\beta_\alpha \in I$ with $\beta_\alpha \geq \alpha$ and $\|x_{\beta_\alpha} - x\| \geq \varepsilon_0$. Put $B = \{\beta_\alpha, \alpha \in I\}$. Then B is a subset of I. Obviously, for arbitrary $\alpha \in B$ we have

$$\|x_\alpha - x\| \geq \varepsilon_0. \tag{3.1}$$

Then for some $j(x) \in J(x)$, there exists $\alpha_1 \in B$ such that

$$|\,\|x_{\alpha_1}\| - \|x\|\,| < 1,$$

$$|\langle x_{\alpha_1} - x, j(x)\rangle| < 1.$$

Hence for the above $j(x) \in J(x)$ and some $j(x_{\alpha_1} - x) \in J(x_{\alpha_1} - x)$, there exists an $\alpha_2 \in B$ such that

$$|\|x_{\alpha_2}\| - \|x\|\,| < \frac{1}{2},$$

$$|\langle x_{\alpha_2} - x, j(x)\rangle| < \frac{1}{2},$$

and

$$|\langle x_{\alpha_2} - x, j(x_{\alpha_1} - x)\rangle| < \frac{1}{2}.$$

Now by mathematical induction, we can find inductive sequence $\{\alpha_n\} \subset B$ such that for given $j(x) \in J(x)$ and $j(x_{\alpha_i} - x) \in J(x_{\alpha_i} - x)$, $i = 1, \ldots, n-1$, we have the following inequalities:

$$|\|x_{\alpha_n}\| - \|x\|\,| < \frac{1}{n},$$

$$|\langle x_{\alpha_n} - x, j(x)\rangle| < \frac{1}{n}, \tag{3.2}$$

and, in addition,

$$|\langle x_{\alpha_n} - x, j(x_{\alpha_i} - x)\rangle| < \frac{1}{n}, \tag{3.3}$$

where $i = 1, \ldots, n-1$. Clearly, $||x_{\alpha_n}|| \to ||x||$ and $\{x_{\alpha_n}\}$ has a weak convergent subsequence $\{x_{\alpha_{n_i}}\}$. We may assume that $x_{\alpha_{n_i}} \rightharpoonup y$. Then $\|y\| \leq \liminf_{i\to+\infty} \|x_{\alpha_{n_i}}\| = \|x\|$. By (3.2), we get $\langle y-x, j(x)\rangle = 0$ which implies $\|y\| \geq \|x\|$. Hence $\|y\| = \|x\|$. Therefore, $x_{\alpha_{n_i}} \rightharpoonup y$ and $\|x_{\alpha_{n_i}}\| \to \|y\|$. By the condition (a), we obtain $x_{\alpha_{n_i}} \to y$. It follows from (3.3) that

$$|\langle x_{\alpha_{n_i}} - x,\ j(x_{\alpha_{n_{i-1}}} - x)\rangle| < \frac{1}{n_i}.$$

Hence

$$\begin{aligned}\|x_{\alpha_{n_{i-1}}} - x\|^2 &= \langle x_{\alpha_{n_{i-1}}} - x,\ j(x_{\alpha_{n_{i-1}}} - x)\rangle \\ &< |\langle x_{\alpha_{n_i}} - x_{\alpha_{n_{i-1}}},\ j(x_{\alpha_{n_{i-1}}} - x)\rangle| + \frac{1}{n_i} \\ &\leq \|x_{\alpha_{n_i}} - x_{\alpha_{n_{i-1}}}\| \cdot \|x_{\alpha_{n_{i-1}}} - x\| + \frac{1}{n_i} \to 0 \quad (i \to +\infty).\end{aligned}$$

This contradicts with (3.1). This completes the proof. □

Lemma 3.2 *If* $\limsup_{s\in G} \limsup_{t\in G} \|T_tT_sx_0 - T_tx_0\| = 0$, *then for all* $f \in L(\Im)$, $\lim_{t\in G} \|T_tx_0 - f\|$ *exists.*

Proof Since

$$\begin{aligned}\|T_tx_0 - f\| &\leq \|T_tx_0 - T_tT_sx_0\| + \|T_tT_sx_0 - T_tf\| + \|T_tf - f\| \\ &\leq \|T_tx_0 - T_tT_sx_0\| + \|T_sx_0 - f\| + R_t(f) + \|T_tf - f\|,\end{aligned}$$

for fixed $s \in G$ and passing the limsup for $t \in G$, we have

$$\limsup_{t\in G} \|T_tx_0 - f\| \leq \limsup_{t\in G} \|T_tx_0 - T_tT_sx_0\| + \|T_sx_0 - f\|.$$

Then

$$\begin{aligned}\limsup_{t\in G} \|T_tx_0 - f\| &\leq \liminf_{s\in G} \limsup_{t\in G} \|T_tx_0 - T_tT_sx_0\| + \liminf_{s\in G} \|T_sx_0 - f\| \\ &\leq \limsup_{s\in G} \limsup_{t\in G} \|T_tx_0 - T_tT_sx_0\| + \liminf_{s\in G} \|T_sx_0 - f\| \\ &= \liminf_{s\in G} \|T_sx_0 - f\|.\end{aligned}$$

This implies that $\lim_{t\in G} \|T_tx_0 - f\|$ exists. This completes the proof. □

Lemma 3.3 *Let* $\lambda \in (0,1)$ *and* $f \in L(\Im)$. *If* $\limsup_{s\in G} \limsup_{t\in G} \|T_tT_sx_0 - T_tx_0\| = 0$, *then for given* $\varepsilon > 0$, *there exists* $s_0 \in G$ *such that*

$$\limsup_{t\in G} \|T_t(\lambda T_sx_0 + (1-\lambda)f) - (\lambda T_tT_sx_0 + (1-\lambda)f)\| < \varepsilon$$

for all $s \geq s_0$.

Proof From Lemma 3.2, $\lim_{t\in G} \|T_tx_0 - f\|$ exists. Put $r = \lim_{t\in G} \|T_tx_0 - f\|$. If $r > 0$, then there exists $d > 0$ such that

$$(r+d)(1 - 2\lambda(1-\lambda)\delta(\frac{\varepsilon}{r+d})) < r - d, \tag{3.4}$$

where δ is the modulus of convexity of the norm, and there exists $s_0 \in G$ such that

$$r - \frac{d}{4} \leq \|T_sx_0 - f\| \leq r + \frac{d}{4} \tag{3.5}$$

and

$$\limsup_{t\in G} \|T_tT_sx_0 - T_tx_0\| < \frac{d}{4} \tag{3.6}$$

for all $s \geq s_0$. Now for fixed $s \geq s_0$, set $z = \lambda T_sx_0 + (1-\lambda)f$. Then from (3.6) there exists $t_0 \in G$ ($t_0 \geq s_0$) such that

$$R_t(z) < \frac{1}{2}\lambda(1-\lambda)d, \qquad \|T_tf - f\| \leq \frac{\lambda d}{4},$$

and

$$\|T_tT_sx_0 - T_tx_0\| < \frac{d}{2} \tag{3.7}$$

for all $t \geq t_0$. Suppose that

$$\|T_t(\lambda T_sx_0 + (1-\lambda)f) - (\lambda T_tT_sx_0 + (1-\lambda)f)\| \geq \varepsilon$$

for some $t \geq t_0$. Put $x = (1-\lambda)(T_tz - f)$ and $y = \lambda(T_tT_sx_0 - T_tz)$. Then

$$\begin{aligned}\|x\| &\leq (1-\lambda)(\|T_tz - T_tf\| + \|T_tf - f\|)\\ &\leq (1-\lambda)(\|z - f\| + R_t(z) + \|T_tf - f\|)\\ &\leq \lambda(1-\lambda)(\|T_sx_0 - f\| + \frac{1}{2}d + \frac{1}{4}d)\\ &\leq \lambda(1-\lambda)(r+d)\end{aligned}$$

and

$$\begin{aligned}\|y\| = \lambda\|T_tT_sx_0 - T_tz\| &\leq \lambda(\|T_sx_0 - z\| + R_t(z))\\ &\leq \lambda(1-\lambda)(\|T_sx_0 - f\| + \frac{1}{2}d) \leq \lambda(1-\lambda)(r+d).\end{aligned}$$

We also have

$$\|x - y\| = \|T_tz - (\lambda T_tT_sx_0 + (1-\lambda)f)\| \geq \varepsilon$$

and

$$\lambda x + (1-\lambda)y = \lambda(1-\lambda)(T_tT_sx_0 - f).$$

So by using the Lemma in [10], we get

$$\begin{aligned}\lambda(1-\lambda)\|T_tT_sx_0 - f\| &= \|\lambda x + (1-\lambda)y\|\\ &\leq \lambda(1-\lambda)(r+d)(1 - 2\lambda(1-\lambda)\delta(\frac{\varepsilon}{r+d}))\end{aligned}$$

and then from (3.5) and (3.7), we have

$$\begin{aligned}r - d &\leq \|T_tx_0 - f\| - \|T_tT_sx_0 - T_tx_0\|\\ &\leq \|T_tT_sx_0 - f\| \leq (r+d)(1 - 2\lambda(1-\lambda)\delta(\frac{\varepsilon}{r+d})).\end{aligned}$$

This contradicts (3.4). In the case $r = 0$, since

$$\begin{aligned}&\|T_tz - (\lambda T_tT_sx_0 + (1-\lambda)f)\|\\ &\leq \lambda\|T_tz - T_tT_sx_0\| + (1-\lambda)\|T_tz - T_tf\| + \|T_tf - f\|\\ &\leq \lambda(R_t(z) + (1-\lambda)\|T_sx_0 - f\|) + (1-\lambda)R_t(z) +\end{aligned}$$

$$\begin{aligned}&\lambda(1-\lambda)\|T_sx_0-f\|+\|T_tf-f\|\\ \le\ & R_t(z)+2\lambda(1-\lambda)\|T_sx_0-f\|+\|T_tf-f\|,\end{aligned}$$

we can get what we desired. This completes the proof. □

Lemma 3.4 *If* $\limsup_{s\in G}\limsup_{t\in G}\|T_tT_sx_0-T_tx_0\|=0$, *then*

$$\lim_{t\in G}\|\lambda T_tx_0+(1-\lambda)f-g\|$$

exists for all $\lambda\in(0,1)$ *and* $f,\,g\in L(\Im)$.

Proof For given $\varepsilon>0$, from Lemma 3.3, there exists $s_0\in G$ such that

$$\limsup_{t\in G}\|T_t(\lambda T_sx_0+(1-\lambda)f)-(\lambda T_tT_sx_0+(1-\lambda)f)\|<\varepsilon$$

for all $s\ge s_0$. Since

$$\begin{aligned}&\|\lambda T_tx_0+(1-\lambda)f-g\|\\ &\quad\le\|T_t(\lambda T_sx_0+(1-\lambda)f)-(\lambda T_tT_sx_0+(1-\lambda)f)\|+\\ &\qquad\|T_t(\lambda T_sx_0+(1-\lambda)f)-T_tg\|+\lambda\|T_tT_sx_0-T_tx_0\|+\|T_tg-g\|\\ &\quad\le\|T_t(\lambda T_sx_0+(1-\lambda)f)-(\lambda T_tT_sx_0+(1-\lambda)f)\|+R_t(g)+\\ &\qquad\|T_sx_0+(1-\lambda)f-g\|+\lambda\|T_tT_sx_0-T_tx_0\|+\|T_tg-g\|,\end{aligned}$$

for fixed $s\ge s_0$ and taking the limsup for $t\in G$, we get

$$\begin{aligned}&\limsup_{t\in G}\|T_tx_0+(1-\lambda)f-g\|\\ &\quad\le\varepsilon+\|T_sx_0+(1-\lambda)f-g\|+\lambda\limsup_{t\in G}\|T_tT_sx_0-T_tx_0\|.\end{aligned}$$

Hence

$$\limsup_{t\in G}\|T_tx_0+(1-\lambda)f-g\|\le\varepsilon+\liminf_{s\in G}\|T_sx_0+(1-\lambda)f-g\|.$$

Since $\varepsilon>0$ is arbitrary, this completes the proof. □

Now we are ready to prove our main theorem.

Theorem 3.1 *Let X be a uniformly convex Banach space such that its dual X^* has the KK property. Let C be a nonempty bounded closed convex subset of X. Let $(G,\le)$ be a directed system and $\Im=\{T_t:t\in G\}$ be asymptotically nonexpansive type mappings on C. Assume that there exists x_0 in C for which*

(a) $\omega_\omega(x_0)\subset L(\Im)$;

(b) $\limsup_{s\in G}\limsup_{t\in G}\|T_tT_sx_0-T_tx_0\|=0$.

Then there exists $p\in L(\Im)$ *such that* $T_tx_0\rightharpoonup p$.

Proof It suffices to show that $\omega_\omega(x_0)$ consists of exactly one point. Since X is reflexive, $\omega_\omega(x_0)$ is nonempty. Let $f,g\in\omega_\omega(x_0)$. By the condition (a), we know $f,g\in L(\Im)$. For any $\lambda\in(0,1)$, from Lemma 3.4, $\lim_{t\in G}\|\lambda T_tx_0+(1-\lambda)f-g\|$ exists. Put

$$h(\lambda)=\lim_{t\in G}\|\lambda T_tx_0+(1-\lambda)f-g\|.$$

Then for given $\varepsilon > 0$, there exists $t_1 \in G$ such that

$$\|\lambda T_t x_0 + (1-\lambda)f - g\| \leq h(\lambda) + \varepsilon$$

for all $t \geq t_1$. Hence

$$\langle \lambda T_t x_0 + (1-\lambda)f - g, j(f-g)\rangle \leq \|f-g\|(h(\lambda)+\varepsilon),$$

for all $t \geq s_1$, where $j(f-g) \in J(f-g)$. Inasmuch as $f \in \bar{\mathrm{co}}\{T_t x_0, t \geq s_1\}$,

$$\langle \lambda f + (1-\lambda)f - g, j(f-g)\rangle \leq \|f-g\|(h(\lambda)+\varepsilon),$$

that is, $\|f-g\| \leq h(\lambda)+\varepsilon$. Since $\varepsilon > 0$ is arbitrary,

$$\|f-g\| \leq h(\lambda). \tag{3.8}$$

It follows from $g \in \omega_\omega(x_0)$ that there exists a subnet $\{t_\alpha, \alpha \in A\}$ of G such that $T_{t_\alpha} x_0 \rightharpoonup g$, where A is a directed system. Put $I = A \times N = \{\beta = (\alpha, n); \alpha \in A, n \in N\}$. For $\beta_i = (\alpha_i, n_i) \in I$, $i = 1, 2$, we define $\beta_1 \leq \beta_2$ if and only if $\alpha_1 \leq \alpha_2$ and $n_1 \leq n_2$. In this case, I is also a directed system. For arbitrary $\beta = (\alpha, n) \in I$, we also define $P_1\beta = \alpha, P_2\beta = n, t_\beta = t_{P_1\beta} = t_\alpha, \varepsilon_\beta = \frac{1}{P_2\beta}$. Then we obtain $T_{t_\beta} x_0 \rightharpoonup g, \varepsilon_\beta \to 0$, $\beta \in I$. From Lemma 2.1, we have

$$\|\lambda T_t x_0 + (1-\lambda)f - g\|^2 \leq \|f-g\|^2 + 2\lambda\langle T_t x_0 - f, j(\lambda T_t x_0 + (1-\lambda)f - g)\rangle.$$

by Lemma 3.4 and (3.8), we get

$$\liminf_{\beta\in I}\langle T_{t_\beta} x_0 - f, j(\lambda T_{t_\beta} x_0 + (1-\lambda)f - g)\rangle \geq 0.$$

Then for arbitrary $\gamma \in I$, there exists $\beta_\gamma \in I$ with $\beta_\gamma \geq \gamma$ and

$$\langle T_{t_{\beta_\gamma}} x_0 - f, j(\varepsilon_\gamma T_{t_{\beta_\gamma}} x_0 + (1-\varepsilon_\gamma)f - g)\rangle \geq -\varepsilon_\gamma. \tag{3.9}$$

Obviously, β_γ is a subset of I, then $T_{t_{\beta_\gamma}} x_0 \rightharpoonup g$. Put

$$j_\gamma = j(\varepsilon_\gamma T_{t_{\beta_\gamma}} x_0 + (1-\varepsilon_\gamma)f - g).$$

Since X is reflexive, X^* is reflexive and the set of all weak limit points of $\{j_\gamma, \gamma \in I\}$ is nonempty. Hence we may assume that, without loss of generality, $\{j_\gamma, \gamma \in I\}$ is weakly convergent to some point $j \in X^*$. Therefore $\|j\| \leq \liminf_{\gamma\in I}\|j_\gamma\| = \|f-g\|$. Since

$$\langle f-g, j_\gamma\rangle = \|\varepsilon_\gamma T_{t_{\beta_\gamma}} x_0 + (1-\varepsilon_\gamma)f - g\|^2 - \varepsilon_\gamma\langle T_{t_{\beta_\gamma}} x_0 - f, j_\gamma\rangle,$$

passing the limit for $\gamma \in I$, we have $\langle f-g, j\rangle = \|f-g\|^2$. Hence $\|j\| \geq \|f-g\|$ and we get $\langle f-g, j\rangle = \|f-g\|^2 = \|j\|^2$. This means $j \in J(f-g)$. Thus we can conclude that $j_\gamma \rightharpoonup j$ and $\|j_\gamma\| \to \|j\|$. Since X^* has KK property, from Lemma 3.1, we have $j_\gamma \to j$. Taking the limit for $\gamma \in I$ in (3.9), we get

$$\langle g-f, j\rangle \geq 0,$$

i.e., $\|f-g\|^2 \leq 0$ which implies $f = g$. This completes the proof. □

Remark 3.1 If $\Im = \{T_t : t \in G\}$ is a right reversible semigroup of asymptotically nonexpansive type mappings on C, then we can get the weak convergence theorem of the right reversible semigroups and the condition (b) in Theorem 3.1 is not necessary (see [10] for more detail).

Remark 3.2 It is well known that if X is a reflexive Banach space with a Fréchet differentiable norm, then its dual X^* has KK property, but not conversely. From Theorem 3.1, we can get the main results in [5,6,7].

From Theorem 3.1, we can get the following corollary.

Corollary 3.1 *Let X be a uniformly convex Banach space such that X^* has KK property. Let C be a nonempty bounded closed convex subset of X and $\Im = \{T_t : t \in G\}$ be a right reversible semigroup of asymptotically nonexpansive type mappings on C. If T_t is weakly continuous and asymptotically regular at x_0 (i.e., $T_{ts}x_0 - T_t x_0 \to 0$ for all $s \in G$). Then $T_t x_0$ converges weakly to a fixed point of $\Im$.*

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