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On the Kazhdan-Lusztig Theory of Dual Extension Quasi-Hereditary Algebras

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Abstract In order to study the representation theory of Lie algebras and algebraic groups, Cline, Parshall and Scott put forward the notion of abstract Kazhdan-Lusztig theory for quasi-hereditary algebras. Assume that a quasi-hereditary algebra B has the vertex set $Q_0 = \{1, \dots, n\}$ such that $\mathrm{Hom}_B(P(i), P(j)) = 0$ for $i > j$. In this paper, it is shown that if the quasi-hereditary algebra B has a Kazhdan-Lusztig theory relative to a length function l, then its dual extension algebra $A = \mathcal{A}(B)$ has also the Kazhdan-Lusztig theory relative to the length function l.

Keywords quasi-hereditary algebra; dual extension algebra; Kazhdan-Lusztig theory.

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1. Introduction

A major unsolved problem in finite group theory centers on determining the characters and degrees of the irreducible modular representations of finite groups of Lie type in their defining characteristic. Lusztig took a significant step toward a solution in 1979 by formulating his celebrated conjecture[1] for the characters of simple modules for semisimple algebraic groups. At the same time, in the study of the representations of Coxeter groups and Hecke algebras, a similar conjecture, by Kazhdan and Lusztig, for the composition factor multiplicities of Verma modules for semisimple complex Lie algebras, has already been settled. By now, Lusztig conjecture has not been completely proved yet. By using the theory of finite dimensional algebra, Cline, Parshall and Scott investigated the above problems and put forward the notion of an abstract Kazhdan-Lusztig theory, which provides conditions on an arbitrary quasi-hereditary algebra which are necessary and sufficient for the validity of the Lusztig conjecture in a Lie-theoretic context, as well as new consequences of the conjecture[2−4]. In this paper a property on the Kazhdan-Lusztig theory of dual extension quasi-hereditary algebras is obtained.

2. Notations and definitions

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We always assume that k is an algebraically closed field, and A is a finite-dimensional associative k-algebra (basic and connected, with an identity), and modules always mean finitely generated right A-modules. Let $S(i), i \in \Lambda$ be all simple A-modules, where the index set Λ is a partially ordered set (abbreviated "poset") with partial ordering $\leq$. In general, let $\Lambda = \{1, \ldots, n\}$, with its natural ordering. For each $i \in \Lambda$, let $P(i)$ be the projective cover, and $I(i)$ the injective envelope of $S(i)$. We denote by $\Delta(i)$ the maximal factor module of $P(i)$ with composition factors of the form $S(j)$ where $j \leq i$. These modules $\Delta(i)$ are called the standard modules, and we set $\Delta = \{\Delta(i) \mid i \in \Lambda\}$. Similarly, denote by $\nabla(i)$ the maximal submodule of $I(i)$ with composition factors of the form $S(j)$ where $j \leq i$. In this way, we obtain the set $\nabla = \{\nabla(i) \mid i \in \Lambda\}$ of costandard modules. Lastly, we denote by $F(\Delta)$ the full subcategory of A-module category, where all modules have a Δ-filtration.

Definition 1[5] *The algebra A (or better the pair (A, Λ)) is called quasi-hereditary if*

(1) $\mathrm{End}_A(\Delta(i)) \cong k$ *for all $i \in \Lambda$, and*

(2) Every projective module belongs to $F(\Delta)$.

For a quasi-hereditary algebra (A, Λ), we call Λ the weight set of A. The category of modules and the set Δ of standard modules over a quasi-hereditary algebra (A, Λ) become the highest weight category (defined by Cline, Parshall and Scott[6]) with the weight poset Λ. Conversely, every highest weight category with a finite weight poset is the category of modules for a certain quasi-hereditary algebra.

Dual extension algebra was introduced in [7] by Xi. He showed that if B has no oriented cycle in its quiver, then its dual extension algebra $A = \mathcal{A}(B)$ is a BGG-algebra. BGG-algebra is a special class of quasi-hereditary algebras[8] which have a duality on their module categories which fixes simple modules. In their study of the representation theory of semisimple complex Lie algebras, Bernstein, Gelfand and Gelfand introduced in [9] the notion of category $\mathcal{O}$, and showed that $\mathcal{O}$ is a highest weight category, over which the standard modules are Verma-modules. Furthermore, every block of category $\mathcal{O}$ is equivalent to the category of modules for a finite-dimensional BGG-algebra. Xi determined in [7] the quivers of representation-finite BGG-algebras, and defined an important class of BGG-algebras, namely, dual extension algebras. Dokin and Reiten determined independently in [10] the quivers and relations of representation-finite BGG-algebras. Schur algebras are important examples of BGG-algebras (see [11], p.32 and p.71).

The concept of the dual extension algebra of a quasi-hereditary algebra is as follows.

Let B be a finite-dimensional basic algebra over k, and B given by a quiver $Q_B = (Q_0, Q_1)$ with relations $\{\rho_i \mid i \in I_B\}$. For each α from i to j in Q_1, let α' be an arrow from j to i. We denote by Q_1' the set of all such α' with $\alpha \in Q_1$. For a path $\alpha_1 \cdots \alpha_m$ in (Q_0, Q_1) we denote by $(\alpha_1 \cdots \alpha_m)'$ the path $\alpha_m' \cdots \alpha_1'$ in (Q_0, Q_1').

Definition 2[7] *Suppose that B is an algebra given by the quiver $Q_B = (Q_0, Q_1)$ with relations $\{\rho_i \mid i \in I_B\}$. Let A be the algebra given by the quiver $\bar{Q} = (Q_0, Q_1 \bigcup Q_1')$ with relations $\{\rho_i \mid i \in I_B\} \bigcup \{\rho_j' \mid j \in I_B\} \bigcup \{\alpha\beta' \mid \alpha, \beta \in Q_1\}$. We say that A is the dual extension of B, denoted by $A = \mathcal{A}(B)$.*

Example Let $B = kQ/\langle\rho\rangle$, where Q is of the form

$$\begin{tikzcd} & 3 \arrow[d, "\beta"] & \\ 4 \arrow[r] & 2 \arrow[r, "\alpha"] & 1 \end{tikzcd}$$

with $\rho = \beta\alpha$. Then $\mathcal{A}(B) = k\bar{Q}/I$, where $\bar{Q}$ is of the form

$$\begin{tikzcd} & 3 \arrow[d, shift left, "\beta"] & \\ 4 \arrow[r, shift left, "\gamma"] & 2 \arrow[l, shift left, "\gamma'"] \arrow[u, shift left, "\beta'"] \arrow[r, shift left, "\alpha"] & 1 \arrow[l, shift left, "\alpha'"] \end{tikzcd}$$

$$I = \langle\beta\alpha, \alpha'\beta', \alpha\alpha', \beta\beta', \gamma\gamma'\rangle$$

Deng and Xi studied systematically the quasi-hereditary algebras which are dual extensions of algebras. They investigated in [12] a special class of quasi-hereditary algebras which are the dual extensions of algebras without oriented cycle in their quivers, for which they also gave a reduction to determine the finiteness of $F(\Delta)$. Xi showed in [13] that gl.dim(A)=2gl.dim(B), where $A = \mathcal{A}(B)$ is the dual extension of an arbitrary finite-dimensional basic k-algebra B and gl.dim(A) denotes the global dimension of the algebra A. They also discussed in [14] the Ringel duals of the dual extensions of the algebra B with a tree-type poset, whose quivers are bipartite. Xi described in [15] the characteristic tilting modules over the dual extension algebras of the direct monomial algebras, especially, the description of the dual extension algebra of an arbitrary hereditary algebra is given there and it is proved that such dual extension algebras have triangular decompositions.

Definition 3[16] *Let (A, Λ) be a quasi-hereditary algebra, and l: $\Lambda \longrightarrow \mathbb{N}$ any map (called a length function). Then algebra A is said to have a Kazhdan-Lusztig theory (relative to l) provided that*

(1) $\mathrm{Ext}_A^n(\Delta(i), S(j)) \neq 0 \Longrightarrow n \equiv l(j) + l(i) (\mathrm{mod}\ 2)$*, and*

(2) $\mathrm{Ext}_A^n(S(j), \nabla(i)) \neq 0 \Longrightarrow n \equiv l(j) + l(i)\ (\mathrm{mod}\ 2)$ *for all weights $i, j \in \Lambda$.*

3. Some properties of dual extension algebras

From now on, we always assume that B has no oriented cycle in its quiver, and $Q_0 = \{1, \ldots, n\}$ such that $\mathrm{Hom}_B(P_B(i), P_B(j)) = 0$ for $i > j$.

Lemma 1[7] *Let B have no oriented cycle in its quiver. Then the dual extension (A, Λ) of B is quasi-hereditary, where $A = \mathcal{A}(B)$, and the index set $\Lambda = Q_0$ is a poset with its natural ordering.*

Suppose that B' is an algebra given by the quiver (Q_0, Q_1') with relations $\{\rho_i' \mid i \in I_B\}$, which is the dual algebra of B. Let $S = kQ_0$. We have the following properties.

Lemma 2[12,13,18] *(1)* $\Delta_B(i) = P_B(i), \nabla_B(i) = S_B(i)$*. There holds*

$$\Delta_{B'}(i) = S_{B'}(i), \nabla_{B'}(i) = I_{B'}(i).$$

(2) B is a subalgebra of A with the same maximal semisimple subalgebra and B is also a factor algebra $B \cong A/\langle \alpha' \mid \alpha \in Q_1\rangle$ of A; B' is a subalgebra of A with the same maximal semisimple subalgebra and B' is also the factor algebra of A with $B' = A/\langle \alpha \mid \alpha \in Q_1\rangle$.

(3) As a left B' and right B module, there holds $A \cong B' \bigotimes_S B$.

(4) ${}_{B'}A$ is a projective left B'-module, A_B is a projective right B-module.

(5) $S_{B'}(i) \bigotimes_{B'} A \cong \Delta_A(i)$, $P_{B'}(i) \bigotimes_{B'} A \cong P_A(i)$. There holds

(6) $I_A(i) = D(A \bigotimes_B D(I_B(i))), \nabla_A(i) = D(A \bigotimes_B D(S_B(i)))$. There holds

(7) top $P_A(i) = S_A(i) = S_{B'}(i)$, *thus for each projective B'-module P, there holds* top $P \cong$ top $(P \bigotimes_{B'} A)$.

(8) soc $I_A(i) = S_A(i) = S_B(i)$, *thus for each injective B-module I, there holds* soc $I \cong$ soc $D(A \bigotimes_B D(I))$.

4. The main result

Theorem *Let B be a quasi-hereditary algebra, and the vertex set $Q_0 = \{1, \ldots, n\}$ such that* $\mathrm{Hom}_B(P(i), P(j)) = 0$ *for $i > j$. Assume that B has a Kazhdan-Lusztig theory relative to a length function l. Then its dual extension $A = \mathcal{A}(B)$ has also a Kazhdan-Lusztig theory relative to the length function l.*

In order to prove the theorem, we require some preparations.

Lemma 3 *$P \xrightarrow{f} M$ is the projective cover of a B'-module $M \Longleftrightarrow P \bigotimes_{B'} A \xrightarrow{f \bigotimes 1_A} M \bigotimes_{B'} A$ is the projective cover of a A-module $M \bigotimes_{B'} A$. Thus* $\ker(f \bigotimes 1_A)$ =$\ker f \bigotimes_{B'} A$.

Proof ($\Longrightarrow$) By the above Lamma 2, we see that $P \bigotimes_{B'} A$ is a projective A-module. By the assumption of Lemma 3 there exists a short exact sequence

$$0 \longrightarrow \ker f \longrightarrow P \xrightarrow{f} M \longrightarrow 0.$$

Applying the exact functor $- \bigotimes_{B'} A$ yields a short exact sequence in mod-A

$$0 \longrightarrow \ker f \bigotimes_{B'} A \longrightarrow P \bigotimes_{B'} A \xrightarrow{f \bigotimes 1} M \bigotimes_{B'} A \longrightarrow 0.$$

Let $Q \longrightarrow M \bigotimes_{B'} A$ be the projective cover of a A-module $M \bigotimes_{B'} A$. By the above Lemma 2, Q is of the form $\bar{P} \bigotimes_{B'} A$, then $\bar{P} \bigotimes_{B'} A$ is a direct summand of $P \bigotimes_{B'} A$. Applying $- \bigotimes_A B'$, we have the fact that $\bar{P}$ is a direct summand of P. Similarly, applying $- \bigotimes_A B'$ to the surjective homomorphism $\bar{P} \bigotimes_{B'} A \xrightarrow{f \bigotimes 1} M \bigotimes_{B'} A \longrightarrow 0$, we have the fact that $\bar{P} \longrightarrow M$ is a surjective homomorphism. Since $P \longrightarrow M$ is a projective cover, we have $\bar{P} \cong P$, hence $Q \cong P \bigotimes_{B'} A$.

($\Longleftarrow$) Let $L \xrightarrow{g} M$ be the projective cover of a B'-module M. By the implication proved above, we have the fact that $L \bigotimes_{B'} A \xrightarrow{g \bigotimes 1_A} M \bigotimes_{B'} A$ is the projective cover of a A-module $M \bigotimes_{B'} A$, hence $P \bigotimes_{B'} A \cong L \bigotimes_{B'} A$. Applying $- \bigotimes_A B'$, we have

$$P \cong (P \bigotimes_{B'} A) \bigotimes_A B' \cong (L \bigotimes_{B'} A) \bigotimes_A B' \cong L.$$

Corollary 1 *Let $M \in$ mod$-B'$, and P_i projective B'-modules. Then*

$$0 \longrightarrow P_s \xrightarrow{f_s} P_{s-1} \xrightarrow{f_{s-1}} \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

is a minimal projective resolution of M if and only if

$$0 \longrightarrow P_s \bigotimes_{B'} A \xrightarrow{f_s \bigotimes 1_A} P_{s-1} \bigotimes_{B'} A \xrightarrow{f_{s-1} \bigotimes 1_A} \cdots$$
$$\longrightarrow P_1 \bigotimes_{B'} A \xrightarrow{f_1 \bigotimes 1_A} P_0 \bigotimes_{B'} A \xrightarrow{f_0 \bigotimes 1_A} M \bigotimes_{B'} A \longrightarrow 0$$

is a minimal projective resolution of $M \bigotimes_{B'} A$ in mod-A.

Lemma 4 $M \xrightarrow{f} I$ *is the injective envelope of a module M in* mod-B *if and only if*

$$D(A \bigotimes_B D(M)) \xrightarrow{D(A \bigotimes D(f))} D(A \bigotimes_B D(I))$$

is the injective envelope of $D(A \bigotimes_B D(M))$ in mod-A. *Thus* coker $D(A \bigotimes D(f)) \cong D(A \bigotimes_B D(\text{coker } f))$.

Proof Since A_B is a projective B-module, we have the fact that $A \bigotimes_B -$ is an exact functor. $M \xrightarrow{f} I$ is an injective envelope in mod-B if and only if $D(I) \xrightarrow{D(f)} D(M)$ is a projective cover in B-mod; if and only if $A \bigotimes_B D(I) \xrightarrow{A \bigotimes_B D(f)} A \bigotimes_B D(M)$ is a projective cover in A-mod; if and only if $D(A \bigotimes_B D(M)) \xrightarrow{D(A \bigotimes_B D(f))} D(A \bigotimes_B D(I))$ is the injective envelope of $D(A \bigotimes_B D(M))$ in mod-A. □

Corollary 2 *Let $M \in$ mod-B, and I_i injective B-modules. Then*

$$0 \longrightarrow M \xrightarrow{g_0} I_0 \longrightarrow \cdots \longrightarrow I_{s-1} \xrightarrow{g_s} I_s \longrightarrow 0$$

is a minimal injective resolution of M in mod-B *if and only if*

$$0 \longrightarrow D(A \bigotimes_B D(M)) \xrightarrow{D(A \bigotimes_B D(g_0))} D(A \bigotimes_B D(I_0)) \longrightarrow \cdots$$
$$\longrightarrow D(A \bigotimes_B D(I_{s-1})) \xrightarrow{D(A \bigotimes_B D(g_s))} D(A \bigotimes_B D(I_s)) \longrightarrow 0$$

is a minimal injective resolution of $D(A \bigotimes_B D(M))$ in mod-A.

Proof $0 \longrightarrow M \xrightarrow{g_0} I_0 \longrightarrow \cdots \longrightarrow I_{s-1} \xrightarrow{g_s} I_s \longrightarrow 0$ is a minimal injective resolution of M in mod-B if and only if

$$0 \longrightarrow D(I_s) \xrightarrow{D(g_s)} D(I_{s-1}) \longrightarrow \cdots \longrightarrow D(I_0) \xrightarrow{D(g_0)} D(M) \longrightarrow 0$$

is a minimal projective resolution of $D(M)$ in $B-$mod; if and only if

$$0 \longrightarrow A \bigotimes_B D(I_s) \xrightarrow{A \bigotimes_B D(g_s)} A \bigotimes_B D(I_{s-1}) \longrightarrow \cdots$$
$$\longrightarrow A \bigotimes_B D(I_0) \xrightarrow{A \bigotimes_B D(g_0)} A \bigotimes_B D(M) \longrightarrow 0$$

is a minimal projective resolution of $A\bigotimes_B D(M)$ in A-mod; if and only if

$$0 \longrightarrow D(A\bigotimes_B D(M)) \xrightarrow{D(A\bigotimes_B D(g_0))} D(A\bigotimes_B D(I_0)) \longrightarrow \cdots$$
$$\longrightarrow D(A\bigotimes_B D(I_{s-1})) \xrightarrow{D(A\bigotimes_B D(g_s))} D(A\bigotimes_B D(I_s)) \longrightarrow 0$$

is a minimal injective resolution of $D(A\bigotimes_B D(M))$ in mod-A. □

Proof of the Theorem Assume that B has a Kazhdan-Lusztig theory relative to l. By [19, Theorem 1], $B'(=B^{op})$ has a Kazhdan-Lusztig theory relative to l. Let

$$0 \longrightarrow P_s \xrightarrow{f_s} P_{s-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} \Delta_{B'}(i) \longrightarrow 0$$

be a minimal projective resolution of B'-module $\Delta_{B'}(i)$. By the above Corollary 1

$$0 \longrightarrow P_s\bigotimes_{B'} A \xrightarrow{f_s\bigotimes 1_A} P_{s-1}\bigotimes_{B'} A \longrightarrow \cdots$$
$$\longrightarrow P_1\bigotimes_{B'} A \xrightarrow{f_1\bigotimes 1_A} P_0\bigotimes_{B'} A \xrightarrow{f_0\bigotimes 1_A} \Delta_A(i) \longrightarrow 0$$

is a minimal projective resolution of A-module $\Delta_A(i)$. Assume that $\mathrm{Ext}_A^n(\Delta_A(i), S(j)) \neq 0$. By [19, Proposition 1], we have the fact that $S_A(j)$ is a direct summand of top $(P_n\bigotimes_{B'} A)$. By the above Lemma 2, we have the fact that $S_{B'}(j)$ is a direct summand of top P_n. Again by [19, Proposition 1], we have $\mathrm{Ext}_{B'}^n(\Delta_{B'}(i), S_{B'}(j)) \neq 0$. Since B' has a Kazhdan-Lusztig theory relative to l, we have $n \equiv l(i)+l(j) (\mathrm{mod}\ 2)$.

On the other hand, let

$$0 \longrightarrow \bigtriangledown_B(i) \xrightarrow{g_0} I_0 \xrightarrow{g_1} I_1 \longrightarrow \cdots \longrightarrow I_{s-1} \xrightarrow{g_s} I_s \longrightarrow 0$$

be a minimal injective resolution of a B-module $\bigtriangledown_B(i)$. By the above Corollary 2, we have the fact that

$$0 \longrightarrow D(A\bigotimes_B D(\bigtriangledown_B(i))) \xrightarrow{D(A\bigotimes_B D(g_0))} D(A\bigotimes_B D(I_0)) \xrightarrow{D(A\bigotimes_B D(g_1))} D(A\bigotimes_B D(I_1))$$
$$\longrightarrow \cdots \longrightarrow D(A\bigotimes_B D(I_{s-1})) \xrightarrow{D(A\bigotimes_B D(g_s))} D(A\bigotimes_B D(I_s)) \longrightarrow 0$$

is a minimal injective resolution of A-module $\nabla_A(i) = D(A\bigotimes_B D(\bigtriangledown_B(i)))$. Assume that $\mathrm{Ext}_A^n(S_A(i), \nabla_A(j)) \neq 0$. By [19, Proposition 1], we have the fact that $S_A(j)$ is a direct summand of soc $D(A\bigotimes_B D(I_n))$. By the above Lemma 2, we have the fact that $S_B(i)$ is a direct summand of soc I_n. Again by [19, Proposition 1], we have $\mathrm{Ext}_B^n(S_B(i), \nabla_B(j)) \neq 0$. Since B has a Kazhdan-Lusztig theory relative to l, $n \equiv l(i)+l(j)$ (mod 2). □

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