Journal of Mathematical Research & Exposition
Jan., 2009, Vol. 29, No. 1, pp. 169–176
DOI:10.3770/j.issn:1000-341X.2009.01.022
Http://jmre.dlut.edu.cn

The Filter Lattices on R_0 Algebras

LUO Qing Jun
(Department of Mathematics, Xi'an Institute of Finance and Economics, Shaanxi 710061, China)
(E-mail: qingjunlou@163.com)

Abstract In the present paper, some basic properties of MP filters of R_0 algebra M are investigated. It is proved that $(\mathcal{F}_{MP}(M), \subset, ', \bar{\wedge}, \bar{\vee}, \{1\}, M)$ is a bounded distributive lattice by introducing the negation operator $'$, the meet operator $\bar{\wedge}$, the join operator $\bar{\vee}$ and the implication operator $\Longrightarrow$ on the set $\mathcal{F}_{MP}(M)$ of all MP filters of M. Moreover, some conditions under which $(\mathcal{F}_{MP}(M), \subset, ', \bar{\vee}, \Longrightarrow, \{1\}, M)$ is an R_0 algebra are given. And the relationship between prime elements of $\mathcal{F}_{MP}(M)$ and prime filters of M is studied. Finally, some equivalent characterizations of prime elements of $\mathcal{F}_{MP}(M)$ are obtained.

Keywords R_0 algebra; MP filter; prime element; equivalent characterization.

Document code A
MR(2000) Subject Classification 03G10; 03G25; 06D99
Chinese Library Classification O141.1

1. Introduction

As is well known, the algebra of logic system as its algebraic semantic is an important research branch in mathematical logics. In the past several years, it has attracted more and more attention and it also has acquired significant development, as pointed out in [5]. Furthermore, different logic algebras such as MV-algebra proposed by Chang in [2], FI-algebra proposed by Wu in [3], Lattice implication algebra introduced by Xu in [4] and BL-algebra invented by Hájek in [5] play the same role in non-classical logics just as Boolean algebra in classical logic.

For trying to provide a logic foundation for fuzzy reasoning and to reduce the gap between fuzzy reasoning and artificial intelligence, Wang proposed a new formal deductive system $\mathcal{L}^*$ for fuzzy propositional calculus in 1997. The corresponding R_0 algebra was introduced subsequently by Wang for the purpose of providing an algebraic proof of the completeness theorem of $\mathcal{L}^*$. In the recent years, the research on R_0 algebras and the formal deductive system $\mathcal{L}^*$ have attracted more and more attention [9, 10, 11, 15]. A series of papers investigating the properties of MP filters of R_0 algebras have been published[12–14]. Unfortunately, in these papers the MP filters are examined individually but not collectively.

In the present paper, the set consisting of all MP filters of an R_0 algebra is taken into account. The operations on this set are introduced, and some basic properties are investigated. Moreover, some new results which are fundamental to study the structure and properties of an R_0 algebra are obtained.

Received date: 2006-12-07; **Accepted date**: 2008-03-08

2. Prelimiaries

Definition 1[1] *Let M be an algebra of type $(\neg, \vee, \to)$. If $(M, \leq, 0, 1)$ is a bounded distributive lattice with a partial order $\leq$ (0 and 1 are the least element and the greatest element of M with respect to $\leq$, respectively), $\vee$ is the supremum operator, and $\neg$ is an order-reserving involution, then M is called an R_0 algebra if for all $a, b, c \in M$ the following conditions are satisfied:*

(M1) $\neg a \to \neg b = b \to a$,

(M2) $1 \to a = a, a \to a = 1$,

(M3) $b \to c \leq (a \to b) \to (a \to c)$,

(M4) $a \to (b \to c) = b \to (a \to c)$,

(M5) $a \to b \wedge c = (a \to b) \wedge (a \to c)$, $a \to b \vee c = (a \to b) \vee (a \to c)$,

(M6) $(a \to b) \vee ((a \to b) \to \neg a \vee b) = 1$.

Example 1[1] Define on [0,1] one unary operator $\neg$ and two binary operators $\vee$ and $\to$ as follows:

$$\neg a = 1 - a, a \vee b = \max\{a, b\}, a \to b = \begin{cases} 1\,, & a \leq b, \\ (1-a) \vee b\,, & a > b. \end{cases}$$

Then ([0,1], $\neg$, $\vee$, $\to$) becomes an R_0 algebra and is called the R_0-interval.

Proposition 1[1] *Let M be an R_0 algebra, and $a, b, c \in M$. Then the following properties hold.*

(P1) $a \to b = 1$ *if and only if* $a \leq b$.

(P2) $a \leq b \to c$ *if and only if* $b \leq a \to c$.

(P3) $a \vee b \to c = (a \to c) \wedge (b \to c)$, $a \wedge b \to c = (a \to c) \vee (b \to c)$.

(P4) $a \vee b \leq ((a \to b) \to b) \wedge ((b \to a) \to a)$.

(P5) $(a \to b) \vee (b \to a) = 1$.

Proposition 2[1] *Let M be an R_0 algebra, and define on M a new binary operator $\otimes$ as follows:*

$$a \otimes b = \neg(a \to \neg b), \quad a, b \in M. \tag{1}$$

Then

(P6) $(M, \otimes, 1)$ *is a commutative semi-group with unit element 1.*

(P7) *If* $b \leq c$, *then* $a \otimes b \leq a \otimes c$.

(P8) $\otimes$ and $\to$ *form an adjoint pair, i.e.,* $a \otimes b \leq c$ *if and only if* $a \leq b \to c$.

(P9) $\neg a \otimes a = 0, a \otimes b \to c = a \to (b \to c)$.

(P10) $a^n = a^2$, *where* $a^n = \underbrace{a \otimes \cdots \otimes a}_{n\text{-times}}$ *for* $n \geq 2$.

(P11) $a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c), (a \vee b)^n = a^n \vee b^n$.

(P12) $a \otimes (a \to b) \leq b$.

3. Filter lattices on R_0 algebras

Definition 2[1] *Let M be an R_0 algebra. A subset F of M is called an MP filter if $1 \in F$, and F is closed under the operation MP, i.e., $a \in F$ and $a \to b \in F$ imply $b \in F$. Moreover, F is said to be proper if $F \neq M$.*

Clearly, $\{1\}$ and M are MP filters. In what follows, we use M and $\mathcal{F}_{MP}(M)$ to denote an R_0 algebra and the set consisting of all MP filters of M respectively unless otherwise explicitly specified.

It should be noted that MP filters are usually different from lattice filters, i.e., every MP filter of M is a lattice filter for M, but the converse is not true in general. In fact, in the R_0-interval, $[\frac{1}{2},1]$ is a lattice filter, but not an MP filter, because $\frac{1}{2}, \frac{1}{2}\to 0=(1-\frac{1}{2})\vee 0=\frac{1}{2}\in[\frac{1}{2},1]$, however, $0\notin[\frac{1}{2},1]$.

Proposition 3[1] *A nonempty subset F of M is an MP filter of M if and only if*

(i) F is an upper set, i.e., $a\in F$ and $a\le b$ imply $b\in F$.

(ii) F is closed under the operation $\otimes$, i.e., $a\in F$ and $b\in F$ imply $a\otimes b\in F$.

Definition 3[1] *Suppose that $A\subset M$. Then the intersection of all MP filters containing A is the smallest MP filter that contains A, and it is called the generated MP filter by A. Formally, it is denoted by $[A)$, and*

$$[A)=\{x\in M|\ a_1\otimes a_2\otimes\cdots\otimes a_n\le x, \textit{for some } a_1,a_2,\ldots,a_n\in A \textit{ and } n\in N\}.$$

In particular, if $A=\{a\}$ with $a\in M$. Let us use $[a)$ to denote the MP filter generated by $\{a\}$, and call $[a)$ a principal MP filter. Clearly,

$$[a)=\{x\in M|a^n\le x, n\in N\}=\{x\in M|a^2\le x, n\in N\}.$$

Proposition 4 *Suppose that $F_1,F_2\subset M$. Then we have the following:*

(i) If $F_1\subset F_2$, then $[F_1)\subset[F_2)$.

(ii) If $x\le y$, then $[y)\subset[x)$.

(iii) If $F_1\in\mathcal{F}_{MP}(M), a\in M$, then $[a)\subset F_1$ if and only if $a\in F_1$.

(iv) If $F_1,F_2\in\mathcal{F}_{MP}(M)$, then $F_1\cap F_2\in\mathcal{F}_{MP}(M)$.

(v) If $F_1,F_2\in\mathcal{F}_{MP}(M)$, then $[F_1\cup F_2)=\{x\in M|a\otimes b\le x,$ for some $a\in F_1$ and $b\in F_2\}$.

Proof The proofs for (i)–(iv) are trivial. Hence we restrict ourselves to the proof of (v). Let $A=\{x\in M|a\otimes b\le x,$ for some $a\in F_1$ and $b\in F_2\}$. Then A is an MP filter. In fact, it follows from $1\in F_1$ and $1\in F_2$ that $1\in A$. Assume that $x, x\to y\in A$. Then there exist $a_1,a_2\in F_1$ and $b_1,b_2\in F_2$ such that $a_1\otimes b_1\le x, a_2\otimes b_2\le x\to y$. It follows from the commutativity and associativity of the operator $\otimes$ and (P12) that $(a_1\otimes b_1)\otimes(a_2\otimes b_2)=(a_1\otimes a_2)\otimes(b_1\otimes b_2)\le x\otimes(x\to y)\le y$. Since $a_1\otimes a_2\in F_1$ and $b_1\otimes b_2\in F_2$, we have $y\in A$. Therefore, A is an MP filter. Now we must show that A is the smallest MP filter containing F_1 and F_2. Assume that F is any MP filter containing F_1 and F_2, and $x\in A$. Then there exist $a\in F_1\subset F$ and $b\in F_2\subset F$ such that $a\otimes b\le x$. Hence it follows from closeness of F under the operator $\otimes$ that $x\in F$. This shows that $A\subset F$. Therefore, $[F_1\cup F_2)=\{x\in M|a\otimes b\le x,$ for some $a\in F_1$ and $b\in F_2\}$.

Theorem 1 *Let us define on $\mathcal{F}_{MP}(M)$ two binary operators $\bar{\wedge}$ and $\bar{\vee}$ as follows:*

$$F_1\bar{\wedge}F_2=F_1\cap F_2,\quad F_1\bar{\vee}F_2=[F_1\cup F_2),\quad F_1,F_2\in\mathcal{F}_{MP}(M). \tag{2}$$

Then $(\mathcal{F}_{MP}(M),\subset,\bar{\wedge},\bar{\vee},\{1\},M)$ is a bounded distributive lattice.

Proof From Proposition 4 and (2) we see that $(\mathcal{F}_{MP}(M),\subset,\bar{\wedge},\bar{\vee},\{1\},M)$ is a bounded lattice with respect to the order of set inclusion. To prove that $(\mathcal{F}_{MP}(M),\subset,\bar{\wedge},\bar{\vee},\{1\},M)$ is distributive, we shall first show that $F_1\cap[F_2\cup F_3)=[F_1\cap(F_2\cup F_3))$ holds in $\mathcal{F}_{MP}(M)$. In fact, since $F_1\cap(F_2\cup F_3)\subset F_1\cap[F_2\cup F_3)$, we have $[F_1\cap(F_2\cup F_3))\subset[F_1\cap[F_2\cup F_3))=F_1\cap[F_2\cup F_3)$. Let x be any element of $F_1\cap[F_2\cup F_3)$. Then $x\in F_1$ and $x\in[F_2\cup F_3)$, hence there exist $a\in F_2$ and $b\in F_3$ such that $a\otimes b\le x$. It follows from $F_1\cap F_2$ and $F_1\cap F_3$ are upper sets that $a\vee x\in F_1\cap F_2$ and $b\vee x\in F_1\cap F_3$. Moreover, since $(a\vee x)\otimes(b\vee x)=(a\otimes b)\vee(x\otimes b)\vee((a\vee x)\otimes x)\le x$, we deduce that $x\in[(F_1\cap F_2)\cup(F_1\cap F_3))=[F_1\cap(F_2\cup F_3))$. This shows that $F_1\cap[F_2\cup F_3)\subset[F_1\cap(F_2\cup F_3))$. Then $F_1\cap[F_2\cup F_3)=[F_1\cap(F_2\cup F_3))$. Hence $F_1\bar{\wedge}(F_2\bar{\vee}F_3)=F_1\cap[F_2\cup F_3)=[F_1\cap(F_2\cup F_3))=[(F_1\cap F_2)\cup(F_1\cap F_3))=(F_1\bar{\wedge}F_2)\bar{\vee}(F_1\bar{\wedge}F_3)$. Therefore Theorem 1 is true.

Lemma 1 *Suppose that $x,y\in M$. Then $[x)\bar{\vee}[y)=[x\otimes y)$ and $[x)\bar{\wedge}[y)=[x\vee y)$.*

Proof By Proposition 4, it follows that $[x)\subset[x\otimes y),[y)\subset[x\otimes y)$, and so $[x)\bar{\vee}[y)\subset[x\otimes y)$. Let $z\in[x\otimes y)$. Then $(x\otimes y)^2=x^2\otimes y^2\le z$. From $x^2\in[x),y^2\in[y)$ and $[x)\bar{\vee}[y)=[[x)\cup[y))$, we have $z\in[x)\bar{\vee}[y)$, and it follows that $[x\otimes y)\subset[x)\bar{\vee}[y)$. Hence $[x)\bar{\vee}[y)=[x\otimes y)$. Next we are to prove that $[x)\bar{\wedge}[y)=[x\vee y)$ holds. $[x\vee y)\subset[x)\bar{\wedge}[y)$ obviously holds. Now assume that $z\in[x)\bar{\wedge}[y)$. Then $z\in[x),z\in[y)$, i.e., $x^2\le z,y^2\le z$. Hence $x^2\vee y^2=(x\vee y)^2\le z$, and so $z\in[x\vee y)$. Thus $[x)\bar{\wedge}[y)=[x\vee y)$.

Proposition 5 *Define on $\mathcal{F}_{MP}(M)$ an implication operator $\Longrightarrow$ as follows:*

$$F_1\Longrightarrow F_2=\{x\in M\mid F_1\bar{\wedge}[x)\subset F_2\},\quad F_1,F_2\in\mathcal{F}_{MP}(M).\tag{3}$$

Then

(i) $F_1\Longrightarrow F_2\in\mathcal{F}_{MP}(M)$.

(ii) $F\in\mathcal{F}_{MP}(M)$, $F\subset F_1\Longrightarrow F_2$ if and only if $F\bar{\wedge}F_1\subset F_2$.

Proof (i) It is clear from (3) that $1\in F_1\Longrightarrow F_2$. Suppose that $x,x\to y\in F_1\Longrightarrow F_2$. Then $F_1\bar{\wedge}[x)\subset F_2$ and $F_1\bar{\wedge}[x\to y)\subset F_2$. Hence it follows from Theorem 1 and Lemma 1 that $(F_1\bar{\wedge}[x))\bar{\vee}(F_1\bar{\wedge}[x\to y))=F_1\bar{\wedge}([x)\bar{\vee}[x\to y))=F_1\bar{\wedge}[x\otimes(x\to y))\subset F_2$, and so we have $F_1\bar{\wedge}[y)\subset F_1\bar{\wedge}[x\otimes(x\to y))\subset F_2$ from (P12) and Proposition 4(ii). Thus $y\in F_1\Longrightarrow F_2$. This shows that $F_1\Longrightarrow F_2$ is an MP filter.

(ii) Suppose $F\subset F_1\Longrightarrow F_2$, and $x\in F$. Then it follows from (3) that $F_1\cap\{x\}\subset F_1\bar{\wedge}[x)\subset F_2$ holds. Hence, $F\bar{\wedge}F_1=F\cap F_1=F_1\cap(\cup_{x\in F}\{x\})=\cup_{x\in F}(F_1\cap\{x\})\subset F_2$. Conversely, assume that $F\bar{\wedge}F_1\subset F_2$. Then for any $x\in F$, we have $[x)\subset F$, and it follows that $F_1\bar{\wedge}[x)\subset F_1\bar{\wedge}F\subset F_2$. Hence $x\in F_1\Longrightarrow F_2$, and so $F\subset F_1\Longrightarrow F_2$.

From Proposition 5, it is easy to deduce the following Corollaries 1 and 2.

Corollary 1 *$(\mathcal{F}_{MP}(M),\subset,\bar{\wedge},\bar{\vee},\Longrightarrow,\{1\},M)$ is a Heyting algebra.*

Corollary 2 *Suppose that $F,F_1,F_2,F_3\in\mathcal{F}_{MP}(M)$. Then the following properties hold.*

(i) $F_1\Longrightarrow F_2=M$ if and only if $F_1\subset F_2$.

(ii) $F_1 \Longrightarrow (F_2 \Longrightarrow F_3) = F_1 \bar{\wedge} F_2 \Longrightarrow F_3 = F_2 \Longrightarrow (F_1 \Longrightarrow F_3)$.

(iii) $M \Longrightarrow F = F, F \Longrightarrow F = M$.

(iv) $F_2 \Longrightarrow F_3 \subset (F_1 \Longrightarrow F_2) \Longrightarrow (F_1 \Longrightarrow F_3)$.

(v) $F_1 \Longrightarrow F_2 \bar{\wedge} F_3 = (F_1 \Longrightarrow F_2) \bar{\wedge} (F_1 \Longrightarrow F_3), F_1 \Longrightarrow F_2 \bar{\vee} F_3 = (F_1 \Longrightarrow F_2) \bar{\vee} (F_1 \Longrightarrow F_3)$.

(vi) $F_1 \bar{\wedge} (F_1 \Longrightarrow F_2) = F_1 \bar{\wedge} F_2, F_2 \bar{\wedge} (F_1 \Longrightarrow F_2) = F_2$.

(vii) $F_1 \bar{\vee} F_2 \subset ((F_1 \Longrightarrow F_2) \Longrightarrow F_2) \bar{\wedge} ((F_2 \Longrightarrow F_1) \Longrightarrow F_1)$.

Definition 4 *Suppose that $F \in \mathcal{F}_{MP}(M)$. Let*

$$F' = F \Longrightarrow \{1\} = \{x \in M \mid F \bar{\wedge} [x) = \{1\}\}, \quad F \in \mathcal{F}_{MP}(M). \tag{4}$$

Then F' is called the negation of F.

It is clear that $F' \in \mathcal{F}_{MP}(M)$ and $F_1'' = F_1' \Longrightarrow \{1\} \in \mathcal{F}_{MP}(M)$. In particular, if $F_1 = [a)$ with $a \in M$, then $[a)' = \{x \in M \mid [a) \bar{\wedge} [x) = \{1\}\} = \{x \in M \mid a \vee x = 1\}$.

Proposition 6 *Suppose that $F, F_1, F_2 \in \mathcal{F}_{MP}(M)$. Then it is easy to prove the following:*

(i) $F \bar{\wedge} F' = F \cap F' = \{1\}, (F_1 \bar{\vee} F_2)' = F_1' \bar{\wedge} F_2'$.

(ii) *If $F_1 \subset F_2$, then $F_2' \subset F_1'$.*

(iii) $F \subset F''$, $F' = F'''$.

(iv) $\{1\}' = M, M' = \{1\}$.

(v) $F_1 \Longrightarrow F_2 \subset F_2' \Longrightarrow F_1'$.

Remark 1 From the Proposition above we see that the operator $'$ is order-reserving, but may be not an involution in $\mathcal{F}_{MP}(M)$, i.e., F is not equal to F'' in general in $\mathcal{F}_{MP}(M)$. For example, consider $M = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ with the usual order $\leq$. Let us define on M one unary operator $\neg$ and two binary operators $\vee$ and $\rightarrow$ on M as follows:

$$\neg a = 1 - a, \quad a \vee b = \max\{a, b\}, \quad a \rightarrow b = \begin{cases} 1\,, & a \leq b, \\ (1-a) \vee b\,, & a > b. \end{cases} \tag{5}$$

Then it is easy to verify that $(M, \leq, \neg, \vee, \rightarrow)$ is an R_0 algebra and $F = \{\frac{2}{3}, 1\}$ is an MP filter of M, however $F'' = M$. Hence $F \neq F''$.

Proposition 7 *The operator $'$ defined by (4) on $\mathcal{F}_{MP}(M)$ is an order-reserving involution if and only if for every $F \in \mathcal{F}_{MP}(M)$, $F \bar{\vee} F' = M$ holds.*

Proof Suppose that the operator $'$ is an order-reserving involution. Then De Morgan's laws hold in $\mathcal{F}_{MP}(M)$, i.e., for all $F_1, F_2 \in \mathcal{F}_{MP}(M)$, $(F_1 \bar{\vee} F_2)' = F_1' \bar{\wedge} F_2'$ and $(F_1 \bar{\wedge} F_2)' = F_1' \bar{\vee} F_2'$ hold. Hence we deduce that $F \bar{\vee} F' = F'' \bar{\vee} F' = (F' \bar{\wedge} F)' = \{1\}' = M$. Conversely, assume that for every $F \in \mathcal{F}_{MP}(M)$, $F \bar{\vee} F' = M$. Since $\mathcal{F}_{MP}(M)$ is distributive, we have that $F'' = F'' \bar{\wedge} (F \bar{\vee} F') = (F'' \bar{\wedge} F) \bar{\vee} (F'' \bar{\wedge} F') = F'' \bar{\wedge} F$, i.e., $F'' \subset F$. By virtue of Proposition 6(iii), we have $F = F''$. Hence the operator $'$ is an order-reserving involution.

Theorem 2 *$(\mathcal{F}_{MP}(M), \subset, ', \bar{\vee}, \Longrightarrow, \{1\}, M)$ is an R_0 algebra if and only if for every $F \in \mathcal{F}_{MP}(M), F \bar{\vee} F' = M$ holds.*

Proof Suppose that $(\mathcal{F}_{MP}(M), \subset, ', \bar{\vee}, \Longrightarrow, \{1\}, M)$ is an R_0 algebra. Then the operator $'$ is an order-reserving involution on $\mathcal{F}_{MP}(M)$. Hence it follows from Proposition 7 that $F \bar{\vee} F' = M$. Conversely, suppose that for every $F \in \mathcal{F}_{MP}(M)$, $F \bar{\vee} F' = M$ holds. Then by Proposition 7, we have that the operator $'$ is an order-reserving involution, and it follows from Proposition 6(vi) that $F_2' \Longrightarrow F_1' \subset F_1' \Longrightarrow F_2' = F_1 \Longrightarrow F_2$. Therefore $F_1 \Longrightarrow F_2 = F_2' \Longrightarrow F_1'$, that is, (M1) in Definition 1 holds. And it can be proved that (M2)–(M5) also hold from Corollary 2. Now we are to prove that (M6) holds. It follows from Corollary 2(ii), (v) and (vi) that

$$
\begin{aligned}
&(F_1 \Longrightarrow F_2) \bar{\vee} ((F_1 \Longrightarrow F_2) \Longrightarrow F_1' \bar{\vee} F_2) \\
&\quad = (F_1 \Longrightarrow F_2) \bar{\vee} (((F_1 \Longrightarrow F_2) \Longrightarrow F_1') \bar{\vee} ((F_1 \Longrightarrow F_2) \Longrightarrow F_2)) \\
&\quad \supset (F_1 \Longrightarrow F_2) \bar{\vee} ((F_1 \Longrightarrow F_2) \Longrightarrow F_1') \\
&\quad = (F_1 \Longrightarrow F_2) \bar{\vee} (F_1 \bar{\wedge} (F_1 \Longrightarrow F_2) \Longrightarrow \{1\}) \\
&\quad = (F_1 \Longrightarrow F_2) \bar{\vee} (F_1 \bar{\wedge} F_2)' = (F_1 \Longrightarrow F_2) \bar{\vee} F_1' \bar{\vee} F_2' \\
&\quad \supset F_2 \bar{\vee} F_2' = M.
\end{aligned}
$$

Thus (M6) is true. This proves that $(\mathcal{F}_{MP}(M), \subset, ', \bar{\vee}, \Longrightarrow, \{1\}, M)$ is an R_0 algebra.

Theorem 3 *$(\mathcal{F}_{MP}(M), \subset, ', \bar{\vee}, \Longrightarrow, \{1\}, M)$ is an R_0 algebra if and only if the mapping $'$: $\mathcal{F}_{MP}(M) \to \mathcal{F}_{MP}(M)$ is surjective ($F^{'} = F \Longrightarrow \{1\} = \{x \in M \mid F \bar{\wedge} [x) = \{1\}\}, F \in \mathcal{F}_{MP}(M)$).*

Proof The necessity is trivial. Now let us prove the sufficiency. Since the mapping $'$ is surjective, for any $F \in \mathcal{F}_{MP}(M)$, there exists $F_1 \in \mathcal{F}_{MP}(M)$ such that $F \bar{\vee} F' = F_1'$. By Proposition 6(i) and (v), it follows that $F_1'' = (F \bar{\vee} F')' = F' \bar{\wedge} F'' = \{1\}$, and so we have $F_1' = F_1''' = \{1\}' = M$, i.e., $F \bar{\vee} F' = M$. Hence from Theorem 2 we see that $(\mathcal{F}_{MP}(M), \subset, ', \bar{\vee}, \Longrightarrow, \{1\}, M)$ is an R_0 algebra.

Theorem 4 *If every MP filter of M is principal and, for every $x \in M$, $\neg x \vee x = 1$, then $(\mathcal{F}_{MP}(M), \subset, ', \bar{\wedge}, \bar{\vee}, \Longrightarrow, \{1\}, M)$ is a Boolean algebra.*

Proof It follows from Theorem 1 that $(\mathcal{F}_{MP}(M), \subset, ', \bar{\wedge}, \bar{\vee}, \Longrightarrow, \{1\}, M)$ is a bounded distributive lattice. To prove that $(\mathcal{F}_{MP}(M), \subset, ', \bar{\wedge}, \bar{\vee}, \Longrightarrow, \{1\}, M)$ is a Boolean algebra, it is only necessary to prove that for every $F \in \mathcal{F}_{MP}(M)$, the complement of F exists in $\mathcal{F}_{MP}(M)$. In fact, since F is principal, there exists $a \in M$ such that $F = [a)$. Therefore, $F' = [a)' = \{x \in M \mid a \vee x = 1\}$. By the condition $\neg a \vee a = 1$, we have $\neg a \in F'$. Hence $0 = \neg a \otimes a \in F \bar{\vee} F'$, then $F \bar{\vee} F' = M$, and $F \bar{\wedge} F' = \{1\}$ from Proposition 6(i). So we see that F' is just the complement of F. Thus, $(\mathcal{F}_{MP}(M), \subset, ', \bar{\wedge}, \bar{\vee}, \Longrightarrow, \{1\}, M)$ is a Boolean algebra.

4. Prime elements of $\mathcal{F}_{MP}(M)$

Definition 5[8] *Let $(L, \leq, 0, 1)$ be a bounded lattice. For any $a \in L$ and $a \neq 1$, a is meet-irreducible if $x \wedge y = a$ implies $x = a$ or $y = a$. a is called a prime element of L if $x \wedge y \leq a$ implies $x \leq a$ or $y \leq a$.*

Proposition 8[8] *Let $(L, \leq, 0, 1)$ be a distributive lattice and $a \in L$. a is meet-irreducible if* *and only if a is prime element of L.*

Definition 6[1] *A proper MP filter F of M is called to be prime if for all $x, y \in M$, $x \vee y \in F$ implies $x \in F$ or $y \in F$. The set consisting of all prime filters of M will be denoted by $\mathcal{PF}_{MP}(M)$.*

Proposition 9[1] *Suppose that $F \in \mathcal{F}_{MP}(M)$. F is prime if and only if for any $x, y \in M$, either $x \to y \in F$ or $y \to x \in F$.*

Proposition 10 *Suppose that F is a prime filter of M. Then*

(i) If F_1 is a proper MP filter such that $F \subset F_1$, then F_1 is also prime.

(ii) If $\{F_i\}_{i \in I}$ is a family of MP filters containing F, that is, $F \subset \cap_{i \in I} F_i$, then $\{F_i\}_{i \in I}$ is a chain.

Proof (i) follows from Proposition 9 and the condition $F \subset F_1$.

(ii) Suppose that F_1 and F_2 are two any MP filters containing F. If one of F_1 and F_2 is M, then it is clear that F_1 and F_2 are comparable. Now assume that neither F_1 nor F_2 is equal to M. Then by $F_1 \bar{\wedge} F_2 = F_1 \cap F_2 \neq M$ and $F \subset F_1 \bar{\wedge} F_2$ and (i), it follows that $F_1 \bar{\wedge} F_2$ is a prime filter. Let $F_1 \not\subset F_2$ and $F_2 \not\subset F_1$. Then there exist $x \in F_1, y \in F_2$ such that $x \notin F_2$ and $y \notin F_1$. Thus $x, y \notin F_1 \bar{\wedge} F_2$. Since $F_1 \bar{\wedge} F_2$ is prime, we have $x \vee y \notin F_1 \bar{\wedge} F_2$, and it follows that $x \vee y \notin F_1$ or $x \vee y \notin F_2$ which contradicts the fact that F_1 and F_2 are upper sets. Hence (ii) is verified.

Theorem 5 *Suppose that $F \subset M$. Then F is a prime element of $(\mathcal{F}_{MP}(M), \subset, \{1\}, M)$ if and only if F is a prime filter of M.*

Proof Suppose that F is a prime element of $(\mathcal{F}_{MP}(M), \subset, \{1\}, M)$. If $x \vee y \in F$, then $[x \vee y) \subset F$. Hence $(F \bar{\vee} [x)) \bar{\wedge} (F \bar{\vee} [y)) = F \bar{\vee} ([x) \bar{\wedge} [y)) = F \bar{\vee} [x \vee y) = F$. Since F is a prime element of $(\mathcal{F}_{MP}(M), \subset)$, we have $F \bar{\vee} [x) = F$ or $F \bar{\vee} [y) = F$, and so $x \in F$ or $y \in F$. We have F is a prime filter of M. Conversely, assume that F is a prime filter of M. If $F_1 \bar{\wedge} F_2 = F$, $F_1, F_2 \in \mathcal{F}_{MP}(M)$, then it follows from Propositiom 10(ii) that $F_1 \subset F_2$ or $F_2 \subset F_1$, and then $F = F_1$ or $F = F_2$ follows. Thus F is a prime element of $(\mathcal{F}_{MP}(M), \subset)$.

Theorem 6 *Suppose that $F \in \mathcal{F}_{MP}(M)$. Then the following conditions are equivalent to each other:*

(i) F is a prime element of $(\mathcal{F}_{MP}(M), \subset, \{1\}, M)$.

(ii) If $x, y \in M$ and $x \vee y = 1$, then $x \in F$ or $y \in F$.

(iii) $M - F$ is an directed set, i.e., if $a, b \in M - F$, then there exists $c \in M - F$ such that $a, b \leq c$.

(iv) For any $F_i \in \mathcal{F}_{MP}(M)$ ($i \in I$, and I is an index set), either $F_i \Longrightarrow F = F$ or $F_i \subset F$.

Proof (i)$\Longrightarrow$(ii) Suppose that F is a prime element of $(\mathcal{F}_{MP}(M), \subset, \{1\}, M)$. It follows from Theorem 5 that F is a prime filter of M. Hence if $x \vee y = 1$, then $x \vee y \in F$. So we have $x \in F$ or $y \in F$.

(ii)$\Longrightarrow$(iii) Assume that $M - F$ is not directed. Then there exist $x, y \in M - F$ such that $x \vee y \notin M - F$, i.e., $x \vee y \in F$. Since F is an upper set, by (P4) $(x \to y) \to y \in F$ and

$(y \to x) \to x \in F$. By (P5) and (ii), we have $x \to y \in F$ or $y \to x \in F$. Hence $y \in F$ or $x \in F$, a contradiction. Thus, $M - F$ is directed.

(iii)$\Longrightarrow$(iv) Suppose on the contrary that $F_i \not\subset F$. Then there exists $y \in F_i$ such that $y \notin F$, i.e., $y \in M - F$. Let $x \in F_i \to F$. If $x \notin F$, then $x \in M - F$. Since $M - F$ is directed, we deduce that $x \vee y \in M - F$. Since the filter is an upper set, by (4) we have $x \vee y \in F_i \bar{\wedge} [x) \subset F$ and then $x \vee y \in F$ follows, a contradiction. Therefore, $x \in F$, i.e., $F_i \Longrightarrow F \subset F$. By $F_i \bar{\wedge} F = F_i \cap F \subset F$ and Proposition 5(ii), it follows that $F \subset F_i \Longrightarrow F$. Thus $F_i \Longrightarrow F = F$.

(iv)$\Longrightarrow$(i) Suppose that $F_1 \bar{\wedge} F_2 = F$, and $F, F_1, F_2 \in \mathcal{F}_{MP}(M)$. Then $F \subset F_1$ and $F \subset F_2$. It follows from Proposition 5(ii) that $F_1 \subset F_2 \Longrightarrow F$. By the hypothesis, we have $F_2 \Longrightarrow F = F$ or $F_2 \subset F$. Then either $F_1 \subset F$ or $F_2 \subset F$, therefore, $F_1 = F$ or $F_2 = F$. Thus, F is a prime element of $(\mathcal{F}_{MP}(M), \subset, \{1\}, M)$.

Corollary 3 *Suppose that F is a prime element of $(\mathcal{F}_{MP}(M), \subset, \{1\}, M)$, and $x, y \in M$. If $[x) \bar{\wedge} [y) \subset F$, then $x \in F$ or $y \in F$.*

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