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The Sum of Standard Generalized Frames in Hilbert W^*-Module

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Abstract In this paper, the sum of standard generalized frames of Hilbert W^*-module is studied intensively by using operator-theoretic-methods, and some conditions are given to assure that the sum of two or more standard generalized frames is a standard generalized frame.

Keywords generalized frame in Hilbert W^*-module; analyzing operator; dual generalized frame; frame operator; disjointness; strong disjointness.

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1. Introduction

Frames have been used for number years by engineers and applied mathematicians for purposes of signal processing and date compression since Duffin and Schaeffer introduced the notion. A lot of results about this subject in finitely or countably generated Hilbert space have been given by Han and Larson in [1], such as the properties of the frame operator and its dual frame, the disjointness of frames and related properties. Also some properties of Hilbert C^*-modular frame were given by Frank and Larson in [2]. As we know, many results which are true in the Hilbert space, have been proved by Frank and Larson in [2], are also true in the Hilbert C*-module. Such as every countably Hilbert C^*-module has a normalized frame $\{f_i\}_{i=1}^{\infty}$, and so on. Recently, the tool of frame in Hilbert C^*-module has been used in [3] to deal with some problems in the index theory which was introduced by Watatani. The purpose of this paper is to study the sum of two or more standard generalized frames in Hilbert W^*-module. For this, we introduce some basic notions first.

Definition 1.1 *Let $\mathcal{A}$ be a C^*-algebra. A pre-Hilbert $\mathcal{A}$-module is a linear space and algebraic (Left) $\mathcal{A}$-module $\mathscr{H}$ together with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle$: $\mathscr{H}\times\mathscr{H}\longrightarrow\mathcal{A}$ that possesses the following properties :*

(1) $\langle x,x\rangle\geq 0$ *for any* $x\in\mathscr{H}$*;*

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(2) $\langle x, x\rangle = 0$ *if and only if* $x = 0$*;*

(3) $\langle x, y\rangle = \langle y, x\rangle^*$ *for any* $x, y \in \mathscr{H}$*;*

(4) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ *for any* $a, b \in \mathcal{A}$*,* $x, y, z \in \mathscr{H}$*.*

To circumvent complications with linearity of the $\mathcal{A}$-valued inner product with respect to imaginary complex numbers, we assume that the linear operations of $\mathcal{A}$ and $\mathscr{H}$ are comparable, i.e., $\lambda(ax) = (\lambda a)x = a(\lambda x)$ for every $\lambda \in \mathbb{C}, a \in A$ and $x \in \mathscr{H}$. The map $x \in H \longrightarrow \|x\| = \|\langle x, x\rangle\|_{\mathcal{A}}^{\frac{1}{2}} \in R^+$ defines a norm on $\mathscr{H}$. If $\mathscr{H}$ is complete with respect to that norm, $\mathscr{H}$ becomes the structure of a Banach $\mathcal{A}$-module. We refer to the pair $\{\mathscr{H};\ \langle\cdot,\cdot\rangle\}$ as a Hilbert $\mathcal{A}$-module. And if $\mathcal{A}$ is a von Neumann algebra, then it will be called Hilbert W^*-module.

Definition 1.2[2] *Let* $\mathcal{A}$ *be a unital* C^**-algebra and* $\mathbb{J}$ *be a finite or countable index subset of* $\mathbb{N}$*. A sequence* $\{x_j\colon j \in \mathbb{J}\}$ *of elements in a Hilbert* $\mathcal{A}$*-module* $\mathscr{H}$ *is said to be a frame if there are real constants* $C, D > 0$ *such that*

$$C \cdot \langle x, x\rangle \le \sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle \le D \cdot \langle x, x\rangle$$

for all $x \in \mathscr{H}$*.*

Let $\mathcal{A}$ be a von Neumann algebra on Hilbert space $\mathscr{X}$, (M, S, μ) be a measure space, and $\mathscr{H}$ be a Hilbert $\mathcal{A}$-module. Let $A_m \in \mathcal{A}$, for all $m \in M$. We say

$$\int_M A_m \mathrm{d}\mu(m) = A \in \mathcal{A}^{[3]}$$

which means $\int_M (A_m x, y)\,\mathrm{d}\mu(m) = (Ax, y)$ for all $x, y \in \mathscr{X}$.

Note In this paper, we will always assume the inner product $(\cdot,\cdot)$ to be in $\mathscr{X}$ while $\langle\cdot,\cdot\rangle$ to be in $\mathscr{H}$ in order to distinguish them from each other.

It is obvious that we can obtain the following Lemma.

Lemma 1.3[3]

a) $\int_N A \cdot A_m \mathrm{d}\mu(m) = A \cdot \int_N A_m \mathrm{d}\mu(m)$*, where* $N \subset M$ *is measurable.*

b) If $\mu(N) < \infty$*, then* $\int_N (A_m - A)\mathrm{d}\mu(m) = \int_N A_m \mathrm{d}\mu(m) - A \cdot \mu(N)$*.*

In this paper, we will say that

$$f = \int_M g(m)\mathrm{d}\mu(m) \in \mathscr{H},$$

by the meaning of

$$\langle f, \phi\rangle = \int_M \langle g(m), \phi\rangle \mathrm{d}\mu(m)$$

for all $\phi \in \mathscr{H}$, where $g : M \longrightarrow \mathscr{H}$ is $\mathcal{A}$-linear.

Also we will denote $\mathcal{L}^2(M, \mathcal{A})$ to be the set of all operators $\varphi : M \longrightarrow \mathcal{A}$ with properties that $(\varphi(m)(x), y)$ is a measurable and integrable function in M for arbitrary choice of $x, y \in \mathscr{X}$ and $\| \int_M \varphi(m)(\varphi(m))^* \mathrm{d}\mu(m) \| < \infty$. The inner product in $\mathcal{L}^2(M, \mathcal{A})$ will be defined by

$$\langle \varphi, \psi\rangle_{\mathcal{L}^2(M,\mathcal{A})} = \int_M \varphi(m)(\psi(m))^* \mathrm{d}\mu(m),$$

and hence one can easily check that $\mathcal{L}^2(M,\mathcal{A})$ is a Hilbert $\mathcal{A}$-module with this inner product.

Definition 1.4 *Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space $\mathscr{X}$, $\mathscr{H}$ be a Hilbert $\mathcal{A}$-module, and let (M,S,μ) be a measure space. A generalized frame in $\mathscr{H}$ indexed by M is a family of vectors $h=\{h_m:\ m\in M\}\subset\mathscr{H}$, for which there exist $0<C\le D<\infty$ such that*

$$C\langle f,f\rangle\le\int_M\langle f,h_m\rangle\langle h_m,f\rangle\mathrm{d}\mu(m)\le D\langle f,f\rangle \tag{1.1}$$

for all $f\in\mathscr{H}$.

The condition (1.1) is called to be frame condition, and the number C and D are called frame bounds.

If M is at most countable and μ is the counting measure, then $h=\{h_m:m\in M\}\subset\mathscr{H}$ is in fact a discrete frame of Hilbert $\mathcal{A}$-module $\mathscr{H}$.

In this paper, we mainly discuss the standard generalized frame of Hilbert $\mathcal{A}$-module which means the middle part in the frame condition (1.1) converges in norm.

If $\{h_m:m\in M\}$ is a standard generalized frame. Let $\bar{f}(m)=\langle f,h_m\rangle\in\mathcal{A}$. Then $\bar{f}\in\mathcal{L}^2(M,\mathcal{A})$, and we will call $T_h:f\mapsto\bar{f}$ the analyzing operator corresponding to generalized frame $\{h_m:\ m\in M\}$, and the adjoint operator $T_h^*:\mathcal{L}^2(M,\mathcal{A})\longrightarrow\mathscr{H}$ is given by

$$T_h^*g=\int_M g(m)h_m\mathrm{d}\mu(m)$$

for any $g\in\mathcal{L}^2(M,\mathcal{A})$.

We will call $\{h_m:m\in M\}$ a generalized tight frame if $C=D$, and a generalized normalized tight one if $C=D=1$.

We will say that two generalized frames $\{h_m:m\in M\}$ and $\{\phi_m:m\in M\}$ are similar if there exists an adjointable invertible $\mathcal{A}$-linear operator T such that $Th_m=\phi_m$ for all $m\in M$, and unitarily equivalent if T is unitary. Suppose $\{h_{1m}:m\in M\},\dots,\{h_{km}:m\in M\}$, $\{\phi_{1m}:m\in M\},\dots,\{\phi_{km}:m\in M\}$ are generalized frames. We say that k-tuple $(\{h_{1m}:m\in M\},\dots,\{h_{km}:m\in M\})$ and k-tuple $(\{\phi_{1m}:m\in M\},\dots,\{\phi_{km}:m\in M\})$ are similar if there are adjointable invertible $\mathcal{A}$-linear operators $T_1,\dots,T_k$ such that $T_1h_{1m}=\phi_{1m},T_2h_{2m}=\phi_{2m},\dots,T_kh_{km}=\phi_{km}$ for all $m\in M$.

For a given standard generalized frame $\{h_m:m\in M\}$ corresponding to the analyzing operator T_h, T_h is injective by condition (1.1). Let $S=T_h^*T_h:\mathscr{H}\longrightarrow\mathscr{H}$, then we have

$$Sf=T_h^*T_hf=\int_M(T_hf)(m)h_m\mathrm{d}\mu(m)=\int_M\langle f,h_m\rangle h_m\mathrm{d}\mu(m). \tag{1.2}$$

Clearly, S is a positive invertible operator on $\mathscr{H}$. For $f\in\mathscr{H}$, we can easily get from (1.2) that

$$f=\int_M\langle f,h_m\rangle S^{-1}h_m\mathrm{d}\mu(m), \tag{1.3}$$

$$S^{-1}f=\int_M\langle f,S^{-1}h_m\rangle S^{-1}h_m\mathrm{d}\mu(m) \tag{1.4}$$

and

$$f=\int_M\langle f,S^{-\frac{1}{2}}h_m\rangle S^{-\frac{1}{2}}h_m\mathrm{d}\mu(m). \tag{1.5}$$

If two Hilbert $\mathcal{A}$-modules $\{\mathscr{H}, \langle\cdot,\cdot\rangle\}$ and $\{\mathscr{K}, \langle\cdot,\cdot\rangle\}$ over a von Neumann algebra $\mathcal{A}$ are given, we define their direct sum $\mathscr{H}\oplus\mathscr{K}$ as the set of all ordered pairs $\{(h,k) : h \in \mathscr{H},\ \in \mathscr{K}\}$ equipped with coordinate-wise operations and with the $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathscr{K}} + \langle\cdot,\cdot\rangle_{\mathscr{K}}$.

Remark 1.5 (a) By (1.5), a standard generalized frame $\{h_m : m \in M\}$ is similar to a standard generalized normalized tight frame $\{S^{-\frac{1}{2}}h_m : m \in M\}$.

(b) We will call $\{S^{-1}h_m : m \in M\}$ the canonical dual frame of $\{h_m : m \in M\}$, and we will denote $h_m^* = S^{-1}h_m$.

(c) If $\{h_m : m \in M\}$ and $\{\phi_m : m \in M\}$ are both standard generalized frames of $\mathscr{H}$ such that for every $f \in \mathscr{H}$, the equality

$$f = \int_M \langle f, h_m\rangle \phi_m \mathrm{d}\mu(m)$$

holds, then we call $\{\phi_m : m \in M\}$ the alternate dual frame of $\{h_m : m \in M\}$.

We will denote by $\mathrm{End}_{\mathcal{A}}(\mathscr{H})$ the set of all $\mathcal{A}$-linear bounded operators on $\mathscr{H}$, by $\mathrm{End}^*_{\mathcal{A}}(\mathscr{H})$ the set of all $\mathcal{A}$-linear adjointable operators on $\mathscr{H}$, and by $\mathrm{End}_{\mathcal{A}}(\mathscr{H},\mathscr{K})$ the set of all bounded $\mathcal{A}$-linear operators from $\mathscr{H}$ to $\mathscr{K}$.

Lemma 1.6 *Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space $\mathscr{X}$, (M, S, μ) be a measure space, and let $\{h_m : m \in M\}$ be a standard generalized frameof a Hilbert $\mathcal{A}$-module$\mathscr{H}$. Then there exists a unique operator $\Phi \in End^*_{\mathcal{A}}(\mathscr{H})$, such that*

$$f = \int_M \langle f, \Phi h_m\rangle h_m \mathrm{d}\mu(m)$$

and Φ is given by $\Phi = A^*A$, where A is an invertible $\mathcal{A}$-linear operator in $\mathrm{End}_{\mathcal{A}}(\mathscr{H},\mathscr{K})$ such that $\{Ah_m : m \in M\}$ is a standard generalized tight frame. In particular, Φ is an invertible $\mathcal{A}$-linear operator on $\mathscr{H}$. Finally, the canonical dual of $\{h_m : m \in M\}$ is standard generalized frame.

We omitted the proof of this lemma here. One can easily check it.

Lemma 1.7 *Suppose that $\{h_m : m \in M\}$ is a standard generalized frame of Hilbert $\mathcal{A}$-module $\mathscr{H}$ with frame operator S_h, and $T : \mathscr{H} \longrightarrow \mathscr{K}$ is an invertible adjointable operator, where $\mathscr{H}$ is also a Hilbert $\mathcal{A}$-module. Then $(Th_m)^* = (T^{-1})^*h_m^*$. In particular, if T is unitary, then $(Th_m)^* = Th_m^*$.*

Proof For any $f \in \mathscr{H}$, we have

$$Tf = \int_M \langle f, h_m^*\rangle Th_m \mathrm{d}\mu(m) = \int_M \langle Tf, ((T^{-1})^*S_h^{-1}T^{-1})Th_m\rangle Th_m \mathrm{d}\mu(m),$$

therefore $(Th_m)^* = ((T^{-1})^*S_h^{-1}T^{-1})Th_m = (T^{-1})^*h_m^*$ by Lemma 1.6. □

Lemma 1.8 *If $\{h_m : m \in M\}$ is a standard generalized frame in Hilbert $\mathcal{A}$-module $\mathscr{H}$, and $C : \mathscr{H} \longrightarrow \mathscr{H}$ is an invertible adjointable operator such that $C^*Ch_m = h_m^*$ for all $m \in M$, then $\{Ch_m : m \in M\}$ is a standard generalized normalized tight frame of $\mathscr{H}$.*

Proof Since $f=\int_M\langle f,h_m^*\rangle h_m\mathrm{d}\mu(m)=\int_M\langle f,C^*Ch_m\rangle h_m\mathrm{d}\mu(m)$, we have

$$C^{-1}f=\int_M\langle C^{-1}f,C^*Ch_m\rangle h_m\mathrm{d}\mu(m)=\int_M\langle f,Ch_m\rangle h_m\mathrm{d}\mu(m),$$

i.e., $f=\int_M\langle f,Ch_m\rangle Ch_m\mathrm{d}\mu(m)$, which implies that $\{Ch_m:m\in M\}$ is a standard generalized normalized tight frame of $\mathscr{H}$. □

Definition 1.9 *A k-tuple $(\{h_{im}:m\in M\};i=1,2,\ldots,k)$ of standard generalized frames of Hilbert $\mathcal{A}$-module $\mathscr{H}$ $(i=1,2,\ldots,k)$, respectively, is called disjoint if $\{h_{1m}\oplus h_{2m}\oplus\cdots\oplus h_{km}:m\in M\}$ is a standard generalized frame of Hilbert $\mathcal{A}$-module $\mathscr{H}\oplus\mathscr{H}\oplus\cdots\oplus\mathscr{H}$.*

Definition 1.10 *A k-tuple $(\{h_{im}:m\in M\};i=1,2,\ldots,k)$ of standard generalized normalized tight frames of Hilbert $\mathcal{A}$-module $\mathscr{H}$ $(i=1,2,\ldots,k)$, respectively, is called strongly disjoint if $\{h_{1m}\oplus h_{2m}\oplus\cdots\oplus h_{km}:m\in M\}$ is a standard generalized normalized tight frame of Hilbert $\mathcal{A}$-module $\mathscr{H}\oplus\mathscr{H}\oplus\cdots\oplus\mathscr{H}$.*

More generally, a k-tuple $(\{h_{im}:m\in M\};i=1,2,\ldots,k)$ of standard generalized frames of Hilbert $\mathcal{A}$-module $\mathscr{H}$ $(i=1,2,\ldots,k)$, respectively, is called strongly disjoint if it is similar to a k-tuple of strongly disjoint standard generalized normalized tight frames of which the direct sum is a standard generalized normalized tight frame of Hilbert $\mathcal{A}$-module $\mathscr{H}\oplus\mathscr{H}\oplus\cdots\oplus\mathscr{H}$, or equivalently we can say it is strongly disjoint if $\{S_{1h}^{-\frac{1}{2}}h_{1m}\oplus S_{2h}^{-\frac{1}{2}}h_{2m}\oplus\cdots\oplus S_{kh}^{-\frac{1}{2}}h_{km}:m\in M\}$ is a standard generalized normalized tight frame of $\mathscr{H}\oplus\mathscr{H}\oplus\cdots\oplus\mathscr{H}$, here, S_{ih} is the frame operator of $\{h_{im}:m\in M\};i=1,2,\ldots,k$, respectively.

Lemma 1.11 *Let $\{h_m:m\in M\}$ and $\{k_m:m\in M\}$ be the standard generalized frames of $\mathscr{H}$ and $\mathscr{K}$, respectively, T_h be the analyzing operator for $\{h_m:m\in M\}$ and T_k for $\{k_m:m\in M\}$. Then we have*

(a) $\{h_m:m\in M\}$ and $\{k_m:m\in M\}$ are strongly disjoint if and only if $T_h^*T_k=0$ (equivalent to $T_k^*T_h=0$).

(b) $\{h_m:m\in M\}$ and $\{k_m:m\in M\}$ are disjoint if and only if $\mathrm{Rng}(T_h)\bigcap\mathrm{Rng}(T_k)=\{0\}$ and $\mathrm{Rng}(T_h)+\mathrm{Rng}(T_k)$ is closed.

We omit the proof of this lemma here. One can easily check this.

2. Main results

Proposition 2.1 *Standard generalized frames $\{h_m:m\in M\}$ and $\{k_m:m\in M\}$ are strongly disjoint if and only if $\{h_m\oplus k_m:m\in M\}$ is a standard generalized frame and $(h_m\oplus k_m)^*=h_m^*\oplus k_m^*$.*

Proof For the sufficiency, assume that $(h_m\oplus k_m)^*=h_m^*\oplus k_m^*$. Suppose A,B are invertible adjointable operators such that $\{Ah_m:m\in M\}$ and $\{Bk_m:m\in M\}$ are standard generalized normalized tight frames and write $\phi_m=Ah_m$ and $\psi_m=Bk_m$ for all $m\in M$. Then $h_m^*=A^*\phi_m,k_m^*=B^*\psi_m$ for all $m\in M$ by Lemma 1.7. Thus $(A^*A\oplus B^*B)(h_m\oplus k_m)=h_m^*\oplus k_m^*=(h_m\oplus k_m)^*$. By Lemma 1.8 we have that $(A\oplus B)(h_m\oplus k_m)$ is a standard generalized normalized tight frame, which implies that $\{h_m : m \in M\}$ and $\{k_m : m \in M\}$ are strongly disjoint by the definition.

Conversely, suppose that $\{h_m : m \in M\}$ and $\{k_m : m \in M\}$ are strongly disjoint. Then let A, B be invertible operators such that $Ah_m = \varphi_m, Bk_m = \psi_m$ for all $m \in M$ with the property that $\{\varphi_m : m \in M\}$ and $\{\psi_m : m \in M\}$ are standard generalized normalized tight frames. We have $\varphi_m \oplus \psi_m = (A \oplus B)(h_m \oplus k_m)$. Thus, by Lemma 1.6,

$$\begin{aligned}(h_m \oplus k_m)^* &= (A \oplus B)^*(\varphi_m \oplus \psi_m) = (A^* \oplus B^*)(\varphi_m \oplus \psi_m)\\ &= A^*\varphi_m \oplus B^*\psi_m = h_m^* \oplus k_m^*.\end{aligned}$$

The proof is completed. □

Corollary 2.2 *A k-tuple $(\{h_{1m} : m \in M\}, \{h_{2m} : m \in M\}, \dots, \{h_{km} : m \in M\})$ of standard generalized frames are strongly disjoint if and only if it is a disjoint k-tuple and the canonical dual of the direct sum frame is equal to the direct sum of their canonical duals.*

Theorem 2.3 *Suppose that $\{h_m : m \in M\}$ and $\{k_m : m \in M\}$ are strongly disjoint standard generalized frames of the same Hilbert $\mathcal{A}$-module $\mathscr{H}$. Then $\{h_m + k_m : m \in M\}$ is a standard generalized frame of $\mathscr{H}$. In particular, if $\{h_m : m \in M\}$ and $\{k_m : m \in M\}$ are strongly disjoint standard generalized normalized tight frames of $\mathscr{H}$, then $\{h_m + k_m : m \in M\}$ is a standard generalized tight frame of $\mathscr{H}$ with frame bound 2.*

Proof For any $f \in \mathscr{H}$, we compute

$$\begin{aligned}&\int_M \langle f, h_m + k_m\rangle\langle h_m + k_m, f\rangle \mathrm{d}\mu(m)+\\ &\quad= \int_M \langle f, h_m\rangle\langle h_m, f\rangle \mathrm{d}\mu(m) + \int_M \langle f, h_m\rangle\langle k_m, f\rangle \mathrm{d}\mu(m)+\\ &\qquad \int_M \langle f, k_m\rangle\langle h_m, f\rangle \mathrm{d}\mu(m) + \int_M \langle f, k_m\rangle\langle k_m, f\rangle \mathrm{d}\mu(m)\\ &\quad= \int_M \langle f, h_m\rangle\langle h_m, f\rangle \mathrm{d}\mu(m) + \int_M \langle f, k_m\rangle\langle k_m, f\rangle \mathrm{d}\mu(m),\end{aligned}$$

where the second equality is available since $\{h_m : m \in M\}$ and $\{k_m : m \in M\}$ are strongly disjoint. If $\{h_m : m \in M\}$ is a standard generalized frame with bounds C_h, D_h, and $\{k_m : m \in M\}$ is a standard generalized frame with bounds C_k, D_k, then we can see from the above that

$$(C_h + C_k)\langle f, f\rangle \le \int_M \langle f, h_m + k_m\rangle\langle h_m + k_m, f\rangle \mathrm{d}\mu(m) \le (D_h + D_k)\langle f, f\rangle$$

as asserted in the first part.

The assertion of the second part is obvious by the equalities above. □

Corollary 2.4 *Suppose that $(\{h_{1m} : m \in M\}, \{h_{2m} : m \in M\}, \dots, \{h_{km} : m \in M\})$ is a strongly disjoint k-tuple of standard generalized frames of the same Hilbert $\mathcal{A}$-module $\mathscr{H}$. Then the sum of these frames is a standard generalized frame of $\mathscr{H}$. In particular, if it is a strongly disjoint k-tuple of standard generalized normalized tight frames of $\mathscr{H}$, then the sum of these* *frames is a tight frame of $\mathscr{H}$ with frame bound k.*

For the weaker notion of disjointness, we have the following result.

Theorem 2.5 *If $\{h_m : m \in M\}$ and $\{k_m : m \in M\}$ are disjoint standard generalized frames of the same Hilbert $\mathcal{A}$-module $\mathscr{H}$, then $\{h_m + k_m : m \in M\}$ is also a standard generalized frame of $\mathscr{H}$.*

Proof By the assumption we see that $\{h_m \oplus k_m : m \in M\}$ is a standard generalized frame of $\mathscr{H} \oplus \mathscr{H}$. We define $\mathcal{A}$-linear operator $L : \mathscr{H} \oplus \mathscr{H} \longrightarrow \mathscr{T}(\mathscr{H}) + \mathscr{T}(\mathscr{H})$ by $L(f \oplus g) = T_h f + T_k g$, where T_h and T_k are the analyzing operators of $\{h_m : m \in M\}$ and $\{k_m : m \in M\}$, respectively. L is obviously well-defined. By Lemma 1.11, we have $\mathrm{Rng}(T_h) \bigcap \mathrm{Rng}(T_k) = \{0\}$ and $\mathrm{Rng}(T_h) + \mathrm{Rng}(T_k)$ is closed. Therefore, L is injective, bounded, and invertible. We say that L is adjointable. L^* is given by $L^*(\varphi + \psi) = T_h^*(\varphi + \psi) \oplus T_k^*(\varphi + \psi)$ for any $\varphi \in \mathrm{Rng}(T_h), \psi \in \mathrm{Rng}(T_k)$. Therefore, we have

$$\| L^{-1} \|^{-2} \langle f \oplus g, f \oplus g\rangle \leq \langle T_h f + T_k g, T_h f + T_k g\rangle = \langle L(f \oplus g), L(f \oplus g)\rangle \leq \| L \|^2 \langle f \oplus g, f \oplus g\rangle.$$

Hence

$$2 \| L^{-1} \|^{-2} \langle f, f\rangle \leq \langle (T_h + T_k) f, (T_h + T_k) f\rangle \leq 2 \| L \|^2 \langle f, f\rangle,$$

which results in

$$2 \| L^{-1} \|^{-2} \langle f, f\rangle \leq \int_M \langle f, h_m + k_m\rangle\langle h_m + k_m, f\rangle \mathrm{d}\mu(m) \leq 2 \| L \|^2 \langle f, f\rangle.$$

Thus, $\{h_m + k_m : m \in M\}$ has the required property. □

Theorem 2.6 *Suppose that $\{h_m : m \in M\}$ and $\{k_m : m \in M\}$ are strongly disjoint standard generalized normalized tight frames of $\mathscr{H}$. Assume that $A, B \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H})$ such that $AA^* + BB^* = I$. Then $\{Ah_m + Bk_m : m \in M\}$ is a standard generalized normalized tight frame of $\mathscr{H}$. In particular, $\{\alpha h_m + \beta k_m : m \in M\}$ is a standard generalized normalized tight frame when α, β are scalars such that $| \alpha |^2 + | \beta |^2 = 1$.*

Proof Let T_h and T_k be the analyzing operators of $\{h_m : m \in M\}$ and $\{k_m : m \in M\}$, respectively. For any $f \in \mathscr{H}$, by Lemma 1.11, we have

$$\begin{aligned}
&\int_M \langle f, Ah_m + Bk_m\rangle\langle Ah_m + Bk_m, f\rangle \mathrm{d}\mu(m) \\
&= \int_M (T_h A^* + T_k B^*)(f)(m)((T_h A^* + T_k B^*)(f)(m))^* \mathrm{d}\mu(m) \\
&= \langle (T_h A^* + T_k B^*)(f), (T_h A^* + T_k B^*)(f)\rangle_{L^2(M,\mathcal{A})} \\
&= \langle (T_h A^* + T_k B^*)^*(T_h A^* + T_k B^*)(f), f\rangle \\
&= \langle (AA^* + BB^*)(f), f\rangle = \langle f, f\rangle.
\end{aligned}$$

Therefore, $\{Ah_m + Bk_m : m \in M\}$ is a standard generalized normalized tight frame of $\mathscr{H}$. For the second part we leave to the reader. □

Corollary 2.7 *Suppose that $(\{h_{1m} : m \in M\}, \{h_{2m} : m \in M\}, \ldots, \{h_{km} : m \in M\})$ is a* *strongly disjoint k-tuple of standard generalized normalized tight frames of the same Hilbert $\mathcal{A}$-module $\mathscr{H}$. Assume $A_i \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H})$; $i = 1, 2, \ldots, k$, such that $\sum_{i=1}^k A_i A_i^* = I$. Then $\{\sum_{i=1}^k A_i h_{im}; m \in M\}$ is a standard generalized normalized tight frame of $\mathscr{H}$.*

Proposition 2.8 *Suppose that $(\{h_{1m} : m \in M\}, \{h_{2m} : m \in M\}, \ldots, \{h_{km} : m \in M\})$ is a strongly disjoint k-tuple of standard generalized frames of the same Hilbert $\mathcal{A}$-module $\mathscr{H}$. Then these k frames have a common alternate dual.*

Proof By the assumption we claim that $(\{h^*_{1m} : m \in M\}, \{h^*_{2m} : m \in M\}, \ldots, \{h^*_{km} : m \in M\})$ is also a strongly disjoint k-tuple, hence by Corollary 2.4, $\{\sum_{i=1}^k h^*_{im}; m \in M\}$ is a standard generalized frame of $\mathscr{H}$. For any $f \in \mathscr{H}$,

$$\int_M \langle f, h^*_{lm}\rangle h_{im} \mathrm{d}\mu(m) = \int_M \langle S_l^{-1} f, h_{lm}\rangle h_{im} \mathrm{d}\mu(m) = 0$$

when $i \neq l$, since $\{h_{lm} : m \in M\}$ and $\{h_{im} : m \in M\}$ are strongly disjoint when $i \neq l$, where S_l is the frame operator of $\{h_{lm} : m \in M\}$ for all $l = 1, 2, \ldots, k$. Thus

$$f = \int_M \langle f, h^*_{lm}\rangle h_{lm} \mathrm{d}\mu(m) = \int_M \langle f, \sum_{i=1}^k h^*_{im}\rangle h_{lm} \mathrm{d}\mu(m)$$

holds for all $f \in \mathscr{H}$ and $l = 1, 2, \ldots, k$, which implies that $\{\sum_{i=1}^k h^*_{im} : m \in M\}$ is a common alternate dual of all $\{h_{im} : m \in M\}; i = 1, 2, \ldots, k$. $\square$

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