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On UR-Rings

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Abstract A ring is said to be UR if every element can be written as the sum of a unit and a regular element. These rings are shown to be a unifying generalization of regular rings, clean rings and $(S,2)$-rings. Some relations of these rings are studied and several properties of clean rings and $(S,2)$-rings are extended. Ring extensions of UR-rings are also investigated.

Keywords clean rings; G-clean rings; regular rings; $(S,2)$-rings; UR-rings.

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1. Introduction

A ring R is (von Neumann) regular provided that for every $a \in R$ there exists $b \in R$ such that $a = aba$. A ring is called an (S,n)-ring if every element is a sum of no more than n units[4]. A ring R is said to be clean if every element of R can be written as the sum of a unit and an idempotent in R. Clean rings were introduced by Nicholson in his fundamental paper[5]. Recently, clean rings were studied by many authors. For example, GM-rings[2], n-clean rings[8], G-clean rings[10]. In this paper, a ring R is called UR if every element of R is the sum of a unit and a regular element. UR-rings are shown to be a unifying generalization of regular rings, $(S,2)$-rings and clean rings.

Some properties of clean rings, $(S,2)$-rings and G-clean rings can be extended to UR-rings. Polynomial rings, (skew) power series rings and triangular matrix rings over a UR-ring are also investigated.

Throughout this paper, R denotes an associative ring with identity and all modules are unitary. $M_n(R)$ is the ring of $n \times n$ matrices over R. The symbols $J(R), N(R), \mathrm{Id}(R)$, $U(R)$ and $\mathrm{Reg}(R)$ will stand for the Jacobson radical, the set of all nilpotent elements, the set of all idempotents, the group of units and the set of all regular elements of R, respectively.

2. UR-rings

A ring R is called G-clean if every element is the sum of a unit regular element and a unit[10]. We generalize regular rings, $(S,2)$-rings, clean rings and G-clean rings as follows

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Definition 2.1 *A ring R is said to be UR if every element $a \in R$ is UR, that is, a is the sum of a regular element and a unit.*

G-clean rings, which are a unifying generalization of clean rings and $(S,2)$-rings[10], are UR. We will give an example in Remark 3.1 (1) below to show that UR-rings need not be G-clean. Hence, the class of G-clean rings is a proper subclass of the class of UR-rings.

Proposition 2.2 *UR-rings have the following properties:*

(1) A homomorphic image of a UR-ring is UR.

(2) Let $R = \Pi R_\alpha$ be a direct product of rings. Then R is UR iff every R_α is UR.

Let $S(R)$ be the nonempty set of all the proper ideals of R generated by central idempotents. Recall that the factor ring R/P is called a Pierce stalk of R if P is a maximal element in $S(R)$.

Proposition 2.3 *For a ring R, the following are equivalent:*

(1) R is a UR-ring;

(2) All factor rings of R are UR;

(3) All indecomposable factor rings of R are UR;

(4) A/I is UR for every proper ideal I of R generated by central idempotents of R;

(5) All Pierce stalks of R are UR.

Proof The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are trivial, and the proofs of (3) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1) are similar. So we only need to prove (5) $\Rightarrow$ (1).

(5) $\Rightarrow$ (1) If R is not a UR-ring, let $\mathbf{S}^*$ be the set of all proper ideals I generated by central idempotents of R such that R/I is not UR. Since $(0) \in \mathbf{S}^*$, $\mathbf{S}^*$ is nonempty. It is easily verified that the union of every ascending chain of ideals from $\mathbf{S}^*$ is contained in $\mathbf{S}^*$. By Zorn's Lemma, $\mathbf{S}^*$ contains a maximal element P. By condition (5), it suffices to prove that R/P is a Pierce stalk. Assume the contrary. Hence, there is a central idempotent e of R such that $P + eR$ and $P + (1-e)R$ are proper ideals of R, $P + eR$ and $P + (1-e)R$ properly contain P and $R/P \cong R/(P + eR) \times R/(P + (1-e)R)$. The ideals $P + eR$ and $P + (1-e)R$ are generated by central idempotents. Hence, $P + eR$ and $P + (1-e)R$ are not in $\mathbf{S}^*$. Therefore, $R/(P + eR)$ and $R/(P + (1-e)R)$ are UR. By Proposition 2.2 (2), R/P is UR, a contradiction. □

For a one-sided ideal I of R, we say that regular elements lift modulo I if whenever $a - aba \in I$ with $a, b \in R$, there exists a regular element d of R such that $a - d \in I$.

Lemma 2.4 *Let I be an ideal of R with $I \subseteq J(R)$. Then idempotents lift modulo I iff regular elements lift modulo I.*

Proof "$\Rightarrow$". Denote $\bar{R} = R/I$ and write $\bar{r} = r + I \in \bar{R}$ for $r \in R$. Let $\bar{a} = \bar{a}\bar{c}\bar{a}$ in $\bar{R}$ for some $c \in R$. Then $(\bar{a}\bar{c})^2 = \bar{a}\bar{c}$. Thus, there exists $e^2 = e \in acR$ such that $e - ac \in I$ by [6, Lemma 5]. So $a - ea \in I$. Write $e = acb$ for some $b \in R$. Let $d = ea$. Then $a - d = a - ea \in I$ and $d(cb)d = ea = d$.

"$\Leftarrow$". Suppose $a^2 - a \in I$. By hypothesis, there exist $d, c \in R$ with $d = dcd$ such that $d - a \in I$. Then $d = ed$ with $e = dc \in \mathrm{Id}(R)$. If $f = e + ed(1-e)$, then $f^2 = f$ and

$\bar{f} = \bar{e} - \bar{e}\bar{d}\bar{e} + \bar{d} = \bar{e}(\bar{e} - \bar{a}\bar{e}) + \bar{a} = \bar{a}$ (since $\bar{e} - \bar{a}\bar{e} = \bar{d}\bar{c} - \bar{a}\bar{d}\bar{c} = (\bar{a} - \bar{a}^2)\bar{c} = \bar{0}$). □

By the above lemma, it is easy to prove the following proposition.

Proposition 2.5 *Let I be an ideal of R with $I \subseteq J(R)$. If R/I is UR and idempotents lift modulo I, then R is UR.*

Recall that a ring R is semiregular if $R/J(R)$ is regular and idempotents lift modulo $J(R)$. By Proposition 2.5 and the fact that regular rings are UR since $a = 1 + (a-1)$ with $a-1$ regular for every $a \in R$, semiregular rings are also UR.

Remark 2.6 The converse of Proposition 2.5 is not true. Since a ring R is exchange iff idempotents can be lifted modulo every left ideals[5, Corollary 1.3], UR-rings need not be exchange. For example, $R = \mathbb{Z}_{(3)} \bigcap \mathbb{Z}_{(5)} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0, 3 \nmid b \text{ and } 5 \nmid b\}$ is UR because each element $x \in R$ can be written in the form $x = u + e$ or $x = u - e$ where $u \in U(R)$ and $e \in \mathrm{Id}(R)$ by [1, Proposition 16]. But idempotents cannot be lifted modulo $J(R)$ because $R/J(R)$ has non-trivial idempotents and $\mathrm{Id}(R) = \{0, 1\}$.

Recall that a ring R is unit regular provided that, for each $a \in R$, there is a unit $u \in R$ such that $a = aua$. Moreover, if $au = ua$, then R is said to be strongly regular.

Proposition 2.7 *Let R be a UR-ring. Then for any $a \in R$, either aR contains a non-zero idempotent or a is the sum of a unit and a right unit.*

Proof Suppose that aR contains no non-zero idempotent. Write $a = d + u$ with $d \in \mathrm{Reg}(R)$ and $u \in U(R)$, and write $d = dcd$ for some $c \in R$. Then $acd = d + ucd$. So $a - acd = (d + u) - (d + ucd) = u(1 - cd)$ with $u(1 - cd)$ unit regular. Thus there exist $f \in \mathrm{Id}(R)$ and $v \in U(R)$ such that $u(1 - cd) = fv$. Hence $f = u(1 - cd)v^{-1} = (a - acd)v^{-1} \in aR$. By the hypothesis of aR, $f = 0$. So $cd = 1$. i.e., d is a right unit. □

3. Some rings close to UR-rings

In this section, some properties of G-clean ring are investigated and some relations of rings mentioned in Section 2 are discussed.

In fact we can obtain a more generalized result using a similar proof as that of [10, Proposition 2.8].

Proposition 3.1 *Let I be an ideal of R with $I \subseteq J(R)$. If idempotents lift modulo I, then R is G-clean iff $\bar{R} = R/I$ is G-clean.*

Proposition 3.2 *If R is G-clean and $2 \in U(R)$, then R is an $(S, 3)$-ring.*

Proof If $a \in R$ is G-clean, then there exist $e \in \mathrm{Id}(R)$ and $u_1, u_2 \in U(R)$ such that $a = eu_1 + u_2$. So $a = 2^{-1}(2e - 1)u_1 + 2^{-1}u_1 + u_2$ is the sum of three units. □

In [4], a ring R is said to be generated by its units if every element is a sum of finitely many units in R. Clearly, an $(S, 3)$-ring is generated by its units.

Remark 3.3 (1) Bergman[3, Example 2] has constructed a directly finite regular ring R with 2 invertible in R and R is not generated by its units. R is UR because R is regular, but R is not G-clean by Proposition 3.2.

(2) The condition " $2 \in U(R)$" in Proposition 3.2 cannot be omitted. Let R be a Boolean ring (i.e., every element is an idempotent) with more than two elements. Then R is G-clean, but not an (S, n)-ring for any positive integer $n > 1$.

Proposition 3.4 *If R is G-clean, then for any $a \in R$, either aR contains a non-zero idempotent or a is the sum of two units.*

Proof For any $a \in R$, suppose that aR contains no non-zero idempotents. Write $a = u + d$ where u is a unit and d is unit regular. Then $d = ev$ where $e \in \mathrm{Id}(R)$ and $v \in U(R)$. If $f = uv^{-1}(1-e)vu^{-1}$, then $f^2 = f$. Because $av^{-1} = uv^{-1} + e$, $f = uv^{-1}(1-e)vu^{-1} = av^{-1}(1-e)vu^{-1} \in aR$. Hence $e = 1$ by hypothesis on aR. So $a = u + v$ is the sum of two units. $\square$

The following fact is easily verified.

Proposition 3.5 *Suppose that the only idempotents of R are 0 and 1. Then R is UR iff R is G-clean iff R is an $(S, 2)$-ring.*

Theorem 3.6 *The following are equivalent for a ring R:*

(1) R is a Boolean ring;

(2) Every $a \in R$ is uniquely the sum of a unit regular element and a unit;

(3) Every $a \in R$ is uniquely the sum of a regular element and a unit.

Proof (1) $\Rightarrow$ (2). For every $x \in R$, $x = (x-1) + 1$. Since R is Boolean, $(x-1)^2 = (x-1)$. So $x - 1$ is regular. Clearly, the representation of x as the sum of a regular element and a unit is unique because $U(R) = \{1\}$.

(1) $\Rightarrow$ (3). The proof is similar to "(1) $\Rightarrow$ (2)" since idempotents are also unit regular.

(3) $\Rightarrow$ (1). If R has a unit u other than 1, then $0 = 1 - 1 = u - u$, a contradiction. Every element $a \in R$ is regular since $a+1$ is uniquely the sum of a regular element and a unit. Moreover, R has no nonzero nilpotent element. Assume not. If $b \in R$ is a nonzero nilpotent element, then $1 - b$ is another unit, a contradiction. So R is unit regular. Since 1 is the only unit, R is a Boolean ring.

(2) $\Rightarrow$ (1). The proof is similar to "(3) $\Rightarrow$ (1)". $\square$

4. Extensions of UR-rings

In this section, we will discuss UR-rings for (skew) power series rings, polynomial rings, matrix rings and so on.

Let φ be an endomorphism of R. We denote the skew power series ring

$$R[[x,\varphi]] = \{\sum_{i=0}^{\infty} a_i x^i \mid a_i \in R\},$$

where addition is naturally defined and multiplication is defined by using the relations $xr = \varphi(r)x$ for all $r \in R$. The power series ring $R[[x]]$ is just the skew power series ring when φ is the identity endomorphism.

Proposition 4.1 *$R[[x,\varphi]]$ is UR iff R is UR. In particular, $R[[x]]$ is UR iff R is UR.*

Proof The "if" part is proven easily because $f(x) \in R[[x,\varphi]]$ is invertible if the constant term is invertible in R. The "only if" part holds since $R \cong R[[x,\varphi]]/(x)$ and by Proposition 2.2 (1).□

The polynomial ring $R[x]$ over any ring R is neither regular nor clean. However, the same result is not true for UR-rings and G-clean rings. For some ring R, $R[x]$ is G-clean. But for some ring R, $R[x]$ is even not UR.

Example 4.2 Let F be a field, and let $R = M_2(F)$. Then $R[x] \cong M_2(F[x])$. Since $F[x]$ is a Euclidean domain, $R[x]$ is G-clean because any proper matrix ring over an elementary divisor ring is an $(S,2)$-ring by [7, Proposition 6].

A ring R satisfies (SI) if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$ and a ring is said to be abelian if all its idempotents are central. Notice that a ring satisfying (SI) is abelian.

Lemma 4.3 *If R is abelian, then the following are equivalent for $f(x) \in R[x]$:*

(1) $f(x)$ is regular;

(1') $f(x)$ is strongly regular;

(2) $f(x)$ is a product of an idempotent in R and a unit in $R[x]$;

(3) $f(x)$ is a product of a regular element in R and a unit in $R[x]$;

(3') $f(x)$ is a product of a strongly regular element in R and a unit in $R[x]$.

Proof Because R is abelian, $r \in \mathrm{Reg}(R)$ iff r is strongly regular in R. Hence (3) $\Leftrightarrow$ (3′) follows. By [9, Lemma 3.6], we have $\mathrm{Id}(R[x]) = \mathrm{Id}(R)$. Thus every regular element in $R[x]$ is strongly regular. Hence, (1) $\Leftrightarrow$ (1′) is proven. (2) $\Rightarrow$ (3) is obvious. So the work here is to prove (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Let $f(x) = e(x)u(x)$ for some $e(x) \in \mathrm{Id}(R[x])$ and $u(x) \in U(R[x])$. By $\mathrm{Id}(R[x]) = \mathrm{Id}(R)$, we have $f(x) = eu(x)$ for some $e \in \mathrm{Id}(R)$ and $u(x) \in U(R[x])$.

(3) $\Rightarrow$ (1). Assume $f(x) = au(x)$ for some $a \in \mathrm{Reg}(R)$ and $u(x) \in U(R[x])$. So $a = aba$ for some $b \in R$. Thus, $f(x) = f(x)(u(x)^{-1}b)f(x)$ is regular in $R[x]$. □

Proposition 4.4 *If R is a ring satisfying (SI), then $R[x]$ is not UR.*

Proof Because R is a ring satisfying (SI), R is abelian. If x is UR, then $x = r(x)+u(x)$ for some $r(x) \in \mathrm{Reg}(R[x])$ and $u(x) \in U(R[x])$. By Lemma 4.3, $r(x) = ev(x)$ for some $e \in \mathrm{Id}(R)$ and $v(x) \in U(R[x])$. Now by [9, Lemma 3.5], $v(x) = a_0+a_1x+\cdots+a_nx^n$, $u(x) = b_0+b_1x+\cdots+b_mx^m$, where $a_0, b_0 \in U(R)$, and $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in N(R), n, m \in \mathbb{N}$. Thus, $x = e(a_0 + a_1x +$

$\cdots + a_n x^n) + (b_0 + b_1 x + \cdots + b_m x^m)$. We have $1 = ea_1 + b_1$. Because $ea_1 = 1 - b_1 \in U(R)$ and $(1-e)ea_1 = 0$, $1 - e = 0$. So $a_1 = ea_1 \in U(R)$, a contradiction. Thus $R[x]$ is not a UR-ring. $\square$

Corollary 4.5[10, Proposition 2.10] *Let R be a commutative ring. Then $R[x]$ is not G-clean.*

Lemma 4.6 *Let $e_1, e_2, \ldots, e_n$ be idempotents of a ring R. If all e_iRe_i are UR-rings, then so is the ring*

$$S_n = \begin{pmatrix} e_1Re_1 & \ldots & e_1Re_n \\ \vdots & \ddots & \vdots \\ e_nRe_1 & \ldots & e_nRe_n \end{pmatrix}.$$

Proof Clearly, the result holds for $n = 1$. Now assume that the result holds for $n = m \geq 1$. For any $A \in S_{m+1}$, write $A = \left(\begin{smallmatrix} A_1 & B_1 \\ C_1 & d_1 \end{smallmatrix}\right)$, where $A_1 \in S_m$, B_1 and C_1 are m-vectors, and $d_1 \in e_{m+1}Re_{m+1}$. By hypothesis, we can find $W_1 \in \mathrm{Reg}(S_m)$ and $U_1 \in U(S_m)$ such that $A_1 = W_1 + U_1$. Because $d_1 - C_1U_1^{-1}B_1 \in e_{m+1}Re_{m+1}$, we have $w_1 \in \mathrm{Reg}(e_{m+1}Re_{m+1})$ and $v_1 \in U(e_{m+1}Re_{m+1})$ such that $d_1 - C_1U_1^{-1}B_1 = w_1 + v_1$. Then $A = \left(\begin{smallmatrix} W_1 & 0 \\ 0 & w_1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} U_1 & B_1 \\ C_1 & C_1U_1^{-1}B_1 + v_1 \end{smallmatrix}\right)$. It is clear that $\left(\begin{smallmatrix} W_1 & 0 \\ 0 & w_1 \end{smallmatrix}\right)$ is regular and $\left(\begin{smallmatrix} U_1 & B_1 \\ C_1 & C_1U_1^{-1}B_1 + v_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} I & 0 \\ C_1U_1^{-1} & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} U_1 & 0 \\ 0 & v_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} I & U_1^{-1}B_1 \\ 0 & 1 \end{smallmatrix}\right)$ is a unit in S_{m+1}. By induction, we conclude that S_n is a UR-ring. $\square$

Theorem 4.7 *The following are equivalent:*

(1) R is a UR-ring.

(2) There exists a complete set $\{e_1, e_2, \ldots, e_n\}$ of orthogonal idempotents such that all e_iRe_i are UR-rings.

Proof (1)⇒(2). It is obvious.

(2)⇒(1). Construct a map

$$\varphi : R \to \begin{pmatrix} e_1Re_1 & \ldots & e_1Re_n \\ \vdots & \ddots & \vdots \\ e_nRe_1 & \ldots & e_nRe_n \end{pmatrix}$$

given by $\varphi(r) = \begin{pmatrix} e_1re_1 & \ldots & e_1re_n \\ \vdots & \ddots & \vdots \\ e_nre_1 & \ldots & e_nre_n \end{pmatrix}$. Since $\{e_1, e_2, \ldots, e_n\}$ is a complete set of orthogonal idempotents, it is easy to prove that φ is a ring isomorphism. By virtue of Lemma 4.6, R is a UR-ring. $\square$

The following results are clear by Theorem 4.7.

Corollary 4.8 *$M_n(R)$ is UR for any positive integer n over every UR-ring R.*

Corollary 4.9 *If $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ are modules and each $\mathrm{End}(M_i)$ is UR, then $\mathrm{End}(M)$ is UR.*

The condition of (2) in Theorem 4.7 cannot be changed to "for every complete set $\{e_1, e_2, \ldots, e_n\}$ of orthogonal idempotents such that all e_iRe_i are UR-rings". For example, $R = M_2(\mathbb{Z})$ and

$e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Although R is UR by [7, Proposition 6], eRe is not UR even if $ReR = R$. Thus, the property UR is not Morita invariant. It also shows that there exists a non-UR ring R such that $M_2(R)$ is UR.

Combining Theorem 4.7 with Propositions 2.2 and 2.5, we obtain

Corollary 4.10 *If A, B are rings and $V = {}_AV_B$ is a bimodule, then the formal triangular matrix ring $\left(\begin{smallmatrix} A & V \\ 0 & B \end{smallmatrix}\right)$ is UR iff A and B are UR.*

Using the corollary above and inductive argument, we can also obtain

Corollary 4.11 *A ring R is a UR-ring iff $n \times n$ upper (low) triangular matrix ring over R is a UR-ring for any positive integer n.*

From the proofs in this section, Lemma 4.6; Proposition 4.1; Theorem 4.7; Corollaries 4.8, 4.9, 4.10 and 4.11 are also true for the cases of G-clean rings.

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