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On the Largest Eigenvalue of Signless Laplacian Matrix of a Graph

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Abstract The signless Laplacian matrix of a graph is the sum of its diagonal matrix of vertex degrees and its adjacency matrix. Li and Feng gave some basic results on the largest eigenvalue and characteristic polynomial of adjacency matrix of a graph in 1979. In this paper, we translate these results into the signless Laplacian matrix of a graph and obtain the similar results.

Keywords signless Laplacian matrix; characteristic polynomial; largest eigenvalue.

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1. Introduction

In this paper, all graphs considered are finite, undirected and loopless. Let G be a graph with vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ and edge set $E(G)$. Its adjacency matrix is defined to be the $n\times n$ matrix $A(G)=(a_{ij})$, where a_{ij} is the number of edges joining v_i to v_j. Let $d_G(v_i)$ denote the degree of v_i in G and $D(G)$ the diagonal matrix of vertex degrees of G, i.e.,

$$D(G)=\mathrm{diag}(d_G(v_1),d_G(v_2),\ldots,d_G(v_n)).$$

Then the signless Laplacian matrix $K(G)$ of G is defined by Haemers and Spence as follows[1]

$$K(G)=D(G)+A(G).$$

It is well known that the Laplacian matrix $L(G)$ of G is defined as follows

$$L(G)=D(G)-A(G).$$

Denote the characteristic polynomials of $A(G)$, $K(G)$ and $L(G)$ by $\phi(G,\lambda)$, $\phi_K(G,\lambda)$ and $\phi_L(G,\lambda)$, or simply by $\phi(G)$, $\phi_K(G)$ and $\phi_L(G)$, respectively. Since $K(G)$ and $L(G)$ are two real symmetric matrices, all of their eigenvalues are real. Write their largest eigenvalues by $\lambda_K(G)$ and $\lambda_L(G)$, respectively.

For a long time, most scholars have been interested in the spectra of adjacency matrix and Laplacian matrix of a graph. Therefore, the two kinds of spectra are studied extensively in the literature. In [2], Cvetković, Doob and Sachs surveyed the properties and applications of

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spectrum of adjacency matrix of a graph. In [3], Merris surveyed the properties and applications of spectrum of Laplacian matrix of a graph. Nowadays the spectrum of signless Laplacian matrix of a graph also attracts many scholars' attention and becomes a heat point studied. In [4], Cvetković, Rowlinson and Simić indicated that the signless Laplacian matrix appears very rarely in published papers and summarized some properties of signless Laplacian matrix of a graph. In [5], Van and Haemers expressed an idea that, among generalized adjacency matrices associated with a graph, the signless Laplacian matrix seems to be the most convenient for use in studying graph properties[4].

This paper has two purposes. On the one hand, we give the calculation formulas on the characteristic polynomials of signless Laplacian matrix of a graph and Laplacian matrix of a bipartite graph. On the other hand, we translate some basic results on the largest eigenvalue and characteristic polynomial of adjacency matrix of a graph obtained by Li and Feng in [6] into the signless Laplacian matrix of a graph and Laplacian matrix of a bipartite graph.

Throughout this paper, we use the following notations. Let P_n denote a path on n vertices, $N_G(v)$ the adjacent vertex set of a vertex v in a graph G, $l(G)$ the line graph of G, $E(Z)$ the set of edges in a subgraph (or an edge sequence) Z, $|E(Z)|$ the cardinality of $E(Z)$ and $\phi_K(P_0,\lambda)\equiv 0$. In particular, for a function $\psi(X,\lambda)$, when $S=\emptyset$, let

$$\sum_{X\in S}\psi(X,\lambda)=0.$$

Lemma 1.1[2] *Let v be a vertex of a graph G and $C_G(v)$ denote the set of all cycles containing v in G. Then*

$$\phi(G,\lambda)=\lambda\phi(G-v,\lambda)-\sum_{u\in N_G(v)}\phi(G-v-u,\lambda)-2\sum_{Z\in C_G(v)}\phi(G-V(Z),\lambda).$$

Lemma 1.2[2] *Let G be a connected graph, G' a proper subgraph of G and H a proper spanning subgraph of G. Then*

$$\lambda_K(G')<\lambda_K(G),\quad \lambda_K(H)<\lambda_K(G).$$

2. On the largest eigenvalue of signless Laplacian matrix of a graph

Let S be a sequence consisting of k distinct edges of a graph G. If the edges in S as vertices of $l(G)$ based on the order in S can form a cycle of length k in $l(G)$, then S is called a line graph cycle of G. If the two line graph cycles S_1 and S_2 of G can form the same cycle in $l(G)$, then S_1 and S_2 are called equal. For an edge e of G, let $E_G(e)$ denote the set of all edges (containing no e) adjacent to e in G and $J_G(e)$ the set of all distinct line graph cycles containing e in G. For instance, to the graph G and its line graph $l(G)$ shown in Figure 1, if we take the edge e_1, then

$$\begin{aligned}J_G(e_1)=\{&e_1e_2e_3,\ e_1e_2e_4,\ e_1e_3e_4,\ e_1e_7e_6;\ e_1e_2e_3e_4,\ e_1e_4e_2e_3,\ e_1e_4e_5e_6;\\&e_1e_3e_4e_5e_6,\ e_1e_4e_5e_6e_7,\ e_1e_2e_4e_5e_6;\ e_1e_2e_3e_4e_5e_6,\ e_1e_3e_2e_4e_5e_6,\\&e_1e_2e_4e_5e_6e_7,\ e_1e_3e_4e_5e_6e_7;\ e_1e_3e_2e_4e_5e_6e_7,\ e_1e_2e_3e_4e_5e_6e_7\}.\end{aligned}$$

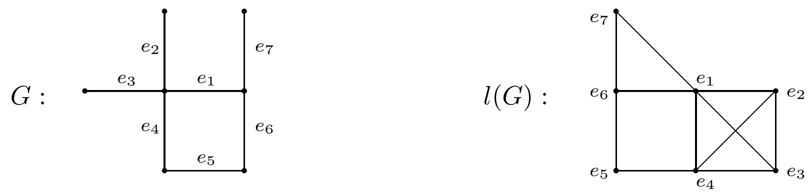

Figure 1 A graph G and its line graph $l(G)$

Theorem 2.1 *Let e be an edge of a graph G. Then*

$$\phi_K(G) = \frac{\lambda-2}{\lambda}\phi_K(G-e) - \sum_{\bar{e}\in E_G(e)} \frac{\phi_K(G-e-\bar{e})}{\lambda^2} - 2\sum_{Z\in J_G(e)} \frac{\phi_K(G-E(Z))}{\lambda^{|E(Z)|}}.$$

Proof A graph is called an (n,m) graph if it exactly has n vertices and m edges. Suppose that $G=(V(G),E(G))$ is an (n,m) graph, where

$$V(G)=\{v_1,v_2,\ldots,v_n\},\quad E(G)=\{e_1,e_2,\ldots,e_m\}.$$

Then the vertex-edge incidence matrix of G is defined to be the $n\times m$ matrix $R(G)=(r_{ij})$, where $r_{ij}=1$ if v_i is an end-vertex of e_j, and $r_{ij}=0$ otherwise. Let $R(G)^t$ denote the transpose of $R(G)$. It is well known that[2]

$$K(G)=R(G)R(G)^t,\quad 2I_m + A(l(G)) = R(G)^tR(G),$$
$$\det(\lambda I_n - R(G)R(G)^t) = \lambda^{n-m}\det(\lambda I_m - R(G)^tR(G)).$$

Therefore, we have

$$\phi_K(G,\lambda) = \lambda^{n-m}\phi(l(G),\lambda-2). \tag{1}$$

For any edge e of G, by the definition of line graph, we have

$$N_{l(G)}(e)=E_G(e),\quad C_{l(G)}(e)=J_G(e).$$

For any $Z\in C_{l(G)}(e)$, let $V(Z)_{l(G)}$ denote the vertex set of Z in $l(G)$. Then from the definition of line graph, we have

$$V(Z)_{l(G)}=E(Z),\quad l(G)-V(Z)_{l(G)} = l(G-E(Z)).$$

Write $l(G)=H$. Then by Lemma 1.1, we have

$$\begin{aligned}\phi(l(G),\mu) &= \mu\phi(H-e,\mu) - \sum_{\bar{e}\in N_H(e)}\phi(H-e-\bar{e},\mu) - 2\sum_{Z\in C_H(e)}\phi(H-V(Z)_H,\mu)\\ &= \mu\phi(l(G-e),\mu) - \sum_{\bar{e}\in E_G(e)}\phi(l(G-e-\bar{e}),\mu) - 2\sum_{Z\in J_G(e)}\phi(l(G-E(Z)),\mu).\end{aligned}$$

Since $G-e$, $G-e-\bar{e}$ and $G-E(Z)$ are $(n,m-1)$ graph, $(n,m-2)$ graph and $(n,m-|E(Z)|)$ graph, respectively, by Equation (1), we have

$$\phi(l(G),\lambda-2) = (\lambda-2)\frac{\phi_K(G-e,\lambda)}{\lambda^{n-m+1}} - \sum_{\bar{e}\in E_G(e)}\frac{\phi_K(G-e-\bar{e},\lambda)}{\lambda^{n-m+2}} -$$

$$2\sum_{Z\in J_G(e)}\frac{\phi_K(G-E(Z),\lambda)}{\lambda^{n-m+|E(Z)|}}. \tag{2}$$

Combining Equations (1) and (2) yields the required result. □

Corollary 2.2 *Let uv and vw be two edges of a graph G such that $d_G(u)=1$ and $d_G(v)=2$. Then we have*

$$\phi_K(G,\lambda)=(\lambda-2)\phi_K(G-u,\lambda)-\phi_K(G-u-v,\lambda).$$

Proof Take $e=uv$ in Theorem 2.1. Then $E_G(e)=\{vw\}$, $J_G(e)=\emptyset$. Note that

$$\phi_K(G-uv,\lambda)=\lambda\phi_K(G-u,\lambda),$$

$$\phi_K(G-uv-vw,\lambda)=\lambda^2\phi_K(G-u-v,\lambda).$$

Therefore, by Theorem 2.1, the result follows. □

Corollary 2.3 *Let e be an edge of a graph G.*

(i) If $\bar{e}$ is an adjacent edge of e, then for $\lambda\geq\lambda_K(G)$, we have

$$\phi_K(G,\lambda)\leq\frac{\lambda-2}{\lambda}\phi_K(G-e,\lambda)-\frac{\phi_K(G-e-\bar{e},\lambda)}{\lambda^2},$$

and the inequality strictly holds if G is connected and e has at least two adjacent edges.

(ii) For $\lambda\geq\lambda_K(G)$, we have

$$\phi_K(G,\lambda)\leq\frac{\lambda-2}{\lambda}\phi_K(G-e,\lambda),$$

and the inequality strictly holds if G is a connected graph with at least two edges.

Proof (i) Let $G_1,G_2,\ldots,G_s$ be all components of G. Without loss of generality, let $e\in E(G_1)$. Then for $\tilde{e}\in E_{G_1}(e)$ and $Z\in J_{G_1}(e)(\neq\emptyset)$, $G_1-e-\tilde{e}$ and $G_1-E(Z)$ are the two proper subgraphs of G_1. By Lemma 1.2, we have

$$\lambda_K(G_1)>\max\{\lambda_K(G_1-e-\tilde{e}),\lambda_K(G_1-E(Z))\}.$$

So for $\lambda\geq\lambda_K(G_1)$, we have $\phi_K(G_1-e-\tilde{e},\lambda)>0$ and $\phi_K(G_1-E(Z),\lambda)\geq 0$. Therefore, from Theorem 2.1, we have

$$\begin{aligned}\phi_K(G_1,\lambda)&\leq\frac{\lambda-2}{\lambda}\phi_K(G_1-e,\lambda)-\sum_{\tilde{e}\in E_{G_1}(e)}\frac{\phi_K(G_1-e-\tilde{e},\lambda)}{\lambda^2}\\&\leq\frac{\lambda-2}{\lambda}\phi_K(G_1-e,\lambda)-\frac{\phi_K(G_1-e-\bar{e},\lambda)}{\lambda^2}.\end{aligned}$$

In particular, if e has at least two adjacent edges, then $E_{G_1}(e)\setminus\{\bar{e}\}\neq\emptyset$. Therefore, when e has at least two adjacent edges, the second inequality in above proof strictly holds. Note that $\lambda_K(G)=\max\{\lambda_K(G_j):j=1,2,\ldots,s\}$. So for $\lambda\geq\lambda_K(G)$, we have

$$\begin{aligned}\phi_K(G,\lambda)&=\phi_K(G_1,\lambda)\phi_K(G_2,\lambda)\cdots\phi_K(G_s,\lambda)\\&\leq[\frac{\lambda-2}{\lambda}\phi_K(G_1-e,\lambda)-\frac{\phi_K(G_1-e-\bar{e},\lambda)}{\lambda^2}]\prod_{j=2}^{s}\phi_K(G_j,\lambda)\end{aligned}$$

$$= \frac{\lambda-2}{\lambda}\phi_K(G-e,\lambda) - \frac{\phi_K(G-e-\bar{e},\lambda)}{\lambda^2}.$$

(ii) In the similar way to (i), we can prove this result.

The proof is completed. □

Theorem 2.4 *Let H be a proper spanning subgraph of a connected graph G. Then for $\lambda \geq \lambda_K(G)$, we have*

$$\phi_K(G,\lambda) < \phi_K(H,\lambda).$$

Proof Without loss of generality, suppose that G has at least two edges. Let

$$E(G) - E(H) = \{e_1, e_2, \ldots, e_t\}.$$

For $1 \leq i \leq t$, set $S_i = \{e_1, e_2, \ldots, e_i\}$. Then $H = G - S_t$. Note that G has at least one edge and $G - S_i$ is a proper spanning subgraph of G. So by Lemma 1.2, we have

$$\lambda_K(G) \geq \max\{\lambda_K(P_2), \lambda_K(G-S_i) : i = 1, 2, \ldots, t\} \geq 2.$$

Hence by Corollary 2.3 (ii), for $\lambda \geq \lambda_K(G)$, we have

$$\phi_K(G,\lambda) < \phi_K(G-S_1,\lambda) \leq \phi_K(G-S_2,\lambda) \leq \cdots \leq \phi_K(G-S_t,\lambda) = \phi_K(H,\lambda).$$

The proof is completed. □

Theorem 2.5 *Let u be a vertex of a connected graph G with at least two vertices. Suppose that $P = a_1a_2\cdots a_k$ and $Q = b_1b_2\cdots b_l$ are two new disjoint paths. Let $G_{k,l}$ denote the graph obtained from G, P and Q by joining u to a_1 with an edge and joining u to b_1 with another edge. If $k \geq l \geq 1$, then*

$$\lambda_K(G_{k+1,l-1}) < \lambda_K(G_{k,l}).$$

Proof Write $k - l = s$, $\phi_K = f$. It is obvious that $G_{s+1,1}$ is a subgraph of $G_{k,l}$ and $G_{s,0}$ is a proper subgraph of $G_{s+1,1}$. So by Lemma 1.2, we have

$$\lambda_K(G_{k,l}) \geq \lambda_K(G_{s+1,1}) > \lambda_K(G_{s,0}). \tag{3}$$

Next we only need show that for $\lambda \geq \lambda_K(G_{k,l})$, $f(G_{k,l}) < f(G_{k+1,l-1})$.

Assume $l \geq 2$. From Corollary 2.2, we have

$$f(G_{k,l}) = (\lambda-2)f(G_{k,l-1}) - f(G_{k,l-2}),$$

$$f(G_{k+1,l-1}) = (\lambda-2)f(G_{k,l-1}) - f(G_{k-1,l-1}).$$

Therefore, we have

$$f(G_{k,l}) - f(G_{k+1,l-1}) = f(G_{k-1,l-1}) - f(G_{k,l-2}).$$

Repeating the above steps, we have

$$f(G_{k,l}) - f(G_{k+1,l-1}) = f(G_{s+1,1}) - f(G_{s+2,0}). \tag{4}$$

Note that Equation (4) hods for $l = 1$. Therefore, Equation (4) always holds for $l \geq 1$.

By Corollary 2.2, we have

$$f(G_{s+2,0}) = (\lambda-2)f(G_{s+1,0}) - f(G_{s,0}).$$

Therefore, by Equation (4), we have

$$f(G_{k,l}) - f(G_{k+1,l-1}) = f(G_{s,0}) + f(G_{s+1,1}) - (\lambda - 2)f(G_{s+1,0}). \tag{5}$$

Since $G_{s+1,1}$ is a connected graph and ub_1 has at least two adjacent edges in $G_{s+1,1}$, from Corollary 2.3 (i), for $\lambda \geq \lambda_K(G_{s+1,1})$, we have

$$\begin{aligned} f(G_{s+1,1}) &< \frac{\lambda - 2}{\lambda} f(G_{s+1,1} - ub_1) - \frac{1}{\lambda^2} f(G_{s+1,1} - ub_1 - ua_1) \\ &= (\lambda - 2)f(G_{s+1,0}) - \frac{1}{\lambda} f(P_{s+1})f(G). \end{aligned} \tag{6}$$

By Equations (3), (5) and (6), for $\lambda \geq \lambda_K(G_{k,l})$, we have

$$f(G_{k,l}) - f(G_{k+1,l-1}) < f(G_{s,0}) - \frac{1}{\lambda} f(P_{s+1})f(G). \tag{7}$$

Case 1 Assume $s = 0$. Then $G_{s,0} = G$ and $f(P_{s+1}) = \lambda$. So by Equation (7), for $\lambda \geq \lambda_K(G_{k,l})$, we have

$$f(G_{k,l}) - f(G_{k+1,l-1}) < 0.$$

Case 2 Assume $s = 1$. By Corollary 2.3 (ii), for $\lambda \geq \lambda_K(G_{s,0})$, we have

$$f(G_{s,0}) \leq \frac{\lambda - 2}{\lambda} f(G_{s,0} - ua_1) = (\lambda - 2)f(G).$$

So combining Equations (3) and (7), for $\lambda \geq \lambda_K(G_{k,l})$, we have

$$f(G_{k,l}) - f(G_{k+1,l-1}) < [(\lambda - 2) - \frac{1}{\lambda} f(P_{s+1})]f(G) = 0.$$

Case 3 Assume $s \geq 2$. Then by Corollary 2.3 (i), for $\lambda \geq \lambda_K(G_{s,0})$, we have

$$\begin{aligned} f(G_{s,0}) &\leq \frac{\lambda - 2}{\lambda} f(G_{s,0} - ua_1) - \frac{1}{\lambda^2} f(G_{s,0} - ua_1 - a_1a_2) \\ &= \frac{\lambda - 2}{\lambda} f(P_s)f(G) - \frac{1}{\lambda} f(P_{s-1})f(G). \end{aligned}$$

Therefore, by Equations (3), (7) and Corollary 2.2, for $\lambda \geq \lambda_K(G_{k,l})$, we have

$$f(G_{k,l}) - f(G_{k+1,l-1}) < \frac{1}{\lambda} f(G)[-f(P_{s+1}) + (\lambda - 2)f(Ps) - f(P_{s-1})] = 0.$$

The proof is completed. □

Theorem 2.6 *Let v and u be two distinct vertices joined by a path of length m in a connected graph G, where $d_G(v) \geq 2$ and $d_G(u) \geq 2$. Suppose that $P = a_1a_2\cdots a_k$ and $Q = b_1b_2\cdots b_l$ are two new disjoint paths. Let $G_{k,l}^{(m)}$ denote the graph obtained from G, P and Q by joining v to a_1 with an edge and joining u to b_1 with another edge. If $k - l \geq m \geq 1$ and $l \geq 1$, then*

$$\lambda_K(G_{k+1,l-1}^{(m)}) < \lambda_K(G_{k,l}^{(m)}).$$

Proof Write $k - l - m = s$, $\phi_K = h$. Next we only need show that for $\lambda \geq \lambda_K(G_{k,l}^{(m)})$, $h(G_{k,l}^{(m)}) < h(G_{k+1,l-1}^{(m)})$. Let $v_0v_1\cdots v_m$ be a path of length m from v to u in G, where $v_0 = v$ and $v_m = u$. In the similar way obtaining Equation (5) in the proof of Theorem 2.5, we have

$$h(G_{k,l}^{(m)}) - h(G_{k+1,l-1}^{(m)}) = h(G_{s+m,0}^{(m)}) + h(G_{s+m+1,1}^{(m)}) - (\lambda - 2)h(G_{s+m+1,0}^{(m)}). \tag{8}$$

Write $E_0 = \emptyset$, $E_i = \{v_m v_{m-1}, v_{m-1}v_{m-2}, \ldots, v_{m-i+1}v_{m-i}\}$, $i = 1, 2, \ldots, m$. Since $G^{(m)}_{s+m+1,1}$ is a connected graph and $v_m b_1$ has at least two adjacent edges in $G^{(m)}_{s+m+1,1}$, by Corollary 2.3 (i), for $\lambda \geq \lambda_K(G^{(m)}_{s+m+1,1})$, we have

$$h(G^{(m)}_{s+m+1,1}) < (\lambda - 2)h(G^{(m)}_{s+m+1,0}) - \frac{1}{\lambda}h(G^{(m)}_{s+m+1,0} - E_1). \tag{9}$$

From Equations (8) and (9), for $\lambda \geq \lambda_K(G^{(m)}_{s+m+1,1})$, we have

$$h(G^{(m)}_{k,l}) - h(G^{(m)}_{k+1,l-1}) < h(G^{(m)}_{s+m,0}) - \frac{1}{\lambda}h(G^{(m)}_{s+m+1,0} - E_1). \tag{10}$$

By Corollary 2.2, we have

$$h(G^{(m)}_{s+m+1,0} - E_1) = (\lambda - 2)h(G^{(m)}_{s+m,0} - E_1) - h(G^{(m)}_{s+m-1,0} - E_1). \tag{11}$$

Again by Corollary 2.3 (i), for $\lambda \geq \lambda_K(G^{(m)}_{s+m,0})$, we have

$$h(G^{(m)}_{s+m,0}) \leq \frac{\lambda - 2}{\lambda}h(G^{(m)}_{s+m,0} - E_1) - \frac{1}{\lambda^2}h(G^{(m)}_{s+m,0} - E_2). \tag{12}$$

Since $G^{(m)}_{s+m+1,1}$ is a subgraph of $G^{(m)}_{k,l}$, $G^{(m)}_{s+m,0}$ is a proper subgraph of $G^{(m)}_{s+m+1,1}$, G is a proper subgraph of $G^{(m)}_{s+m,0}$ and P_2 is a proper subgraph of G, by Lemma 1.2, we have

$$\lambda_K(G^{(m)}_{k,l}) \geq \lambda_K(G^{(m)}_{s+m+1,1}) > \lambda_K(G^{(m)}_{s+m,0}) > \lambda_K(G) > \lambda_K(P_2) = 2.$$

Therefore, by Equations (10)–(12), for $\lambda \geq \lambda_K(G^{(m)}_{k,l})$, we have

$$h(G^{(m)}_{k,l}) - h(G^{(m)}_{k+1,l-1}) < \frac{1}{\lambda}[h(G^{(m)}_{s+m-1,0} - E_1) - \frac{1}{\lambda}h(G^{(m)}_{s+m,0} - E_2)]. \tag{13}$$

In the similar discussion to Equations (10)–(13), for $\lambda \geq \lambda_K(G^{(m)}_{k,l})$, we have

$$h(G^{(m)}_{k,l}) - h(G^{(m)}_{k+1,l-1}) < \frac{1}{\lambda^2}[h(G^{(m)}_{s+m-2,0} - E_2) - \frac{1}{\lambda}h(G^{(m)}_{s+m-1,0} - E_3)].$$

Repeating the above steps, for $\lambda \geq \lambda_K(G^{(m)}_{k,l})$, we have

$$h(G^{(m)}_{k,l}) - h(G^{(m)}_{k+1,l-1}) < \frac{1}{\lambda^{m-1}}[h(G^{(m)}_{s+1,0} - E_{m-1}) - \frac{1}{\lambda}h(G^{(m)}_{s+2,0} - E_m)]. \tag{14}$$

Let $G^{(m)}_{s+1,0} - E_{m-1} = U$, $G^{(m)}_{s+1,0} - E_m = F$, $G^{(m)}_{s+2,0} - E_m = B$ and $G - E_m = M$.

Case 1 Assume $s = 0$.

By Corollary 2.2, we have

$$\begin{aligned} h(G^{(m)}_{s+2,0} - E_m) &= (\lambda - 2)h(B - a_2) - h(B - a_2 - a_1) \\ &= (\lambda - 2)h(F) - h(M). \end{aligned} \tag{15}$$

By Corollary 2.3 (i), for $\lambda \geq \lambda_K(G^{(m)}_{k,l})$, we have

$$\begin{aligned} h(G^{(m)}_{s+1,0} - E_{m-1}) &\leq \frac{\lambda - 2}{\lambda}h(U - v_0v_1) - \frac{1}{\lambda^2}h(U - v_0v_1 - va_1) \\ &= \frac{\lambda - 2}{\lambda}h(F) - \frac{1}{\lambda}h(M). \end{aligned} \tag{16}$$

By Equations (14)–(16), for $\lambda \geq \lambda_K(G^{(m)}_{k,l})$, we have $h(G^{(m)}_{k,l}) - h(G^{(m)}_{k+1,l-1}) < 0$.

Case 2 Assume $s \geq 1$.

By Theorem 2.1, we have

$$\begin{aligned} h(G_{s+2,0}^{(m)} - E_m) &= \frac{\lambda-2}{\lambda}h(B - va_1) - \frac{1}{\lambda^2}h(B - va_1 - a_1a_2) - \Delta_1 - \Delta_2 \\ &= \frac{1}{\lambda}h(M)[(\lambda-2)h(P_{s+2}) - h(P_{s+1})] - \Delta_1 - \Delta_2 \\ &= \frac{1}{\lambda}h(M)h(P_{s+3}) - \Delta_1 - \Delta_2, \end{aligned} \tag{17}$$

where

$$\Delta_1 = 2\sum_{Z\in J_B(va_1)} \frac{h(B - E(Z))}{\lambda^{|E(Z)|}}, \quad \Delta_2 = \frac{h(P_{s+2})}{\lambda^2}\sum_{w\in N_M(v)} h(M - vw).$$

By Corollary 2.3 (i) and Theorem 2.1, for $\lambda \geq \lambda_K(G_{k,l}^{(m)})$, we have

$$\begin{aligned} &h(G_{s+1,0}^{(m)} - E_{m-1}) \leq \frac{\lambda-2}{\lambda}h(U - v_0v_1) - \frac{1}{\lambda^2}h(U - v_0v_1 - va_1) \\ &= \frac{\lambda-2}{\lambda}h(F) - \frac{1}{\lambda^2}h(F - va_1) \\ &= \frac{\lambda-2}{\lambda}[\frac{\lambda-2}{\lambda}h(F - va_1) - \frac{h(F - va_1 - a_1a_2)}{\lambda^2} - \Delta_3 - \Delta_4] - \frac{h(M)h(P_{s+1})}{\lambda^2} \\ &= \frac{\lambda-2}{\lambda}[\frac{\lambda-2}{\lambda}h(M)h(P_{s+1}) - \frac{h(M)h(P_s)}{\lambda} - \Delta_3 - \Delta_4] - \frac{h(M)h(P_{s+1})}{\lambda^2} \\ &= \frac{1}{\lambda^2}h(M)[(\lambda-2)^2h(P_{s+1}) - (\lambda-2)h(P_s) - h(P_{s+1})] - \frac{\lambda-2}{\lambda}(\Delta_3 + \Delta_4) \\ &= \frac{1}{\lambda^2}h(M)h(P_{s+3}) - \frac{\lambda-2}{\lambda}(\Delta_3 + \Delta_4), \end{aligned} \tag{18}$$

where

$$\Delta_3 = 2\sum_{Z\in J_F(va_1)} \frac{h(F - E(Z))}{\lambda^{|E(Z)|}}, \quad \Delta_4 = \frac{h(P_{s+1})}{\lambda^2}\sum_{w\in N_M(v)} h(M - vw).$$

By $s \geq 1$, we have $J_B(va_1) = J_F(va_1)$. By Corollary 2.3 (ii), for $\lambda \geq \lambda_K(B - E(Z))$, we have

$$h(B - E(Z)) \leq (\lambda-2)h(F - E(Z)).$$

Note that $\lambda_K(G_{k,l}^{(m)}) > \lambda_K(B - E(Z))$. So for $\lambda \geq \lambda_K(G_{k,l}^{(m)})$, we have

$$\Delta_1 - (\lambda-2)\Delta_3 \leq 2\sum_{Z\in J_B(va_1)} \frac{(\lambda-2)h(F - E(Z))}{\lambda^{|E(Z)|}} - (\lambda-2)\Delta_3 = 0. \tag{19}$$

By Corollary 2.2, for $\lambda \geq \lambda_K(G_{k,l}^{(m)})$, we have

$$\begin{aligned} \Delta_2 - (\lambda-2)\Delta_4 &= \frac{1}{\lambda^2}[h(P_{s+2}) - (\lambda-2)h(P_{s+1})]\sum_{w\in N_M(v)} h(M - vw) \\ &= -\frac{h(P_s)}{\lambda^2}\sum_{w\in N_M(v)} h(M - vw) \leq 0. \end{aligned} \tag{20}$$

By Equations (14), (17)–(20), for $\lambda \geq \lambda_K(G_{k,l}^{(m)})$, we have $h(G_{k,l}^{(m)}) - h(G_{k+1,l-1}^{(m)}) < 0$.

The proof is completed. $\square$

Theorem 2.7 *Let vu be an edge of a connected graph G such that $d_G(v) \geq 2$ and $d_G(u) \geq 2$.*

Suppose that $P = a_1a_2\cdots a_k$ and $Q = b_1b_2\cdots b_l$ are two new disjoint paths. Let $G^{(1)}_{k,l}$ denote the graph obtained from G, P and Q by joining v to a_1 with an edge and joining u to b_1 with another edge. If $k \geq l \geq 1$, then $\lambda_K(G^{(1)}_{k+1,l-1}) < \lambda_K(G^{(1)}_{k,l})$.

Proof Set $\phi_K = h$. We need show that for $\lambda \geq \lambda_K(G^{(1)}_{k,l})$, $h(G^{(1)}_{k,l}) < h(G^{(1)}_{k+1,l-1})$. Note that the case $k > l$ is a special case of Theorem 2.6 when $m = 1$. Therefore, next assume $k = l$. Write $G^{(1)}_{1,0} - uv = B$ and $G - vu = M$. Since $G^{(1)}_{0,0} = G$, in the similar way obtaining Equation (10) in the proof of Theorem 2.6, for $\lambda \geq \lambda_K(G^{(1)}_{k,l})$, we have

$$h(G^{(1)}_{k,l}) - h(G^{(1)}_{k+1,l-1}) < h(G) - \frac{1}{\lambda}h(B). \tag{21}$$

By Theorem 2.1, we have

$$\begin{aligned} h(G) - \frac{1}{\lambda}h(B) &= \frac{\lambda-2}{\lambda}h(G-vu) - \alpha_1 - \alpha_2 - \frac{1}{\lambda}[\frac{\lambda-2}{\lambda}h(B-va_1) - \alpha_3 - \alpha_4] \\ &= \frac{\lambda-2}{\lambda}h(M) - \alpha_1 - \alpha_2 - \frac{1}{\lambda}[(\lambda-2)h(M) - \alpha_3 - \alpha_4] \\ &= [\frac{1}{\lambda}\alpha_3 - \alpha_1] + [\frac{1}{\lambda}\alpha_4 - \alpha_2], \end{aligned} \tag{22}$$

where

$$\alpha_1 = 2\sum_{Z\in J_G(vu)} \frac{h(G-E(Z))}{\lambda^{|E(Z)|}}, \quad \alpha_2 = \frac{1}{\lambda^2}\sum_{e\in E_G(vu)} h(M-e).$$

$$\alpha_3 = 2\sum_{Z\in J_B(va_1)} \frac{h(B-E(Z))}{\lambda^{|E(Z)|}}, \quad \alpha_4 = \frac{1}{\lambda}\sum_{w\in N_M(v)} h(M-vw).$$

It is easy to find that $N_M(u) \neq \emptyset$ and

$$\sum_{e\in E_G(vu)} h(M-e) = \sum_{w\in N_M(v)} h(M-vw) + \sum_{w\in N_M(u)} h(M-uw).$$

Therefore, for $\lambda \geq \lambda_K(G^{(1)}_{k,l})$, we have

$$\frac{1}{\lambda}\alpha_4 - \alpha_2 = -\frac{1}{\lambda^2}\sum_{w\in N_M(u)} h(M-uw) < 0. \tag{23}$$

For each $Z \in J_B(va_1)$, let $\bar{Z}$ denote the edge sequence obtained from Z and vu by replacing va_1 of Z with vu. Then we have

$$h(B-E(Z)) = h(G^{(1)}_{1,0} - vu - E(Z)) = h((G-E(\bar{Z}))\bigcup\{a_1\}) = \lambda h(G-E(\bar{Z})).$$

Write $J_1 = \{\bar{Z} : Z \in J_B(va_1)\}$ and $J_2 = J_G(vu) - J_1$. Then we have

$$\alpha_1 = 2\sum_{\bar{Z}\in J_1} \frac{h(G-E(\bar{Z}))}{\lambda^{|E(\bar{Z})|}} + 2\sum_{Z\in J_2} \frac{h(G-E(Z))}{\lambda^{|E(Z)|}} = \frac{1}{\lambda}\alpha_3 + 2\sum_{Z\in J_2} \frac{h(G-E(Z))}{\lambda^{|E(Z)|}}.$$

Therefore, for $\lambda \geq \lambda_K(G^{(1)}_{k,l})$, we have

$$\frac{1}{\lambda}\alpha_3 - \alpha_1 = -2\sum_{Z\in J_2} \frac{h(G-E(Z))}{\lambda^{|E(Z)|}} \leq 0. \tag{24}$$

By Equations (21)–(24), for $\lambda \geq \lambda_K(G^{(1)}_{k,l})$, we have $h(G^{(1)}_{k,l}) - h(G^{(1)}_{k+1,l-1}) < 0$.

The proof is completed. □

3. On the largest eigenvalue of Laplacian matrix of a bipartite graph

For a bipartite graph G, $K(G)$ and $L(G)$ have the same spectrum[2]. Therefore, by Theorems 2.1, 2.4, 2.5, 2.6 and 2.7, we immediately obtain the following corollaries.

Corollary 3.1 *Let e be an edge of a bipartite graph G. Then*

$$\phi_L(G) = \frac{\lambda-2}{\lambda}\phi_L(G-e) - \sum_{\bar{e}\in E_G(e)} \frac{\phi_L(G-e-\bar{e})}{\lambda^2} - 2\sum_{Z\in J_G(e)} \frac{\phi_L(G-E(Z))}{\lambda^{|E(Z)|}}.$$

Corollary 3.2 *Let H be a proper spanning subgraph of a connected bipartite graph G. Then for $\lambda \geq \lambda_L(G)$, we have $\phi_L(G,\lambda) < \phi_L(H,\lambda)$.*

Corollary 3.3[7] *Let u be a vertex of a connected bipartite graph G with at least two vertices. Suppose that $P = a_1a_2\cdots a_k$ and $Q = b_1b_2\cdots b_l$ are two new disjoint paths. Let $G_{k,l}$ denote the graph obtained from G, P and Q by joining u to a_1 with an edge and joining u to b_1 with another edge. If $k \geq l \geq 1$, then $\lambda_L(G_{k+1,l-1}) < \lambda_L(G_{k,l})$.*

Corollary 3.4 *Let v and u be two distinct vertices joined by a path of length m in a connected bipartite graph G, where $d_G(v) \geq 2$ and $d_G(u) \geq 2$. Suppose that $P = a_1a_2\cdots a_k$ and $Q = b_1b_2\cdots b_l$ are two new disjoint paths. Let $G_{k,l}^{(m)}$ denote the graph obtained from G, P and Q by joining v to a_1 with an edge and joining u to b_1 with another edge. If $k-l \geq m \geq 1$ and $l \geq 1$, then $\lambda_L(G_{k+1,l-1}^{(m)}) < \lambda_L(G_{k,l}^{(m)})$.*

Corollary 3.5 *Let vu be an edge of a connected bipartite graph G such that $d_G(v) \geq 2$ and $d_G(u) \geq 2$. Suppose that $P = a_1a_2\cdots a_k$ and $Q = b_1b_2\cdots b_l$ are two new disjoint paths. Let $G_{k,l}^{(1)}$ denote the graph obtained from G, P and Q by joining v to a_1 with an edge and joining u to b_1 with another edge. If $k \geq l \geq 1$, then $\lambda_L(G_{k+1,l-1}^{(1)}) < \lambda_L(G_{k,l}^{(1)})$.*

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