Green’s Relations on Semigroups of Transformations
Preserving Two Equivalence Relations

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Abstract Let \( T_X \) be the full transformation semigroup on a set \( X \). For a non-trivial equivalence \( F \) on \( X \), let \( T_F(X) = \{ f \in T_X : \forall (x, y) \in F, (f(x), f(y)) \in F \} \). Then \( T_F(X) \) is a subsemigroup of \( T_X \). Let \( E \) be another equivalence on \( X \) and \( T_{FE}(X) = T_F(X) \cap T_E(X) \). In this paper, under the assumption that the two equivalences \( F \) and \( E \) are comparable and \( E \subseteq F \), we describe the regular elements and characterize Green’s relations for the semigroup \( T_{FE}(X) \).

Keywords transformation semigroup; equivalence; regular element; Green’s relations.

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1. Introduction

Green’s relations are five equivalences that have played an important role in the development of semigroup theory[1]. Let \( X \) be a set with \( |X| \geq 3 \) and \( T_X \) be the full transformation semigroup on the set \( X \). In [2], the author observed a kind of transformation semigroup determined by an equivalence \( F \) on \( X \), that is,

\[
T_F(X) = \{ f \in T_X : \forall (x, y) \in F, (f(x), f(y)) \in F \}.
\]

It is easy to see that \( T_F(X) = T_X \) if \( F = \{(x, x), x \in X \} \) or \( F = X \times X \). Some interesting properties for \( T_F(X) \) were studied in some papers. For example, in [3] and [4], the author observed some subsemigroups of \( T_F(X) \) which induce certain lattices. In [5] and [6] some special congruences on \( T_F(X) \) were investigated, and Green’s relations on \( T_F(X) \) were described in [7] and so on.

Let \( E \) be another equivalence on \( X \). In [2] the author also studied the semigroup

\[
T_{FE}(X) = T_F(X) \cap T_E(X),
\]

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and determined the suitable lattice of $T_{FE}(X)$.

The regular elements and Green’s relation on the semigroups $T(X, \rho, R)$ consisting of transformations preserving an equivalence relation and a cross-section, and on the semigroups $O_E(X)$ consisting of transformations preserving order and an equivalence relation were considered in [8] and [9], respectively. Clearly, $T_{FE}(X)$ is a subsemigroup of both $T_F(X)$ and $T_E(X)$ and so $f \in T_{FE}(X)$ should preserve two equivalences on $X$. Naturally, we may ask how to describe the regular elements and Green’s relations on $T_{FE}(X)$? However, this is a difficult problem, mainly because we have great difficulty in constructing the desired maps. In this paper, we consider a special case, that is, $F$ and $E$ are comparable. For convenience, we assume, in the remainder, that $T_{FE}(X)$ will denote $T_F(X) \cap T_E(X)$ and that $E \subseteq F$, which is crucial for all that follows. Under the above assumption, each $E$-class is contained in some $F$-class, while each $F$-class is a union of some $E$-classes.

This paper is organized as follows. In Section 2, we observe the conditions under which an element $f \in T_{FE}(X)$ is regular. In Section 3, Green’s relations on $T_{FE}(X)$ are considered and the relations $L, R, H, D$ and $J$ are completely characterized for arbitrary elements.

Now we recall some concepts and notations which will be used in the sequel. Denote by $X/F$ the quotient set. The symbol $\pi(f)$ will denote the partition of $X$ induced by $f \in T_X$, namely,

$$\pi(f) = \{f^{-1}(y) : y \in f(X)\}.$$

Also, for a subset $A \subseteq X$, we denote

$$\pi_A(f) = \{M \in \pi(f) : M \cap A \neq \emptyset\}.$$

Lemma 1.1\[7\] Let $f \in T_X$. Then $f \in T_F(X)$ if and only if for each $B \in X/F$, there exists some $B' \in X/F$ such that $f(B) \subseteq B'$. Consequently, if $f \in T_F(X)$, then for each $A \in X/F$, the set $f^{-1}(A)$ is a union of some $F$-classes or $f^{-1}(A) = \emptyset$.

For each $f \in T_F(X)$, let

$$F(f) = \{f^{-1}(A) : A \in X/F \text{ and } f^{-1}(A) \neq \emptyset\}.$$

Then $F(f)$ is also a partition of $X$. It is clear that $\pi(f)$ refines $F(f)$ and that $x, y \in V \in F(f)$ if and only if $(f(x), f(y)) \in F$. Moreover, for each $V \in F(f)$, there exists some $A \in X/F$ such that $f(V) = A \cap f(X)$. We have similar notations for $f \in T_E(X)$. For standard terms and concepts in semigroup theory, one may consult [1].

2. The regular elements of $T_{FE}(X)$

In this section, we observe when an element $f \in T_{FE}(X)$ is regular.

Theorem 2.1 Let $f \in T_{FE}(X)$. Then $f$ is regular if and only if for each $A \in X/F$, there exists some $B \in X/F$ such that $A \cap f(X) \subseteq f(B)$, while for each E-class $A' \subseteq A$, there exists some E-class $B' \subseteq B$ such that $A' \cap f(X) \subseteq f(B')$.

Proof Suppose that $f$ is regular in $T_{FE}(X)$. Then there exists $g \in T_{FE}(X)$ such that $f = fgf$. 


Let $A \subseteq X/F$. If $A \cap f(X) = \emptyset$, then $A \cap f(X) \subseteq f(B)$ for some $F$-class $B$. If $A \cap f(X) \neq \emptyset$, take $y \in A \cap f(X)$ and $x \in X$ so that $y = f(x)$. Let $g(A) \subseteq B \subseteq X/F$. Then

$$y = f(x) = fgf(x) = fg(y) \in fg(A) \subseteq f(B)$$

and it follows that $A \cap f(X) \subseteq f(B)$. Let $A' \subseteq X/E$ with $A' \subseteq A$. If $A' \cap f(X) = \emptyset$, then $A' \cap f(X) \subseteq f(B')$ for some $B' \subseteq X/E$ with $B' \subseteq B$. Now suppose that $A' \cap f(X) \neq \emptyset$. Let $y' \in A' \cap f(X)$. Then there exists some $x' \in X$ such that $y' = f(x')$. Assume $g(A') \subseteq B' \subseteq X/E$. Then

$$y' = f(x') = fgf(x') = fg(y') \in fg(A') \subseteq f(B'),$$

so $A' \cap f(X) \subseteq f(B')$. Noticing that $A' \subseteq A$, $g(A') \subseteq B'$ and $g(A) \subseteq B$, we have

$$g(A') \subseteq g(A) \subseteq B.$$

By the hypothesis $E \subseteq F$, we can deduce that $B' \subseteq B$ and the necessity follows.

Conversely, suppose the condition holds and we need to find some $g \in T_{FE}(X)$ such that $f = fgf$. Let $A \subseteq X/F$ and $A \cap f(X) \subseteq f(B)$ for some $B \subseteq X/F$. Suppose $B = \cup_{i \in I} B_i$ where $B_i \subseteq X/E$. Thus $A \cap f(X) \subseteq f(\cup_{i \in I} B_i)$. If $A \cap f(X) = \emptyset$, then we define $g(x) = x$ for each $x \in A$. If $A \cap f(X) \neq \emptyset$, fix $b \in B$ and $b_i \in B_i$ for each $i$. For each $x \in A$, there exists some $A' \subseteq X/E$ such that $x \in A' \subseteq A$. Moreover, by the hypothesis, there exists $E$-class $B_i \subseteq B$ such that $A' \cap f(X) \subseteq f(B_i)$. We first consider the case that $A' \cap f(X) \neq \emptyset$. If $x \in A \cap f(X)$, then $x = f(b_i)$ for some $b_i \in B_i$ and define $g(x) = b_i$. If $x \notin A \cap f(X)$, then define $g(x) = b_i$. Secondly, if $A' \cap f(X) = \emptyset$, then we define $g(x) = b$ for each $x \in A'$. Thus we have defined $g$ on each $A \subseteq X/F$, consequently, on all of $X$. One routinely verifies that $g \in T_{FE}(X)$. To see that $f = fgf$, take any $x \in X$ and let $y = f(x) \in A' \cap f(X) \subseteq f(A \cap f(X))$ where $A \subseteq X/F$ and $A' \subseteq X/E$. By the definition of $g$, we have $g(y) = b_i$ where $b_i \in B_i \subseteq B$ with $f(b_i) = y$. Thus $f(g(f(x))) = f(g(y)) = f(b_i) = y = f(x)$, which implies $f = fgf$ and $f$ is regular in $T_{FE}(X)$. The proof is completed.

3. Green’s relations on $T_{FE}(X)$

In this section, we characterize Green’s relations on $T_{FE}(X)$ and begin with the relation $\mathcal{L}$.

Recall that, in [7], a map $\phi : Y \rightarrow Z$ where $Y, Z \subseteq X$ is said to be $F$-preserving if $F$ is an equivalence on $X$ and $(\phi(y), \phi(y')) \in F$ for each $(y, y') \in F$ with $y, \, y' \in Y$. If $\phi$ satisfies that $(\phi(y), \phi(y')) \in F$ if and only if $(y, y') \in F$, then $\phi$ is said to be $F^*$-preserving.

**Definition 3.1** If $\phi$ is both $F$-preserving and $E$-preserving, then $\phi$ is said to be $FE$-preserving. If $\phi$ is both $F^*$-preserving and $E^*$-preserving, then $\phi$ is said to be $F^*E^*$-preserving.

**Remark 1** An element $f \in T_{FE}(X)$ being either $E^*$-preserving and $F$-preserving, or $F^*$-preserving and $E$-preserving, is not necessarily $F^*E^*$-preserving. For example, let $X = \{1, 2, \ldots\}$, $X/F = \{A_1, A_2\}$ and $X/E = \{A_1, B_1, B_2, B_3, \ldots\}$, where $A_1 = \{1, 2\}$, $A_2 = \{3, 4, \ldots\}$, $B_1 = \ldots$. Then...
Then both $g$ and $h$ are $FE$-preserving. It is not hard to verify that $g$ is $E^*$-preserving, but not $F^*$-preserving while $h$ is $F^*$-preserving, but not $E^*$-preserving. So both $g$ and $h$ are not $F^*E^*$-preserving.

**Theorem 3.2** Let $f, g \in T_{FE}(X)$. Then the following statements are equivalent:

1. $(f, g) \in \mathcal{L}$;
2. $\pi(f) = \pi(g)$, $F(f) = F(g)$ and $E(f) = E(g)$;
3. There exists an $F^*E^*$-preserving bijection $\phi : f(X) \to g(X)$ such that $g = \phi f$.

**Proof** (1)$\Rightarrow$(2). Suppose $(f, g) \in \mathcal{L}$ in $T_{FE}(X)$. Then $(f, g) \in \mathcal{L}$ in both $T_F(X)$ and $T_E(X)$. By Theorem 3.1 in [7], it follows readily that $\pi(f) = \pi(g)$, $F(f) = F(g)$ and $E(f) = E(g)$.

(2)$\Rightarrow$(3). Define $\phi : f(X) \to g(X)$ by $\phi(x) = g(f^{-1}(x))$ for each $x \in f(X)$. Then $\phi$ is well-defined (since $\pi(f) = \pi(g)$) and $g = \phi f$. It is routine to show $\phi$ is $F^*E^*$-preserving.

(3)$\Rightarrow$(1). Suppose that (3) holds. We need to find some $h$, $k \in T_{FE}(X)$ such that $g = hf$ and $f = kg$. For $A \in X/F$, assume $A = \cup_{i \in I} B_i$ where $B_i \in X/E$. Denote $A' = A \cap f(X)$. If $A' = \emptyset$, then define $h(x) = x$ for each $x \in A$. Now assume $A' \neq \emptyset$. Since $\phi$ is $F^*$-preserving, there exists $D \in X/F$ such that $\phi(A') \subseteq D \cap g(X)$. Fix $d \in D$. Notice that $\phi$ is also $E'$-preserving. For each $i \in I$ with $B_i \cap f(X) \neq \emptyset$, there exists some $C_i \in X/E$ such that $\phi(B_i \cap f(X)) \subseteq C_i \subseteq D$. Fix $c_i \in C_i$ for each $i \in I$ with $B_i \cap f(X) \neq \emptyset$ and define

$$h(x) = \begin{cases} \phi(x), & x \in A', \\ c_i, & x \in A - A', x \in B_i \in X/E \text{ and } B_i \cap f(X) \neq \emptyset, \\ d, & x \in A - A', x \in B_i \in X/E \text{ and } B_i \cap f(X) = \emptyset. \end{cases}$$

In this way, we have defined the map $h$ on each $F$-class $A$ and, consequently, on all of $X$. It is not difficult to check that $h \in T_F(X)$ and $h \in T_E(X)$, namely, $h \in T_{FE}(X)$. Finally, we verify that $g = hf$. Let $x \in X$ and assume $f(x) \in A \cap f(X)$ for $A \in X/F$. Then $h(f(x)) = \phi(f(x)) = g(x)$ and $g = hf$. Similarly, one may find some $k \in T_{FE}(X)$ such that $f = kg$. So $(f, g) \in \mathcal{L}$.

In what follows we investigate the relation $\mathcal{R}$. We need some preparations before stating the conclusion. Let $f, g \in T_{FE}(X)$. Recall that a map $\psi : \pi(f) \to \pi(g)$ is said to be $F$-admissible, if for each $A \in X/F$, there exists some $B \in X/F$ such that $B \cap \psi(P) \neq \emptyset$ for each $P \in \pi_A(f)$. If $\psi$ is bijective and both $\psi$ and $\psi^{-1}$ are $F$-admissible, then $\psi$ is said to be $F^*$-admissible. This concept was useful in describing the relation $\mathcal{R}$ on $T_F(X)$ in [7]. To describe the relation $\mathcal{R}$ on $T_{FE}(X)$, we need the following terminology.

**Definition 3.3** Let $\psi : \pi(f) \to \pi(g)$ be a map with $f, g \in T_{FE}(X)$. Suppose for each $A \in X/F$, there exists $B \in X/F$ such that $B \cap \psi(P) \neq \emptyset$ for each $P \in \pi_A(f)$, while for each $A' \in X/E$ with $A' \subseteq A$, there exists $B' \in X/E$ with $B' \subseteq B$ such that $B' \cap \psi(P') \neq \emptyset$ for each $P' \in \pi_{A'}(f)$. Then $\psi$ is said to be $FE$-admissible. If $\psi$ is bijective and both $\psi$ and $\psi^{-1}$ are $FE$-admissible,
then $\psi$ is said to be $F^*E^*$-admissible.

**Remark 2** If $\psi : \pi(f) \to \pi(g)$ is $FE$-admissible, then $\psi$ is both $F$-admissible and $E$-admissible. However, the converse is not, in general, true. For example, let $X = \{1, 2, \ldots\}, X/F = \{A_1, A_2\}$ and $X/E = \{B_1, B_2, B_3, \ldots\}$, where $A_1 = \{1, 2, 3, 4\}, A_2 = \{5, 6, \ldots\}, B_1 = \{1, 2\}, B_2 = \{3, 4\}, B_3 = \{5, 6, \ldots\}$. It is clear that $E \subseteq F$. Let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \cdots \\ 5 & 6 & 7 & 8 & 9 & 10 \cdots \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 12 \cdots \\ 5 & 6 & 7 & 8 & 9 & 10 \cdots \end{pmatrix}.$$ 

Clearly, $f, g \in T_{FE}(X)$, while $\pi(f) = \{\{1\}, \{2\}, \ldots\}$ and $\pi(g) = \{\{1, 5, 6\}, \{2, 7, 8\}, \{3, 4, 9\}, \{10\}, \{11\}, \ldots\}$. Define $\psi : \pi(f) \to \pi(g)$ as follows:

$$\psi(\{1\}) = \{1, 5, 6\}, \quad \psi(\{2\}) = \{2, 7, 8\}, \quad \psi(\{3\}) = \{3, 4, 9\},$$
$$\psi(\{4\}) = \{10\}, \quad \psi(\{5\}) = \{11\}, \quad \psi(\{6\}) = \{12\}, \ldots.$$ 

It is not hard to verify that $\psi$ is both $F$-admissible and $E$-admissible, but not $FE$-admissible. In fact, for $E$-classes $B_1$ and $B_2$ which are contained in the $F$-class $A_1$, there exist $E$-classes $B_1$ and $B_3$ such that $\psi(\pi_B(f)) \subseteq \pi_B(g)$ and $\psi(\pi_B(f)) \subseteq \pi_B(g)$. While there is no $E$-class $B \neq B_3$ such that $\psi(\pi_B(f)) \subseteq \pi_B(g)$. Note that $B_1$ and $B_2$ are contained in the different $F$-classes. By Definition 3.3, $\psi$ is not $FE$-admissible.

For each $h \in T_X$, let $h_*$ denote the map from $\pi(h)$ into $\pi(h)$ defined by $h_*(P) = h(P)$ for $P \in \pi(h)$.

**Theorem 3.4** Let $f, g \in T_{FE}(X)$. Then the following statements are equivalent:

1. $(f, g) \in R$;
2. For each $A \in X/F$, there exist $B, C \in X/F$ such that $f(A) \subseteq g(B)$, $g(A) \subseteq f(C)$ and for each $A' \subseteq A$, there exist $B', C' \subseteq C$ such that $f(A') \subseteq g(B')$ and $g(A') \subseteq f(C')$;
3. There exists an $F^*E^*$-admissible bijection $\psi : \pi(f) \to \pi(g)$ such that $f_* = g_*\psi$.

**Proof**  (1) $\implies$ (2). By the hypothesis, we have $f(X) = g(X)$. Define $\psi : \pi(f) \to \pi(g)$ by $\psi(P) = g^{-1}(f_*(P))$ for each $P \in \pi(f)$. Obviously, $\psi$ is well-defined and $f_* = g_*\psi$ and, by Theorem 3.2 of [7], $\psi$ is $F$-admissible. What remains for us is to show that $\psi$ is $E$-admissible. Now for each $A' \subseteq X/F$ with $A' \subseteq A \in X/F$, by the hypothesis, there exists $B' \subseteq B \in X/F$ such that $f(A') \subseteq g(B')$. Let $\pi_{A'}(f) = \{P_i : i \in I\}$ and $\{x_i'\} = f_*(P_i)(i \in I)$. Then $x_i' \in f(A') \subseteq g(B')$, so $B' \cap g^{-1}(x_i') \neq \emptyset$. Consequently,

$$B' \cap \psi(P_i) = B' \cap g^{-1}(f_*(P_i)) = B' \cap g^{-1}(x_i') \neq \emptyset$$

for each $P_i \in \pi_{A'}(f)$ which means that $\psi$ is $FE$-admissible. Similarly, one may show that $\psi^{-1}$ is also $FE$-admissible. And $\psi : \pi(f) \to \pi(g)$ is $F^*E^*$-admissible, as required.

(2) $\implies$ (1). Suppose that (3) holds. We need to find $h, k \in T_{FE}(X)$ such that $f = gh$ and $g = fk$. Since $\psi$ is $F$-admissible, for each $A \subseteq X/F$, there exists $B \subseteq X/F$ such that $B \cap \psi(P) \neq \emptyset$
for each $P \in \pi_A(f)$. Assume $A = \cup_{i \in I} A_i$ where $A_i \in X/E$ and let $P_x = f^{-1}(f(x))$ for every $x \in A_i$. Then $x \in P_x \in \pi_A(f)$. Therefore there exists some $B_i \in X/E$ with $B_i \subset B$ such that $B_i \cap \psi(P_x) \neq \emptyset$ for each $P_x \in \pi_A(f)$. Choose $y \in B_i \cap \psi(P_x)$ and define $h(x) = y$. Then $gh(x) = g(y) = g_\ast(\psi(P_x))$ and $\psi(P_x) = g^{-1}(gh(x))$. Now we have defined the map $h$ on each $F$-class $A$, consequently, on all of $X$. It is clear that $h \in T_{FE}(X)$ and $f = gh$. Similarly, one can find some $k \in T_{FE}(X)$ such that $g = fk$. Consequently, $(f,g) \in \mathcal{R}$.

Using Theorems 3.2 and 3.4, we can establish the next result.

**Theorem 3.5** Let $f, g \in T_{FE}(X)$. Then the following statements are equivalent:

1. $(f,g) \in \mathcal{H}$;
2. $\pi(f) = \pi(g), F(f) = F(g), E(f) = E(g)$. For each $A \in X/F$, there exist $B, C \in X/F$ such that $f(A) \subseteq g(B)$ and $g(A) \subseteq f(C)$, while for each $A' \in X/E$ with $A' \subset A$, there exist $B', C' \in X/E$ with $B' \subseteq B$, $C' \subset C$ such that $f(A') \subseteq g(B')$, $g(A') \subseteq f(C')$;
3. There exist an $F^*E^*$-preserving bijection $\phi: f(X) \rightarrow g(X)$ and an $F^*E^*$-admissible bijection $\psi: \pi(f) \rightarrow \pi(g)$ such that $g = \phi f$ and $f_\ast = g_\ast \psi$.

Next we consider the relation $\mathcal{D}$.

**Theorem 3.6** Let $f, g \in T_{FE}(X)$. Then the following statements are equivalent:

1. $(f,g) \in \mathcal{D}$;
2. There exist an $F^*E^*$-admissible bijection $\psi: \pi(f) \rightarrow \pi(g)$ and an $F^*E^*$-preserving bijection $\phi: f(X) \rightarrow g(X)$ such that $\phi f_\ast = g_\ast \psi$.

The proof is similar to that of Theorem 3.4 of [7] and it is omitted.

Now we discuss the last relation $\mathcal{J}$. Recall that, in a semigroup $S$, $J_a \leq J_b$ means that $S^1aS^1 \subseteq S^1bS^1$ where $J_x$ denotes the $J$-class containing $x \in S$.

**Lemma 3.7** Let $f, g \in T_{FE}(X)$. Then $J_f \leq J_g$ if and only if there exists an $FE$-preserving surjection $\phi: g(X) \rightarrow f(X)$ such that for each $A \in X/F$, there exists $B \in X/F$ such that $f(A) \subseteq \phi(g(B))$, while for each $C \in X/E$ with $C \subseteq A$, there exists $D \in X/E$ with $D \subseteq B$ such that $f(C) \subseteq \phi(g(D))$.

**Proof** Suppose $J_f \leq J_g$. Then there exist $h, k \in T_{FE}(X)$ such that $f = h g k$. Take $A \in X/F$ with $A \cap g(X) \neq \emptyset$. Assume $A = \cup_{i \in I} B_i$, where $B_i \in X/E$. Denote $A' = A \cap g(X), A'' = A \cap gk(X), B_i' = B_i \cap g(X)$ and $B_i'' = B_i \cap gk(X)$. Fix $a \in h(A'') \subseteq f(X)$ and $x_i \in B_i''$ for each $i$ with $B_i'' \neq \emptyset$. Define

$$
\phi(x) = \begin{cases} 
    h(x), & x \in A' \\
    h(x_i), & x \in A' - A'' \text{ and } B_i'' \neq \emptyset \\
    a, & x \in A' - A'' \text{ and } B_i'' = \emptyset.
\end{cases}
$$

In this way, we can define the map $\phi$ on $g(X)$. To see $\phi(g(X)) \subseteq f(X)$, for each $x \in g(X)$, if $x \in gk(X)$, then $\phi(x) = h(x) \in h gk(X) = f(X)$; if $x \in g(X) - gk(X)$, $x \in B_i'$ and $B_i'' \neq \emptyset$ for some $i$, then $\phi(x) = h(x_i) \in h gk(X) = f(X)$, too. So $\phi$ indeed maps $g(X)$ into $f(X)$. One routinely verifies that $\phi$ is $FE$-preserving. For each $A \in X/F$, let $k(A) \subseteq B$ for some $B \in X/F$. 


Thus
\[ f(A) = h(gk(A)) = \phi(gk(A)) \subseteq \phi(g(B)), \]
which implies that \( \phi \) is surjective. Similarly, for each \( C \in X/E \) with \( C \subseteq A \), there exists some \( D \in X/E \) such that \( k(C) \subseteq D \) and \( f(C) \subseteq \phi(g(D)) \). By the hypothesis \( E \subseteq F \) and \( C \subseteq A \), it follows that \( k(C) \subseteq k(A) \) and \( D \subseteq B \).

Conversely, suppose there exists such a map \( \phi \). We shall construct some \( h, k \in T_{FE}(X) \) such that \( f = h g k \). Let \( A \in X/F \) and assume \( A = \bigcup_{i \in I} B_i \) where \( B_i \in X/E \). Denote \( A' = A \cap g(X) \).

If \( A' = \emptyset \), then define \( h(x) = x \) for each \( x \in A \). If \( A' \neq \emptyset \), let
\[ B = \{ B_i : B_i \cap g(X) \neq \emptyset \}. \]
Fix \( x_i \in B_i \cap g(X) \) for each \( B_i \in B \). Since \( \phi \) is \( FE \)-preserving, there exists some \( D \in X/F \) such that \( \phi(A') \subseteq D \). Fix \( b \in D \) and define
\[ h(x) = \begin{cases} \phi(x), & x \in A', \\ \phi(x_i), & x \in A - A' \text{ and } x \in B_i \in B, \\ b, & x \in A - A' \text{ and } x \in B_i \notin B. \end{cases} \]

It is not hard to verify that \( h \in T_{FE}(X) \).

Now we construct \( k \). By the hypothesis, for each \( A \in X/F \) there exists \( B \in X/F \) such that \( f(A) \subseteq \phi(g(B)) \), while for each \( C \in X/E \) with \( C \subseteq A \), there exists \( D \in X/E \) with \( D \subseteq B \) such that \( f(C) \subseteq \phi(g(D)) \). Thus, for each \( x \in C \subseteq A \), there exists some \( y \in D \subseteq B \) such that \( f(x) = \phi(g(y)) \). Define \( k(x) = y \). Clearly, \( k \in T_{FE}(X) \). One may routinely verify that \( f = h g k \).

This completes the proof.

As an immediate consequence of Lemma 3.7, we have the following

**Theorem 3.8** Let \( f, g \in T_{FE}(X) \). Then \( (f, g) \in \mathcal{J} \) if and only if there exist \( FE \)-preserving surjections \( \phi : g(X) \to f(X) \) and \( \psi : f(X) \to g(X) \) such that for each \( A \in X/F \), there exists \( B, B' \in X/F \) such that \( f(A) \subseteq \phi(g(B)), g(A) \subseteq \psi(f(B')) \), while for each \( C \in X/E \) with \( C \subseteq A \), there exists \( D, D' \in X/E \) with \( D \subseteq B \) and \( D' \subseteq B' \) such that \( f(C) \subseteq \phi(g(D)), g(C) \subseteq \psi(f(D')) \).

**Remark** In general, \( \mathcal{J} \neq \mathcal{D} \) in the semigroup \( T_{FE}(X) \). For example, let \( X = \{0, 1, 2, 3, \ldots\} \) and \( X/F = \{A_1, B_1, A_2, B_2, A_3, B_3, \ldots\} \), \( X/E = \{C_1, C_2, B_1, A_2, B_2, A_3, B_3, \ldots\} \), where \( A_1 = \{0, 2, 4, 6\}, B_1 = \{1, 3, 5\}, A_2 = \{8, 10\}, B_2 = \{7, 9\}, A_3 = \{12, 14\}, B_3 = \{11, 13\}, \ldots \), \( C_1 = \{0, 2\}, C_2 = \{4, 6\} \). Then \( E \subseteq F \). Let \( f, g \in T_X \) be such that
\[ f(C_1) = f(C_2) = C_1, f(B_1) = C_2, f(A_2) = A_2, f(B_2) = A_3, \]
\[ f(A_3) = A_4, f(B_3) = A_5, \ldots \]
and
\[ g(C_1) = g(C_2) = \{1, 3\}, g(B_1) = \{5\}, g(A_2) = B_2, \]
\[ g(B_2) = B_3, g(A_3) = B_4, g(B_3) = B_5, \ldots \]
Clearly, $f, g \in T_{FE}(X)$. Now we define $\psi: f(X) \to g(X)$ and $\phi: g(X) \to f(X)$, respectively, as follows:

$$
\psi(C_1) = \{1, 3\}, \psi(C_2) = \{5\}, \psi(A_2) = B_2, \psi(A_3) = B_3, \ldots
$$

and

$$
\phi(\{1, 3\}) = C_1, \phi(\{5\}) = \{2\}, \phi(B_2) = C_2, \phi(B_3) = A_2, \phi(B_4) = A_3, \ldots.
$$

Then $\psi$ and $\phi$ satisfy the conditions in Theorem 3.8. Therefore, $(f, g) \in J$. However, since $f(X) = \bigcup\{A_i : i = 1, 2, \ldots\}$ and $g(X) = \bigcup\{B_i : i = 1, 2, \ldots\}$, there is no $F^*E^*$-preserving bijection from $f(X)$ onto $g(X)$. In fact, suppose there exists such one, say $\rho$, then $\rho(A_1) = B_i$ for some $i$. Note that $|A_1| = 4$ and $|B_i| \leq 3$ for each $i$. So $\rho$ is impossible to be bijective. Thus, by Theorem 3.6, $(f, g) \notin D$.

References