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The Lattices of Congruences on Regular Semigroups with Q-Inverse Transversals

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Abstract In this paper, a complete congruence on the congruence lattice of regular semigroups with Q-inverse transversals is analysed. The classes of this complete congruence which are intervals are discussed and their least and greatest elements are presented clearly.

Keywords Q-inverse transversal; congruence; congruence pairs; congruence lattices; congruence relations on the congruence lattice.

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1. Preliminaries

We shall use the notations and terminologies of [1] and [2] in this paper. Let S be a regular semigroup. An inverse subsemigroup S° of S is an inverse transversal if $|V(x)\cap S^\circ|=1$ for each $x\in S$, where $V(x)$ denotes the set of inverses of x. In this case, the unique element of $V(x)\cap S^\circ$ is denoted by x° and $(x^\circ)^\circ$ is denoted by $x^{\circ\circ}$. We have $x^{\circ\circ\circ}=x^\circ$ for each $x\in S$. E° denotes the semilattice of idempotents of S°, while $I(S)=\{e\in S|ee^\circ=e\}$, $\Lambda(S)=\{g\in S|g^\circ g=g\}$, $R(S)=\{x\in S|x^\circ x=x^\circ x^{\circ\circ}\}$ and $L(S)=\{a\in S|aa^\circ=a^{\circ\circ}a^\circ\}$. The above signs are denoted by I,Λ,R and L if no confusion is possible. For every $x\in S$, we define $x_I=xx^\circ, x_\Lambda=x^\circ x, x_R=xx^\circ x^{\circ\circ}$ and $x_L=x^{\circ\circ}x^\circ x$. Obviously, for each $x\in S, x_R\in R, x_L\in L, (x_R)^\circ=x^\circ$ and $(x_L)^\circ=x^\circ$. For every $e\in I, g\in\Lambda, a\in R$ and $x\in L$, we have $e_I=e, g_\Lambda=g, a_R=a$ and $x_L=x$. If an inverse transversal S° of S is a quasi-ideal of S (that is $S^\circ SS^\circ\subseteq S^\circ$), S° is called a Q-inverse transversal of S. Throughout this paper, S will denote a regular semigroup with a Q-inverse transversal S° if no special mention is made. Each x in S can be written uniquely in the form $x=ea$, where $e\in I, a\in L$. Thus there is a mapping $x\longmapsto(xx^\circ, x^{\circ\circ}x^\circ x)$ from S onto the set

$$\{(e,a)\in I\times L|e^\circ=aa^\circ\}.$$

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By using this mapping, S can be coordinatized by pairs. A band B is left [resp. right] normal if $efg = egf$ [resp. $efg = feg$] for every $e, f, g \in B$. A non-empty subset A of S is called a left ideal [resp. right ideal] if $SA \subseteq A$ [resp. $AS \subseteq A$]. A semigroup is called orthodox if it is regular and if its idempotents form a subsemigroup.

We list several known results, which will be used frequently without special reference in this paper.

(1.1)[9] $(xy)^\circ = (x^\circ xy)^\circ x^\circ = y^\circ(xyy^\circ)^\circ$ for every $x, y \in S$.

(1.2)[3,5,7−9] $I(S)$ [resp. $\Lambda(S)$] is a left [resp. right] normal band with a Q-inverse transversal E°. Furthermore, $R(S)$ and $L(S)$ are orthodox semigroups with a Q-inverse transversal S° which is a right ideal of R and a left ideal of L.

(1.3)[5,8] $R(S) \cap L(S) = S^\circ, I(S) \cap \Lambda(S) = E^\circ, E(R(S)) = I(S)$ and $E(L(S)) = \Lambda(S)$.

(1.4)[6] S is orthodox if and only if for any $x, y \in S, (xy)^\circ = y^\circ x^\circ$.

(1.5)[9] For a regular semigroup S with an inverse transversal S°, S° is a Q-inverse transversal if and only if for every $x, y \in S, x_\Lambda.y_I \in S^\circ$, and if and only if for any $s, t \in S^\circ, x \in S, sxt = sx^{\circ\circ}t$, and if and only if for any $s, t \in S^\circ, x \in S, (sxt)^\circ = t^\circ x^\circ s^\circ$.

(1.6)[9] For $\rho \in \mathrm{Con}(S)$, let $\rho^\circ = \rho|_{S^\circ}$. Then for $x, y \in S, x\rho y$ implies $x^\circ \rho y^\circ$.

(1.7)[9] For any congruence π on S°, there exists $\rho \in \mathrm{Con}(S)$ such that $\pi = \rho|_{S^\circ}$.

For a regular semigroup $S, E(S)$ denotes the set of idempotents. The complete lattice of congruences on S is denoted by $\mathrm{Con}(S)$. For any $\rho \in \mathrm{Con}(S)$, define $\rho^\circ, \rho_I, \rho_\Lambda, \rho_R$ and ρ_L as follows:

$$\rho^\circ = \rho|_{S^\circ}, \quad \rho_I = \rho|_I, \quad \rho_\Lambda = \rho|_\Lambda, \quad \rho_R = \rho|_R, \quad \rho_L = \rho|_L.$$

If $\rho \in \mathrm{Con}(S)$, then the trace of ρ is $\mathrm{tr}\rho = \rho|_{E(S)}$ and the kernel of ρ is $\ker\rho = \{s \in S | s\rho s^2\}$. We present the following notions and results due to Pastijn and Petrich[4]. For any $\rho, \sigma \in \mathrm{Con}(S)$, define T_l, T_r, U and V as follows:

$$\rho T_l \sigma \Leftrightarrow tr(\rho \vee \mathcal{L})^\flat = \mathrm{tr}(\sigma \vee \mathcal{L})^\flat, \quad \rho T_r \sigma \Leftrightarrow tr(\rho \vee \mathcal{R})^\flat = \mathrm{tr}(\sigma \vee \mathcal{R})^\flat,$$

$$\rho U \sigma \Leftrightarrow \rho \cap \leq = \sigma \cap \leq, \quad \rho V \sigma \Leftrightarrow \rho U \sigma, \rho K \sigma,$$

where $(\ )^\flat$ denotes the greatest congruence on S contained in the relation $(\ )$, $\leq$ denotes the natural partial order on $E(S)$, and K is a relation on $\mathrm{Con}(S)$ such that $\rho K \sigma$ if and only if $\ker\rho = \ker\sigma$. Then these relations are complete congruences on the lattice $\mathrm{Con}(S)$. The congruence class ρT_r [resp. $\rho T_l, \rho U, \rho V$] is an interval of $\mathrm{Con}(S)$ with greatest and smallest element to be denoted by ρ^{T_r} [resp. $\rho^{T_l}, \rho^{T_U}, \rho^{T_V}$] and ρ_{T_r} [resp. $\rho_{T_l}, \rho_{T_U}, \rho_{T_V}$], respectively. Let τ be a relation on S. The congruence generated by τ is denoted by τ^*.

In [9], congruences were coordinatized abstractly by triples which consist of congruences on S°, I and Λ satisfying certain conditions. Five complete congruences V, T, T_r, T_l and U on the congruences lattices are discussed and their least and greatest elements are presented in terms of congruence triples. We present the following notions and results due to Wang[9]. Let π be a congruence on S°. Define relations $\mu_I(\pi)$ on I as follows:

$$e\mu_I(\pi)f \Leftrightarrow (\exists p^\circ \in E^\circ)ep^\circ = fp^\circ, p^\circ \pi e^\circ \pi f^\circ.$$

Furthermore, $\mu_I(\pi)$ is a congruence on I and $\mu_I(\pi)|_{E^\circ} = \pi|_{E^\circ}$. If $\pi = \rho|_{E^\circ}$, then $\mu_I(\pi) \subseteq \rho_I$. A congruence τ on I is normal if τ satisfies the following condition:

$$(\forall e, f \in I)(\forall a \in S^\circ) e\tau f \Rightarrow aea^\circ \tau afa^\circ.$$

Denote by $C_N(I)$ the set of normal congruences on I. Let $\tau_I \in C_N(I)$. Define a relation $\tau^{\circ t}$ on S° by

$$a\tau^{\circ t}b \Leftrightarrow (\forall e^\circ \in E^\circ) ae^\circ a^\circ \tau_I be^\circ b^\circ.$$

$\tau^{\circ t}$ is the greatest congruence in $\mathrm{Con}(S^\circ)$ such that $\tau^{\circ t}|_{E^\circ} = \tau^\circ|_{E^\circ}^{[2]}$. Furthermore, $\tau^\circ \subseteq \tau^{\circ t}$. For any $\rho, \lambda \in \mathrm{Con}(S)$, by Theorems 4.4, 4.9 and the dual of Theorem 4.9 in [9], we have

$$\rho V \lambda \Leftrightarrow \rho^\circ = \lambda^\circ, \quad \rho T_r \lambda \Leftrightarrow \rho_I = \lambda_I, \quad \rho T_l \lambda \Leftrightarrow \rho_\Lambda = \lambda_\Lambda.$$

Shang and Wang[10] have shown that congruences on regular semigroups with Q-inverse transversals can also be characterized abstractly by congruence pairs which consist of congruences on I and L satisfying certain conditions.

Definition 1.1[10] *Let τ_I and τ_L be congruences on I and L, respectively. If they satisfy the following conditions, (τ_I, τ_L) is called a congruence pair for S.*

(i) $(\tau_I)|_{E^\circ} = (\tau_L)|_{E^\circ}$;

(ii) $(\forall e, f \in I, x \in L) e\tau_I f \Rightarrow xe\tau_L xf$, $(\forall x, y \in L, e \in I) x\tau_L y \Rightarrow xe\tau_L ye$.

Clearly, (ii) is equivalent to the following condition (iii),

(iii) $(\forall e, f \in I, x, y \in L) e\tau_I f, x\tau_L y \Rightarrow xe\tau_L yf$. *Define a relation $\rho_{(\tau_I,\ \tau_L)}$ on S by the following rule,*

$$x\rho_{(\tau_I,\ \tau_L)} y \Leftrightarrow x_I \tau_I y_I, \quad x_L \tau_L y_L.$$

Lemma 1.1[10] *For any $\rho, \sigma \in \mathrm{Con}(S)$, $\rho \subseteq \sigma \Leftrightarrow \rho_I \subseteq \sigma_I$, $\rho_L \subseteq \sigma_L$. Therefore, $\rho = \sigma \Leftrightarrow \rho_I = \sigma_I$, $\rho_L = \sigma_L$.*

Lemma 1.2[10] *For every congruence pair (τ_I, τ_L) for S, the relation $\rho_{(\tau_I,\ \tau_L)}$ is the unique congruence on S whose restrictions to I and L are τ_I and τ_L, respectively. Conversely, every congruence on S can be represented in this way.*

In this paper, we study the complete congruence Q on the congruence lattice of regular semigroups with Q-inverse transversals. And we go one step further to give the least and the greatest elements of complete congruence T_r in terms of congruence pairs.

2. The congruence relation Q on $\mathrm{Con}(S)$

In this section, we investigate the congruence relation Q on the lattice $\mathrm{Con}(S)$. The classes of the congruence relation Q are intervals of $\mathrm{Con}(S)$. We present the least and the greatest elements of each classes of the congruence relation Q clearly.

One may observe from Lemma 1.2 that for every $\rho \in \mathrm{Con}(S)$, there exists a congruence pair $J_\rho = (\rho_I, \rho_L)$ and vice versa for every congruence pair J, a congruence ρ_J. By Lemma 1.2, $\rho \longmapsto (\rho_I, \rho_L)$ and $J \longmapsto \rho_J$ are mutually inverse mappings satisfying $\rho_{J_\rho} = \rho, J_{\rho_J} = J$.

Denote by $CP(S)$ the set of congruence pairs for S. Define an order $\leq$ on $CP(S)$ by componentwise inclusion. It is clear that $\leq$ is a partial order on $CP(S)$. From Lemma 1.1, we have

$$(\tau_I, \tau_L) \leq (\tau'_I, \tau'_L) \Leftrightarrow \rho_{(\tau_I,\ \tau_L)} \subseteq \rho_{(\tau'_I,\ \tau'_L)}.$$

By Lemma 1.2, we know that $\mathrm{Con}(S)$ and $CP(S)$ are isomorphic as partially ordered sets and therefore as (complete) lattices. In what follows we determine joins and meets in the lattice $CP(S)$.

Lemma 2.1 *Let Ψ be a family of congruences on S. For $\rho \in \Psi$, denote $J_\rho = (\rho_I, \rho_L)$. Then*

$$J_{\cap_{\rho\in\Psi}\rho} = (\cap_{\rho\in\Psi}\rho_I,\ \ \cap_{\rho\in\Psi}\ \rho_L),\quad J_{\vee_{\rho\in\Psi}\rho} = (\vee_{\rho\in\Psi}\rho_I,\ \ \vee_{\rho\in\Psi}\ \rho_L).$$

Proof The first equality is obvious. In order to show the second, it suffices to prove $(\vee_{\rho\in\Psi}\rho)_I = \vee_{\rho\in\Psi}\rho_I$ and $(\vee_{\rho\in\Psi}\rho)_L = \vee_{\rho\in\Psi}\rho_L$. Suppose $e(\vee_{\rho\in\Psi}\rho)_I f$ for $e, f \in I$, then

$$\begin{aligned} e(\vee_{\rho\in\Psi}\rho)_I f &\Rightarrow e(\vee_{\rho\in\Psi}\rho) f \\ &\Rightarrow (\exists \rho_i \in \Psi, g_i \in S) e\rho_1 g_1 \rho_2 g_2 \rho_3 \ldots \rho_{n-1} g_{n-1} \rho_n f \\ &\Rightarrow e = ee^\circ \rho_{1I} g_1 g_1^\circ \rho_{2I} g_2 g_2^\circ \rho_{3I} \ldots \rho_{n-1 I} g_{n-1} g_{n-1}^\circ \rho_{nI} f f^\circ = f \\ &\Rightarrow e(\vee_{\rho\in\Psi}\rho_I) f. \end{aligned}$$

So $(\vee_{\rho\in\Psi}\rho)_I \subseteq \vee_{\rho\in\Psi}\rho_I$. The reverse inclusion is obvious.

Next assume $x(\vee_{\rho\in\Psi}\rho)_L y$ for $x, y \in L$, then

$$\begin{aligned} x(\vee_{\rho\in\Psi}\rho)_L y &\Rightarrow x(\vee_{\rho\in\Psi}\rho) y \\ &\Rightarrow (\exists \rho_i \in \Psi, z_i \in S) x\rho_1 z_1 \rho_2 z_2 \rho_3 \ldots \rho_{n-1} z_{n-1} \rho_n y \\ &\Rightarrow x = x^{\circ\circ} x^\circ x \rho_{1L} z_1^{\circ\circ} z_1^\circ z_1 \rho_{2L} z_2^{\circ\circ} z_2^\circ z_2 \rho_{3L} \ldots \rho_{n-1L} z_{n-1}^{\circ\circ} z_{n-1}^\circ z_{n-1} \rho_{nL} y^{\circ\circ} y^\circ y = y \\ &\Rightarrow x(\vee_{\rho\in\Psi}\rho_L) y. \end{aligned}$$

Thus $(\vee_{\rho\in\Psi}\rho)_L \subseteq \vee_{\rho\in\Psi}\rho_L$. The reverse inclusion is obvious. □

Lemma 2.2 *Let Γ be a nonempty family of congruence pairs for S and denote $J = (\tau_I, \tau_L) \in \Gamma$. Then*

$$\cap_{J\in\Gamma}\rho_J = \rho_{(\cap_{J\in\Gamma}\tau_I,\ \cap_{J\in\Gamma}\tau_L)},\quad \vee_{J\in\Gamma}\rho_J = \rho_{(\vee_{J\in\Gamma}\tau_I,\ \vee_{J\in\Gamma}\tau_L)}.$$

Proof Denote simply $\rho = \rho_J = \rho_{(\tau_I,\ \tau_L)}$. From Lemma 2.1,

$$J_{\cap_{J\in\Gamma}\rho} = (\cap_{J\in\Gamma}\tau_I,\ \ \cap_{J\in\Gamma}\ \tau_L).$$

From Lemma 1.2, $\rho_{J_{\cap\rho}} = \cap\rho$. Hence the first equality holds. The second one may be proved similarly.

Lemma 2.3 *$CP(S)$ is a lattice under the partial order $\leq$. The lattice operations in $CP(S)$ are given as follows:*

$$(\tau_I,\ \tau_L) \cap (\tau_I',\ \tau_L') = (\tau_I \cap \tau_I',\ \tau_L \cap \tau_L'),$$
$$(\tau_I,\ \tau_L) \vee (\tau_I',\ \tau_L') = (\tau_I \vee \tau_I',\ \tau_L \vee \tau_L').$$

Proof We omit the proof since it is easy. □

Lemma 2.4 *Let π be a congruence on S°. Then we have*

$$(\forall a, b \in S^\circ)(\forall e \in I, x \in L) a\pi b \Rightarrow xae\pi xbe.$$

Proof Suppose $a\pi b$ for $a, b \in S^\circ$. Then for $e \in I, x \in L$, by (1.5),

$$xae = x^{\circ\circ}x^\circ xaee^\circ = x^{\circ\circ}x^\circ x^{\circ\circ}ae^{\circ\circ}e^\circ \pi x^{\circ\circ}x^\circ x^{\circ\circ}be^{\circ\circ}e^\circ = xbe.$$

Let π be a congruence on S°. Define a relation $\xi_I(\pi)$ on I with respect to π by

$$e\xi_I(\pi)f \Leftrightarrow (\forall x \in L)xe\pi xf.$$

We set

$$C_N(L) = \{\sigma \in \mathrm{Con}(L) | \exists \rho \in \mathrm{Con}(S), \rho|_L = \sigma\}.$$

Theorem 2.1 *Define a mapping ψ from* $\mathrm{Con}(S)$ *into $C_N(L)$ by*

$$\psi : \rho \longmapsto \rho_L.$$

Then the following statements are true.

(i) ψ is a complete homomorphism from $\mathrm{Con}(S)$ *onto $C_N(L)$;*

(ii) The complete congruence Q on $\mathrm{Con}(S)$ *induced by ψ is $V \cap T_l$;*

(iii) For any $\rho \in \mathrm{Con}(S)$, the Q-class ρQ is an interval of $\mathrm{Con}(S)$ *such that*

$$\rho Q = [\rho_Q, \rho^Q],$$

where $\rho_Q = \rho_{(\mu_I(\rho^\circ),\ \rho_L)}$ and $\rho^Q = \rho_{(\xi_I(\rho^\circ),\ \rho_L)}$.

Proof (i) By the definition of $C_N(L)$, ψ is surjective; ψ is a complete homomorphism by Lemma 2.2.

(ii) We need to prove that $Q = V \cap T_l$. If $\rho(V \cap T_l)\sigma$, then by Theorem 4.4 and the dual of Theorem 4.9 in [9], $\rho|_{S^\circ} = \sigma|_{S^\circ}$ and $\rho|_\Lambda = \sigma|_\Lambda$. Let $x, y \in L$ with $x\rho y$. Then $x^\circ \rho y^\circ$, and so $x^{\circ\circ}\rho y^{\circ\circ}$. Hence $x^\circ x\rho y^\circ y$. Thus $x^\circ x\sigma y^\circ y$ and $x^{\circ\circ}\sigma y^{\circ\circ}$, and so $x = x^{\circ\circ}x^\circ x\sigma y^{\circ\circ}y^\circ y = y$. Therefore $\rho_L \subseteq \sigma_L$. Similarly, the reverse inclusion holds. So $\rho Q\sigma$. That is to say, $V \cap T_l \subseteq Q$. Obviously, the reverse inclusion also holds.

(iii) By Theorem 4.4 in [9], we have $\mu_I(\rho^\circ)|_{E^\circ} = \rho^\circ|_{E^\circ} = (\rho_L)|_{E^\circ}$. Assume $e\mu_I(\rho^\circ)f$ for $e, f \in I$. Since $\mu_I(\rho^\circ) \subseteq \rho_I$, we have $e\rho f$. For any $x \in L$, by (1.3), we have $xe\rho_L xf$. Let $x, y \in L$ with $x\rho_L y$. Then for any $e \in I$, we have $xe\rho ye$. So $xe\rho_L ye$. Hence $(\mu_I(\rho^\circ), \rho_L)$ is a congruence pair such that $\rho_{(\mu_I(\rho^\circ),\ \rho_L)} \in \rho Q$. We have also $\rho_{(\mu_I(\rho^\circ),\ \rho_L)} \subseteq \rho_{(\rho_I,\ \rho_L)} = \rho$. Since the definition of $\mu_I(\rho^\circ)$ depends only on $\rho^\circ, \rho_{(\mu_I(\rho^\circ),\ \rho_L)}$ is the smallest element of ρQ.

Let π be a congruence on S°. Clearly, $\xi_I(\pi)$ is an equivalence relation on I. Let $e\xi_I(\pi)f$. Then for any $x \in L, xe\pi xf$, and for any $g \in I$, by Lemma 2.4, we have

$$xeg = x^{\circ\circ}x^\circ xeg\pi x^{\circ\circ}x^\circ xfg = xfg$$

so $eg\xi_I(\pi)fg$. We also have $g^\circ e\pi g^\circ f$. Since $xg \in S^\circ$, we get

$$xge = xgg^\circ e\pi xgg^\circ f = xgf,$$

and thus $ge\xi_I(\pi)gf$. Hence $\xi_I(\pi)$ is a congruence on I. If for $e^\circ, f^\circ \in E^\circ, e^\circ\pi f^\circ$, then for any $x \in L$, by Lemma 2.4, we have

$$xe^\circ = xe^\circ e^\circ \pi x f^\circ e^\circ = xe^\circ f^\circ \pi x f^\circ f^\circ = xf^\circ.$$

So $e^\circ \xi_I(\pi) f^\circ$. Thus $\pi|_{E^\circ} \subseteq (\xi_I(\pi))|_{E^\circ}$. Conversely, if $e^\circ(\xi_I(\pi))|_{E^\circ} f^\circ$, then

$$e^\circ = e^\circ e^\circ \pi e^\circ f^\circ = f^\circ e^\circ \pi f^\circ f^\circ = f^\circ.$$

Thus $(\xi_I(\pi))|_{E^\circ} \subseteq \pi|_{E^\circ}$, and so $(\xi_I(\pi))|_{E^\circ} = \pi|_{E^\circ}$. Let $\pi = \rho^\circ$. Then $\xi_I(\rho^\circ)|_{E^\circ} = \rho^\circ|_{E^\circ} = (\rho_L)|_{E^\circ}$. Assume $e\xi_I(\rho^\circ)f$ for $e, f \in I$. For any $x \in L$, by the definition of $\xi_I(\rho^\circ)$ we have $xe\rho^\circ xf$. Noticing $xe,\ xf \in S^\circ \subseteq L$, we get $xe\rho_L xf$. If for $x, y \in L$, $x\rho_L y$, then $x\rho y$. For any $e \in I$, we have $xe\rho ye$, and thus $xe\rho_L ye$. Hence $(\xi_I(\rho^\circ), \rho_L)$ is a congruence pair such that $\rho_{(\xi_I(\rho^\circ),\ \rho_L)} \in \rho Q$. Noticing $\rho_I \subseteq \xi_I(\rho^\circ)$, we also have $\rho = \rho_{(\rho_I,\ \rho_L)} \subseteq \rho_{(\xi_I(\rho^\circ),\ \rho_L)}$. Since the definition of $\xi_I(\rho^\circ)$ depends only on ρ°, it follows that $\rho_{(\xi_I(\rho^\circ),\ \rho_L)}$ is the greatest element of ρQ.

Therefore, the Q-class ρQ is $[\rho_Q, \rho^Q]$.

3. The congruence relation T_r on $\mathrm{Con}(S)$

Using congruence pairs, we present the least and the greatest element in each T_r- class in this section, which is in a different approach to [9].

Let π be a congruence on S°. Define relations $\xi_L(\pi)$ and $\nu_L(\pi)$ on L with respect to π respectively, as follows.

$$x\xi_L(\pi)y \Leftrightarrow (\forall e \in I) xe\pi ye,$$

$$x\nu_L(\pi)y \Leftrightarrow (\exists p^\circ \in E^\circ) p^\circ x\pi^* p^\circ y, \quad p^\circ \pi x^{\circ\circ} x^\circ \pi y^{\circ\circ} y^\circ,$$

where π^* is the congruence generated by π on S. Noticing π^* is the smallest element of $\{\rho \in \mathrm{Con}(S) : \rho|_{S^\circ} = \pi\}$. This is because $\pi = \pi|_{S^\circ} \subseteq (\pi^*)|_{S^\circ}$. By (1.7), there exists a $\xi \in \mathrm{Con}(S)$ such that $\xi|_{S^\circ} = \pi$. Thus $(\pi^*)|_{S^\circ} \subseteq \xi|_{S^\circ} = \pi$, and so $(\pi^*)|_{S^\circ} = \pi$. If $\rho \in \mathrm{Con}(S)$, $\rho|_{S^\circ} = \pi$, then $\pi \subseteq \rho$. So $\pi^* \subseteq \rho$.

Let $\tau_I \in C_N(I) = \{\sigma \in \mathrm{Con}(I) | \exists \rho \in \mathrm{Con}(S), \rho|_I = \sigma\}$. Define a relation τ_n° on S° by $\tau_n^\circ = \beta^*$, where $\beta = \{(xe, xf) | e\tau_I f, x \in L\}$.

Theorem 3.1 *For $\rho \in \mathrm{Con}(S)$, we have $\rho_{T_r} = \rho_{(\rho_I,\ \nu_L(\rho_n^\circ))}$ and $\rho^{T_r} = \rho_{(\rho_I,\ \xi_L(\rho^{\circ t}))}$.*

Proof Let π be a congruence on S°. We need to prove that $\nu_L(\pi)$ is a congruence on L. For $x \in L$, there is $p^\circ = x^{\circ\circ} x^\circ \in E^\circ$ such that $x^{\circ\circ} x^\circ x \pi^* x^{\circ\circ} x^\circ x$ and $x^{\circ\circ} x^\circ \pi x^{\circ\circ} x^\circ \pi x^{\circ\circ} x^\circ$. Hence $\nu_L(\pi)$ is reflexive. Clearly, $\nu_L(\pi)$ is symmetric. For $x, y, z \in L$ with $x\nu_L(\pi)y$ and $y\nu_L(\pi)z$, there exist $p^\circ, q^\circ \in E^\circ$ such that

$$p^\circ x\pi^* p^\circ y, \quad p^\circ \pi x^{\circ\circ} x^\circ \pi y^{\circ\circ} y^\circ,$$

$$q^\circ y\pi^* q^\circ z, \quad q^\circ \pi y^{\circ\circ} y^\circ \pi z^{\circ\circ} z^\circ,$$

hence $q^\circ p^\circ x\pi^* q^\circ p^\circ z$ and $q^\circ p^\circ \pi y^{\circ\circ} y^\circ \pi x^{\circ\circ} x^\circ \pi z^{\circ\circ} z^\circ$. Therefore $\nu_L(\pi)$ is transitive, and so $\nu_L(\pi)$ is an equivalence relation on L. Assume $x\nu_L(\pi)y$ for $x, y \in L$. Then $p^\circ x\pi^* p^\circ y$ for some $p^\circ \in E^\circ$

such that $p^\circ \pi x^{\circ\circ} x^\circ \pi y^{\circ\circ} y^\circ$. For any $z \in L$, we have $z^{\circ\circ} p^\circ z^\circ \pi z^{\circ\circ} x^{\circ\circ} x^\circ z^\circ \pi z^{\circ\circ} y^{\circ\circ} y^\circ z^\circ$. Since L is an orthodox semigroup with Q-inverse transversal S°, by (1.4) we have

$$\begin{aligned} z^{\circ\circ} p^\circ z^\circ . zx &= z^{\circ\circ} p^\circ z^\circ z x^{\circ\circ} x^\circ x = z^{\circ\circ} p^\circ z^\circ z^{\circ\circ} x^{\circ\circ} x^\circ x \\ &= z^{\circ\circ} z^\circ z^{\circ\circ} p^\circ x \pi^* z^{\circ\circ} z^\circ z^{\circ\circ} p^\circ y = z^{\circ\circ} p^\circ z^\circ z^{\circ\circ} y = z^{\circ\circ} p^\circ z^\circ . zy, \end{aligned}$$

$$z^{\circ\circ} p^\circ z^\circ \pi (zx)^{\circ\circ} (zx)^\circ \pi (zy)^{\circ\circ} (zy)^\circ .$$

Thus, by the definition of $\nu_L(\pi)$, $zx \nu_L(\pi) zy$. As $p^\circ \pi x^{\circ\circ} x^\circ \pi y^{\circ\circ} y^\circ$, we have

$$p^\circ x^{\circ\circ} z^{\circ\circ} z^\circ x^\circ \pi x^{\circ\circ} z^{\circ\circ} z^\circ x^\circ = (xz)^{\circ\circ} (xz)^\circ ,$$

$$p^\circ y^{\circ\circ} z^{\circ\circ} z^\circ y^\circ \pi y^{\circ\circ} z^{\circ\circ} z^\circ y^\circ = (yz)^{\circ\circ} (yz)^\circ .$$

Since $p^\circ \in E^\circ$ and $p^\circ x \pi^* p^\circ y$, by (1.4) and (1.6), we have $x^\circ p^\circ = x^\circ p^{\circ\circ} \pi y^\circ p^{\circ\circ} = y^\circ p^\circ$, hence $p^\circ x^{\circ\circ} \pi p^\circ y^{\circ\circ}$. Noticing $x^{\circ\circ} z^{\circ\circ} z^\circ x^\circ \in E^\circ$, we also have

$$\begin{aligned} p^\circ x^{\circ\circ} z^{\circ\circ} z^\circ x^\circ &= p^\circ p^\circ x^{\circ\circ} z^{\circ\circ} z^\circ x^\circ \\ &= p^\circ x^{\circ\circ} z^{\circ\circ} z^\circ x^\circ p^\circ \pi p^\circ y^{\circ\circ} z^{\circ\circ} z^\circ y^\circ p^\circ = p^\circ y^{\circ\circ} z^{\circ\circ} z^\circ y^\circ . \end{aligned}$$

Hence, by the proof above and the definition of π^*,

$$p^\circ x^{\circ\circ} z^{\circ\circ} z^\circ x^\circ xz = p^\circ x^{\circ\circ} z^{\circ\circ} z^\circ x^\circ x z^{\circ\circ} z^\circ z = p^\circ x^{\circ\circ} z^{\circ\circ} z^\circ x^\circ x^{\circ\circ} z^{\circ\circ} z^\circ z = p^\circ x^{\circ\circ} z$$

$$\pi^* p^\circ y^{\circ\circ} z = p^\circ y^{\circ\circ} z^{\circ\circ} z^\circ y^\circ yz = p^\circ y^{\circ\circ} z^{\circ\circ} z^\circ y^\circ p^\circ yz \pi^* p^\circ x^{\circ\circ} z^{\circ\circ} z^\circ x^\circ p^\circ yz = p^\circ x^{\circ\circ} z^{\circ\circ} z^\circ x^\circ yz,$$

$$p^\circ x^{\circ\circ} z^{\circ\circ} z^\circ x^\circ \pi (xz)^{\circ\circ} (xz)^\circ \pi (yz)^{\circ\circ} (yz)^\circ .$$

Thus $xz \nu_L(\pi) yz$. Therefore $\nu_L(\pi)$ is a congruence on L. Assume $x \nu_L(\pi) y$ for $x, y \in S^\circ$. Then $p^\circ x \pi^* p^\circ y$ for some $p^\circ \in E^\circ$ such that $p^\circ \pi x^{\circ\circ} x^\circ \pi y^{\circ\circ} y^\circ$. Thus $x \pi p^\circ x \pi^* p^\circ y \pi y$, and so $\nu_L(\pi)|_{S^\circ} \subseteq \pi$. For $x, y \in S^\circ$ with $x \pi y$, we have $x^\circ \pi y^\circ$. Thus $x^{\circ\circ} x^\circ = x x^\circ \pi y y^\circ = y^{\circ\circ} y^\circ$, and so $x^{\circ\circ} x^\circ y^{\circ\circ} y^\circ \pi y^{\circ\circ} y^\circ$. Noticing $\pi^*|_{S^\circ} = \pi$ and $x \pi y$, we also have

$$x^{\circ\circ} x^\circ y^{\circ\circ} y^\circ . x \pi^* x^{\circ\circ} x^\circ y^{\circ\circ} y^\circ . y,$$

$$x^{\circ\circ} x^\circ y^{\circ\circ} y^\circ \pi x^{\circ\circ} x^\circ \pi y^{\circ\circ} y^\circ ,$$

and thus $x \nu_L(\pi) y$. Hence $\pi \subseteq \nu_L(\pi)|_{S^\circ}$, so $\pi = \nu_L(\pi)|_{S^\circ}$. Let $\pi = \rho_n^\circ$. Then $\nu_L(\rho_n^\circ)|_{S^\circ} = \rho_n^\circ$. We need to prove that $(\rho_I)|_{E^\circ} = \rho_n^\circ|_{E^\circ}$. For $e^\circ, f^\circ \in E^\circ$ with $e^\circ \rho_I f^\circ$, by the definition of ρ_n° we have $e^\circ = e^\circ e^\circ \beta e^\circ f^\circ = f^\circ e^\circ \beta f^\circ f^\circ = f^\circ$. So $e^\circ \rho_n^\circ f^\circ$. Thus $(\rho_I)|_{E^\circ} \subseteq \rho_n^\circ|_{E^\circ}$. To get the reverse inclusion, assume $e^\circ \rho_n^\circ f^\circ$ for $e^\circ, f^\circ \in E^\circ$. Then there exist $e_i, f_i \in I, x_i \in L, s_i, t_i \in S^{o1}$ $(i = 1, 2, \ldots, n)$ such that

$$e^\circ = s_1 x_1 e_1 t_1, \quad s_1 x_1 f_1 t_1 = s_2 x_2 e_2 t_2, \quad \ldots \quad , \quad s_n x_n f_n t_n = f^\circ, \quad e_i \rho_I f_i .$$

By $e_i \rho_I f_i$, we get $e_i \rho f_i$, and thus $e^\circ \rho f^\circ$. Hence $e^\circ (\rho_I)|_{E^\circ} f^\circ$, and so $\rho_n^\circ|_{E^\circ} \subseteq (\rho_I)|_{E^\circ}$. Thus

$$\nu_L(\rho_n^\circ)|_{E^\circ} = \rho_n^\circ|_{E^\circ} = (\rho_I)|_{E^\circ} .$$

Therefore $(\rho_I, \nu_L(\rho_n^\circ))$ satisfies Definition 1.1(i). For $e, f \in I, x \in L$ with $e \rho_I f$, by the definition of ρ_n°, we have $xe \rho_n^\circ xf$. By $xe, xf \in S^\circ$ and $\nu_L(\rho_n^\circ)|_{S^\circ} = \rho_n^\circ$, we get $xe \nu_L(\rho_n^\circ) xf$. Assume $x \nu_L(\rho_n^\circ) y$

for $x, y \in L$. Then $q^\circ x(\rho_n^\circ)^* q^\circ y$ for some $q^\circ \in E^\circ$ such that $q^\circ \rho_n^\circ x^{\circ\circ} x^\circ \rho_n^\circ y^{\circ\circ} y^\circ$. Thus

$$x(\rho_n^\circ)^* q^\circ x(\rho_n^\circ)^* q^\circ y(\rho_n^\circ)^* y,$$

and so $x(\rho_n^\circ)^* y$. For any $e \in I$, we have $xe(\rho_n^\circ)^* ye$. By $xe, ye \in S^\circ$ and $(\rho_n^\circ)^*|_{S^\circ} = \rho_n^\circ = \nu_L(\rho_n^\circ)|_{S^\circ}$, we get $xe\nu_L(\rho_n^\circ)ye$. Therefore $(\rho_I, \nu_L(\rho_n^\circ))$ is a congruence pair such that $\rho_{(\rho_I, \nu_L(\rho_n^\circ))} \in \rho T_r$. For $a, b \in S^\circ$ with $a\rho_n^\circ b$, there exist $e_i, f_i \in I, x_i \in L, s_i, t_i \in S^{o1}$ $(i = 1, 2, \ldots, n)$ such that

$$a = s_1 x_1 e_1 t_1, \quad s_1 x_1 f_1 t_1 = s_2 x_2 e_2 t_2, \quad \ldots \quad , \quad s_n x_n f_n t_n = b, \quad e_i \rho_I f_i.$$

Hence $a\rho b$, so $a\rho^\circ b$. Therefore $\rho_n^\circ \subseteq \rho^\circ$. Thus, by the definition of $\nu_L(\pi), \nu_L(\rho_n^\circ) \subseteq \nu_L(\rho^\circ) \subseteq \rho_L$. Hence $\rho_{(\rho_I, \nu_L(\rho_n^\circ))} \subseteq \rho_{(\rho_I, \rho_L)} = \rho$. Since the definition of $\nu_L(\rho_n^\circ)$ depends only on ρ_n° and the definition of ρ_n° depends only on ρ_I, $\rho_{(\rho_I, \nu_L(\rho_n^\circ))}$ is the smallest element of ρT_r.

From Theorem III. 2.5 in [2], $\rho^{\circ t}$ is the greatest congruence in $\mathrm{Con}(S^\circ)$ such that $\rho^{\circ t}|_{E^\circ} = \rho^\circ|_{E^\circ}$. Let π be a congruence on S°. $\xi_L(\pi)$ is also an equivalence relation on L. Let $x\xi_L(\pi)y$. Then for any $e \in I, xe\pi ye$, and for any $z \in L$, by Lemma 2.4, we have

$$zxe = zxee^\circ \pi zyee^\circ = zye$$

so $zx\xi_L(\pi)zy$. Noticing $xz^{\circ\circ}z^\circ \pi yz^{\circ\circ}z^\circ$ and $ze \in S^\circ$, we have

$$xze = xz^{\circ\circ}z^\circ ze\pi yz^{\circ\circ}z^\circ ze = yze,$$

and thus $xz\xi_L(\pi)yz$. Hence $\xi_L(\pi)$ is also a congruence on L.

If for $x^\circ, y^\circ \in S^\circ, x^\circ \pi y^\circ$, then for any $e \in I$, by Lemma 2.4, we have

$$x^\circ e = x^\circ x^{\circ\circ} x^\circ e\pi x^\circ x^{\circ\circ} y^\circ e = x^\circ x^{\circ\circ} y^\circ y^{\circ\circ} y^\circ e = y^\circ y^{\circ\circ} x^\circ x^{\circ\circ} y^\circ e\pi y^\circ y^{\circ\circ} y^\circ y^{\circ\circ} y^\circ e = y^\circ e.$$

So $x^\circ \xi_L(\pi) y^\circ$. Thus $\pi \subseteq (\xi_L(\pi))|_{S^\circ}$. Conversely, if $x^\circ(\xi_L(\pi))|_{S^\circ} y^\circ$, then $x^{\circ\circ}(\xi_L(\pi))|_{S^\circ} y^{\circ\circ}$, and so $x^{\circ\circ}x^\circ(\xi_L(\pi))|_{S^\circ} y^{\circ\circ}y^\circ$. Thus, by the definition of $\xi_L(\pi)$,

$$x^\circ = x^\circ x^{\circ\circ} x^\circ \pi y^\circ x^{\circ\circ} x^\circ = y^\circ y^{\circ\circ} y^\circ x^{\circ\circ} x^\circ = y^\circ x^{\circ\circ} x^\circ y^{\circ\circ} y^\circ \pi y^\circ y^{\circ\circ} y^\circ y^{\circ\circ} y^\circ = y^\circ.$$

Hence $(\xi_L(\pi))|_{S^\circ} \subseteq \pi$. Consequently $(\xi_L(\pi))|_{S^\circ} = \pi$. Let $\pi = \rho^{\circ t}$. Then

$$\xi_L(\rho^{\circ t})|_{E^\circ} = \rho^{\circ t}|_{E^\circ} = \rho^\circ|_{E^\circ} = \rho_I|_{E^\circ}.$$

For $e, f \in I, e\rho_I f$, then for any $x \in L$, we have $xe\rho^\circ xf$. As $\rho^\circ \subseteq \rho^{\circ t}$ and $\xi_L(\rho^{\circ t})|_{S^\circ} = \rho^{\circ t}$, we have $xe\xi_L(\rho^{\circ t})xf$. Assume $x\xi_L(\rho^{\circ t})y$ for $x, y \in L$. Then for any $e \in I$, by the definition of $\xi_L(\rho^{\circ t})$, $xe\rho^{\circ t}ye$. Since $xe, ye \in S^\circ$ and $\xi_L(\rho^{\circ t})|_{S^\circ} = \rho^{\circ t}$, we have $xe\xi_L(\rho^{\circ t})ye$. Therefore $(\rho_I, \xi_L(\rho^{\circ t}))$ is a congruence pair and $\rho_{(\rho_I,\, \xi_L(\rho^{\circ t}))} \in \rho T_r$. By the definition of $\xi_L(\pi)$, we have $\rho_L \subseteq \xi_L(\rho^\circ) \subseteq \xi_L(\rho^{\circ t})$ and

$$\rho = \rho_{(\rho_I,\, \rho_L)} \subseteq \rho_{(\rho_I,\, \xi_L(\rho^{\circ t}))}.$$

Since the definition of $\xi_L(\rho^{\circ t})$ depends only on $\rho^{\circ t}$ and the definition of $\rho^{\circ t}$ depends only on $(\rho_I)|_{E^\circ}$, $\rho_{(\rho_I,\, \xi_L(\rho^{\circ t}))}$ is the greatest element of ρT_r.

With respect to any congruence ρ on S, there exist two congruence classes containing ρ, and there are four extremal values related to these congruence classes. Further, we describe the fine relations among these extremal congruences for a fixed congruence on S.

Theorem 3.2 *Let ρ be a congruence on S. Then $T_r \cap Q = \epsilon_{\mathrm{Con}(S)}$, $\rho = \rho_{T_r} \vee \rho_Q = \rho^{T_r} \cap \rho^Q$, $(\rho^{T_r})^{T_r} = \rho^{T_r}$, $(\rho_{T_r})_{T_r} = \rho_{T_r}$, $(\rho^{T_r})_{T_r} = \rho_{T_r}$, $(\rho_{T_r})^{T_r} = \rho^{T_r}$, $(\rho^Q)^Q = \rho^Q$, $(\rho_Q)_Q = \rho_Q$, $(\rho^Q)_Q = \rho_Q$, $(\rho_Q)^Q = \rho^Q$.*

Proof Only $\rho = \rho_{T_r} \vee \rho_Q = \rho^{T_r} \cap \rho^Q$ and $(\rho^{T_r})^{T_r} = \rho^{T_r}$ are proved below.

$\nu_L(\rho_n^\circ) \subseteq \rho_L$ has been shown in the proof of Theorem 3.1. Combining $\mu_I(\rho^\circ) \subseteq \rho_I$, Lemma 2.2, Theorems 2.1 and 3.1, we have $\rho = \rho_{T_r} \vee \rho_Q$. Similarly, $\rho_I \subseteq \xi_I(\rho^\circ)$, $\rho_L \subseteq \xi_L(\rho^{\circ t})$, and then $\rho = \rho^{T_r} \cap \rho^Q$. Finally, $(\rho^{T_r})^{T_r} = \rho_{(\rho_I, \xi_L((\rho^{\circ t})^{\circ t}))} = \rho^{T_r}$ from the fact that $(\rho^{T_r})_I = \rho_I, \xi_L(\rho^{\circ t})|_{S^\circ} = \rho^{\circ t}$ and $(\rho^{\circ t})^{\circ t} = \rho^{\circ t}$.

References

[1] HOWIE J M. *An Introduction to Semigroup Theory* [M]. Academic Press, London-New York, 1976.

[2] PETRICH M. *Inverse Semigroups* [M]. John Wiley & Sons, Inc., New York, 1984.

[3] MCALISTER D B, MCFADDEN R B. *Regular semigroups with inverse transversals* [J]. Quart. J. Math. Oxford Ser. (2), 1983, **34**(136): 459–474.

[4] PASTIJN F, PETRICH M. *Congruences on regular semigroups* [J]. Trans. Amer. Math. Soc., 1986, **295**(2): 607–633.

[5] SAITO T. *A note on regular semigroups with inverse transversals* [J]. Semigroup Forum, 1986, **33**(1): 149–152.

[6] SAITO T. *Construction of regular semigroups with inverse transversals* [J]. Proc. Edinburgh Math. Soc. (2), 1989, **32**(1): 41–51.

[7] SAITO T. *Structure of regular semigroups with a quasi-ideal inverse transversal* [J]. Semigroup Forum, 1985, **31**(3): 305–309.

[8] TANG Xilin. *Regular semigroups with inverse transversals* [J]. Semigroup Forum, 1997, **55**(1): 24–32.

[9] WANG Limin. *On congruence lattices of regular semigroups with Q-inverse transversals* [J]. Semigroup Forum, 1995, **50**(2): 141–160.

[10] SHANG Yu, WANG Limin. *Congruences on regular semigroups with Q-inverse transversals* [J]. J. Lanzhou Univ. Nat. Sci., 2007, **43**(5): 84–87. (in Chinese)