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Inequalities Relative to Vilenkin-Like System

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Abstract For bounded Vilenkin-Like system, the inequality is also true:

$$(\sum_{k=1}^{\infty} k^{p-2}|\hat{f}(k)|^p)^{1/p} \le C\|f\|_{H_p}, \quad 0 < p \le 2, \tag{*}$$

where $\hat{f}(\cdot)$ denotes the Vilenkin-Like Fourier coefficient of f and the Hardy space $H_p(G_m)$ is defined by means of maximal functions. As a consequence, we prove the strong convergence theorem for bounded Vilenkin-Like Fourier series, i.e.,

$$(\sum_{k=1}^{\infty} k^{p-2}\|S_k f\|_p^p)^{1/p} \le C\|f\|_{H_p}, \quad 0 < p < 1. \tag{**}$$

Keywords Hardy space; Vilenkin-Like systems; strong convergence.

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1. Introduction

It is well known that the inequality $(*)$ is true for Walsh-Paley system. It was proved first by Ladhawala[1] and another proof was given in the book[2] written by Schipp, Wade, Simon and Pál. For Vilenkin system, it was proved by Fridli and Simon[3]. In this paper, we will discuss the theorem about Vilenkin-Like system. In fact Vilenkin-Like system is a more generalized orthonormal system in Vilenkin space G_m. It has the corresponding definition in Walsh-Paley system, p-series Field and Vilenkin system even in noncommutative martingale theory. We will prove the inequality $(*)$ is also true for the bounded Vilenkin-Like system.

It is well known that Vilenkin system, especially Walsh-Paley system, does not form a Schauder basis in L_1. Moreover, there exists a function in H_1 such that its partial sums are not bounded in L_1. Hence it is of interest that certain means of the partial sums of function from H_1 can be convergent. Simon[4] proved that in the Walsh case

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f\|_1}{k} = \|f\|_1, \quad f \in H_1. \tag{1}$$

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Furthermore, it was proved that (1) follows from the next statement on strong convergence:

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|S_kf-f\|_1}{k}=0,\quad f\in H_1. \tag{2}$$

It is not hard to see that (1) is also equivalent to (2). Moreover for (1) it is enough to show that

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|S_kf\|_1}{k}\le C\|f\|_1,\quad f\in H_1. \tag{3}$$

The Vilenkin analogue of (1)–(3) can be found in Gát[5]. In Weisz[6] a certain extension of (3) to H_p $(0<p\le 1)$ space was given with respect to Walsh system. As a consequence of inequality $(*)$, we prove the strong convergence theorem for bounded Vilenkin-Like Fourier series. The result is a generalization for Walsh-Paley system[4], even more for Vilenkin system[5].

2. Definitions and notation

We denote by $\mathbb{N}$ the set of nonnegative integers and $\mathbb{P}$ the set of positive integers. Let $m:=(m_0,m_1,\ldots,m_k,\ldots)$ be sequence of natural numbers such that $m_k\ge 2$ $(k\in\mathbb{N})$. For all $k\in\mathbb{N}$ we denote by Z_{m_k} the m_k-th discrete cyclic group. Let Z_{m_k} be represented by $\{0,1,\ldots,m_k-1\}$. Suppose that each (coordinate) set has the discrete topology and the measure μ_k which maps ever singleton of Z_{m_k} to $1/m_k$ $(u_k(Z_{m_k})=1)$ for $k\in\mathbb{N}$. Let G_m denote the complete direct product of Z'_{m_k}s equipped with product topology and product measure μ. Then G_m forms a compact Abelian group with Haar measure 1. The elements of G_m are sequences of the form $(x_0,x_1,\ldots,x_k,\ldots)$, where $x_k\in Z_{m_k}$ for every $k\in\mathbb{N}$ and the topology of the group G_m is completely determined by the sets

$$I_n(0):=\{(x_0,x_1,\ldots,x_k,\ldots)\in G_m: x_k=0\ (k=0,\ldots,n-1)\}$$

$(I_0(0):=G_m)$. Let $I_n(x):=I_n(0)+x$ $(n\in\mathbb{N})$. The Vilenkin space G_m is said to be bounded if the generating system m is bounded. Throughout this paper we assume m is bounded.

Let $M_0:=1$ and $M_{k+1}:=m_kM_k$ for $k\in\mathbb{N}$, it is so-called the generalized powers. Then every $n\in\mathbb{N}$ can be uniquely expressed as $n=\sum_{k=0}^{\infty}n_kM_k, 0\le n_k<m_k, n_k\in\mathbb{N}$. The sequence $(n_0,n_1,\ldots)$ is called the expansion of n with respect to m. We often use the following notations: $|n|:=\max\{k\in\mathbb{N}: n_k\ne 0\}$ (that is, $M_{|n|}\le n<M_{|n|+1}$) and $n^{(k)}=\sum_{j=k}^{\infty}n_jM_j$. Next we introduce an orthonormal system on G_m which we call a Vilenkin-Like system.

A complex-valued function $r_k^n: G_m\longrightarrow C$ is called a generalized Rademacher function if it has the following properties:

(i) r_k^n is Σ_{k+1}-measurable (i.e., r_k^n depends only on $x_0,x_1,\ldots,x_k(x\in G_m)$), for all $k,n\in\mathbb{N}$, and $r_k^0=1$.

(ii) If M_k is a divisor of n and l and $n^{(k+1)}=l^{(k+1)}$ $(k,l,n\in\mathbb{N})$, then

$$E_k(r_k^n\bar{r}_k^l)=\begin{cases}1, & \text{if } n_k=l_k,\\ 0, & \text{if } n_k\ne l_k,\end{cases}$$

where E_k is the conditional expectation with respect to Σ_k and $\bar{z}$ is the complex conjugate of z.

(iii) If M_{k+1} is a divisor of n (that is, $n = n_{k+1}M_{k+1} + \cdots + n_{|n|}M_{|n|}$), then

$$\sum_{j=0}^{m_k-1} |r_k^{jM_k+n}(x)|^2 = m_k$$

for all $x \in G_m$.

(iv) There exists a $\delta > 1$ for which $\|r_k^n\|_\infty \leq \sqrt{m_k/\delta}$.

Define Vilenkin-Like systems $\psi = (\psi_n : n \in \mathbb{N})$ as follows:

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}}, \quad n \in \mathbb{N}.$$

(Since $r_k^0 = 1$, we have $\psi_n = \prod_{k=0}^{|n|} r_k^{n^{(k)}}$.)

If $f \in L_1(G_m)$, the maximal function can also be given by

$$f^* = \sup_n |I_n(x)|^{-1} |\int_{I_n(x)} f(t)\mathrm{d}\mu(t)|,$$

where the supremum is taken over all intervals I containing $x \in G_m$.

The martingale Hardy space $H_p(G_m)$ for $0 < p \leq \infty$ is the space of martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

A measurable function a is called a p-atom, if a is identically equal to 1 or there exists an interval I such that

1) $\int_I a\mathrm{d}\mu = 0$;

2) $\|a\|_\infty \leq \mu(I)^{-\frac{1}{p}}, \quad 0 < p \leq q, 1 < q \leq \infty$;

3) $\operatorname{supp} a \subset I$.

For $f \in L_1(G_m)$, we define the Fourier coefficients and partial sums by

$$\hat{f}(k) := \int_{G_m} f\bar{\psi}_k \mathrm{d}\mu, \quad k \in \mathbb{N},$$

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k, \quad n \in \mathbb{P}, S_0 f := 0$$

and the Dirichlet kernels by:

$$D_n(y,x) := \sum_{k=0}^{n-1} \psi_k(y)\bar{\psi}_k(x), \quad n \in \mathbb{P}, D_0 := 0.$$

It is clear that

$$S_n f(y) = \int_{G_m} f(x)D_n(y,x)\mathrm{d}\mu(x).$$

3. Formulation of main results

Our main results in this paper are as follows:

Theorem 1 *There exists an absolute constant $C > 0$ such that for any $f \in H_p(G_m)$ $(0 < p \leq 2)$,* *we have*

$$(\sum_{k=1}^{\infty} k^{p-2}|\hat{f}(k)|^p)^{1/p} \le C\|f\|_{H_p}.$$

Theorem 2 *There exists an absolute constant $C > 0$ such that for any $f \in H_p(G_m)$ $(0 < p < 1)$, we have*

$$(\sum_{k=1}^{\infty} k^{p-2}\|S_k f\|_p^p)^{1/p} \le C\|f\|_{H_p}.$$

The results as above are based on the following lemmas.

Lemma 1[8]

$$D_{M_n}(y,x) = \begin{cases} M_n, & \text{if } y \in I_n(x) \\ 0, & \text{if } y \in G_m \backslash I_n(x). \end{cases} \tag{4}$$

Set $\psi_{k,n} := \prod_{s=n}^{\infty} r_s^{k^{(s)}}$, *we have*

Lemma 2[8] *Let $x, y \in G_m, n \in N$. Then*

$$D_n(y,x) = \sum_{s=0}^{\infty} \psi_{n,s+1}(y)\bar{\psi}_{n,s+1}(x) D_{M_s}(y,x) \sum_{j=0}^{n_s-1} r_s^{n^{(s+1)}+jM_s}(y)\bar{r}_s^{n^{(s+1)}+jM_s}(x). \tag{5}$$

Lemma 3[9] *If $f \in H_p(G_m)$ $(0 < p \le 1)$, then there exist sequences $\{\lambda_j\}$ (of positive numbers) and $\{a_j\}$ (of p-atom), such that*

$$f = \sum_1^{\infty} \lambda_j a_j \quad \text{in } H_p \text{ norm and pointwise and} \quad \|f\|_{H_p}^p \sim \sum_1^{\infty} \lambda_j^p.$$

4. Proofs of the results

Proof of Theorem 1 (1) First suppose that $0 < p \le 1$. Since $f \in H_p(G_m)$, by Lemma 3, we have $f = \sum_1^{\infty} \lambda_j a_j$, where a_j is p-atoms and $\sum_1^{\infty} \lambda_j^p < \infty$. So,

$$\sum_{k=1}^{\infty} k^{p-2}|\hat{f}(k)|^p = \sum_{k=1}^{\infty} k^{p-2}|\sum_{j=1}^{\infty} \lambda_j \hat{a}_j(k)|^p \le \sum_{j=1}^{\infty} \lambda_j^p \sum_{k=1}^{\infty} k^{p-2}|\hat{a}_j(k)|^p,$$

that is the reason why it suffices to show that there exists an absolute constant $C > 0$ such that for all p-atoms

$$\sum_{k=1}^{\infty} k^{p-2}|\hat{a}(k)|^p \le C.$$

Let a be an arbitrary p-atom. If $a \equiv 1$, then

$$\begin{aligned} \hat{a}(k) &= \int_{G_m} \bar{\psi}_k(x)\mathrm{d}\mu(x) = E_0(\bar{\psi}_k) = E_0(\prod_{j=1}^{|k|} \bar{r}_j^{k^{(j)}}) \\ &= E_0(E_{|k|}(\prod_{j=1}^{|k|} \bar{r}_j^{k^{(j)}})) = E_0(\prod_{j=1}^{|k|-1} E_{|k|}(r_{|k|}^0 \bar{r}_{|k|}^{k^{(|k|)}})) \\ &= 0, \end{aligned}$$

because $k^{(|k|)} = k_{|k|}M_{|k|} \neq 0$ if $k \in \mathbb{P}$ and $E_k(r_k^n \bar{r}_k^l) = 0$ if $n_k \neq l_k$. In this case the statement of the theorem is trivial.

Suppose a is a p-atom with support $I_N(u)$ for some N and $u \in G_m$. We have

$$\hat{a}(k) = \int_{G_m} a(x)\bar{\psi}_k(x)\mathrm{d}\mu(x) = \int_{I_N(u)} a(x)\bar{\psi}_k(x)\mathrm{d}\mu(x).$$

For $k = 0, \ldots, M_N - 1$, $\psi_k(x)$ depends only on the first N coordinates of x, hence the function $\psi_k(x)$ on the set $I_N(u)$ is invariable

$$\hat{a}(k) = \int_{G_m} a(x)\bar{\psi}_k(x)\mathrm{d}\mu(x) = c\int_{I_N(u)} a(x)\mathrm{d}\mu(x) = 0$$
$$\Rightarrow \sum_{k=1}^{\infty} k^{p-2}|\hat{a}(k)|^p = \sum_{k=M_N}^{\infty} k^{p-2}|\hat{a}(k)|^p.$$

Using the Cauchy-Buniakovski-Schwarz inequality

$$\begin{aligned}\sum_{k=M_N}^{\infty} k^{p-2}|\hat{a}(k)|^p &\leq (\sum_{k=M_N}^{\infty} k^{(p-2)\alpha})^{\frac{1}{\alpha}}(\sum_{k=M_N}^{\infty} |\hat{a}(k)|^{p\beta})^{\frac{1}{\beta}} \\ &= (\sum_{k=M_N}^{\infty} k^{(p-2)\frac{2}{2-p}})^{\frac{2-p}{2}}(\sum_{k=M_N}^{\infty} |\hat{a}(k)|^2)^{\frac{p}{2}} \\ &= (\sum_{k=M_N}^{\infty} k^{-2})^{\frac{2-p}{2}}(\sum_{k=M_N}^{\infty} |\hat{a}(k)|^2)^{\frac{p}{2}} \\ &\leq (\frac{C}{\sqrt{M_N}})^{2-p}(\sum_{k=M_N}^{\infty} |\hat{a}(k)|^2)^{\frac{p}{2}}\end{aligned}$$

where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $\beta \cdot p = 2$ and Bessel's inequality

$$(\sum_{k=M_N}^{\infty} |\hat{a}(k)|^2)^{\frac{1}{2}} \leq \|a\|_2,$$

we get

$$\begin{aligned}\sum_{k=M_N}^{\infty} k^{p-2}|\hat{a}(k)|^p &\leq (\frac{C}{\sqrt{M_N}})^{2-p}(\int_{I_N(u)} |a(x)|^2\mathrm{d}\mu(x))^{\frac{p}{2}} \\ &\leq (\frac{C}{\sqrt{M_N}})^{2-p}\|a\|_\infty^p \mu(I_N)^{\frac{p}{2}} \\ &\leq (\frac{C}{\sqrt{M_N}})^{2-p}(\mu(I_N))^{-1+\frac{p}{2}} \\ &\leq (\frac{C}{M_N})^{\frac{2-p}{2}} M_N^{-1+\frac{p}{2}} \\ &\leq C.\end{aligned}$$

(2) Secondly let $1 < p \leq 2$. Introduce on $\mathbb{P}$ the measure $\eta(n) := 1/n^2$. If

$$Tf(n) = n\hat{f}(n),$$

then it follows from Parseval's formula and from the previous theorem (for $p = 1$) that both operators

$$T : L_2 \to L_2(\mathbb{P}, \eta) \text{ and } T : H_1 \to L_1(\mathbb{P}, \eta)$$

are bounded. By the Marcinkiewicz interpolation theorem, the operator

$$T : (H_1, L_2)_{\theta,p} \to (L_1(\mathbb{P}, \eta), L_2(\mathbb{P}, \eta))_{\theta,p}$$

is bounded where $0 < \theta < 1$ and $1/p = (1-\theta) + \theta/2$. That is to say the operator T is bounded from H_p to $L_p(\mathbb{P}, \eta)$. Thus we complete the proof of Theorem 1. □

Proof of Theorem 2 Let us estimate the sum in the theorem as follows:

$$\begin{aligned}\sum_{k=1}^{\infty} k^{p-2}\|S_k f\|_p^p &= \sum_{n=0}^{\infty}\sum_{j=1}^{m_n-1}\sum_{k=jM_n}^{(j+1)M_n-1}\frac{\|S_k f\|_p^p}{k^{2-p}} \\ &\le \sum_{n=0}^{\infty}\sum_{j=1}^{m_n-1}\frac{1}{(jM_n)^{2-p}}\sum_{k=jM_n}^{(j+1)M_n-1}\|S_k f\|_p^p.\end{aligned}$$

By Lemma 3, it is enough to prove that for all p-atom we have

$$\sum_{n=0}^{\infty}\sum_{j=1}^{m_n-1}\frac{1}{(jM_n)^{2-p}}\sum_{k=jM_n}^{(j+1)M_n-1}\|S_k a\|_p^p \le C. \tag{6}$$

If $a \equiv 1$, similar to the proof of Theorem 1, we have $\hat{a}(k) = 0$ for all $k \in \mathbb{N}$, i.e., $S_k a = \sum_{i=0}^{k-1}\hat{a}(i)\psi_i = 0$. In this case the statement of the theorem is trivial.

So, assume a is an arbitrary atom with support $I_N(u)$ for some N and $u \in G_m$. For $k = 0, \ldots, M_N - 1$, $\psi_k(x)$ depends only on the first N coordinates of x, hence the function $\psi_k(x)$ on the set $I_N(u)$ is invariable

$$\hat{a}(k) = \int_{G_m} a(x)\bar{\psi}_k(x)\mathrm{d}\mu(x) = c\int_{I_N(u)} a(x)\mathrm{d}\mu(x) = 0.$$

This means that we need to show the inequality

$$\sum_{n=N}^{\infty}\sum_{j=1}^{m_n-1}\frac{1}{(jM_n)^{2-p}}\sum_{k=jM_n}^{(j+1)M_n-1}\|S_k a\|_p^p \le C_p. \tag{7}$$

For this purpose let $\|S_k a\|_p^p$ $(k = M_N, M_N + 1, \ldots)$ be decomposed in the following way:

$$\|S_k a\|_p^p = \int_{I_N(u)} |S_k a(y)|^p \mathrm{d}\mu(y) + \int_{G_m \setminus I_N(u)} |S_k a(y)|^p \mathrm{d}\mu(y). \tag{8}$$

Applying Holder's and Parseval's inequalities, we get the estimation:

$$\begin{aligned}\int_{I_N(u)} |S_k a(y)|^p \mathrm{d}\mu(y) &\le (\int_{I_N(u)} |S_k a(y)|^2 \mathrm{d}\mu(y))^{p/2}\mu(I_N)^{1-p/2} \\ &\le \|a\|_2^p \mu(I_N)^{1-p/2} \\ &\le \|a\|_\infty^p \mu(I_N)^{p/2}\mu(I_N)^{1-p/2} \\ &\le 1.\end{aligned}$$

To estimate the second integral in (8), let $\mathbb{N} \ni k \geq M_N$. By Lemmas 1 and 2, we get

$$
\begin{aligned}
&\int_{G_m \setminus I_N(u)} |S_k a(y)|^p \mathrm{d}\mu(y) \\
&= \sum_{i=0}^{N-1} \int_{I_i(u)\setminus I_{i+1}(u)} |S_k a(y)|^p \mathrm{d}\mu(y) \\
&= \sum_{i=0}^{N-1} \int_{I_i(u)\setminus I_{i+1}(u)} |\int_{I_N(u)} a(x) D_k(y,x) \mathrm{d}\mu(x)|^p \mathrm{d}\mu(y) \\
&= \sum_{i=0}^{N-1} \int_{I_i(u)\setminus I_{i+1}(u)} |\int_{I_N(u)} a(x)[\sum_{s=0}^{i-1} M_s \prod_{l=s+1}^{i-1} |r_l^{k^{(l)}}(x)|^2 \psi_{k,i}(y)\bar{\psi}_{k,i}(x) \cdot \\
&\quad \sum_{j=0}^{k_s-1} |r_s^{k^{(s+1)}+jM_s}(x)|^2 + M_i \psi_{k,i+1}(y)\bar{\psi}_{k,i+1}(x) \cdot \\
&\quad \sum_{j=0}^{k_i-1} r_i^{k^{(i+1)}+jM_i}(y) \bar{r}_i^{k^{(i+1)}+jM_i}(x)] \mathrm{d}\mu(x)| \mathrm{d}\mu(y).
\end{aligned}
$$

From the definition of generalized Rademacher functions and Jensen inequality, we have

$$
\begin{aligned}
&\int_{G_m \setminus I_N(u)} |S_k a(y)|^p \mathrm{d}\mu(y) \\
&\leq \sum_{i=0}^{N-1} \int_{I_i(u)\setminus I_{i+1}(u)} (|\hat{a}(k)| \sum_{s=0}^{i-1} M_s \prod_{l=s+1}^{i-1} \frac{m_l}{\delta} |\psi_{k,i}(y)| k_s \frac{m_s}{\delta} + M_i |\psi_{k,i+1}(y)| k_i \frac{m_i}{\delta})^p \mathrm{d}\mu(y) \\
&\leq C|\hat{a}(k)|^p \sum_{i=0}^{N-1} \int_{I_i(u)\setminus I_{i+1}(u)} |\psi_{k,i}(y)|^p (\sum_{s=0}^{i-1} M_s \frac{M_i}{M_s \delta^{i-s}} + M_i)^p \mathrm{d}\mu(y) \\
&\leq C|\hat{a}(k)|^p M_i^p \sum_{i=0}^{N-1} \int_{I_i(u)\setminus I_{i+1}(u)} |\psi_{k,i}(y)|^p \mathrm{d}\mu(y) \\
&= C|\hat{a}(k)|^p \sum_{i=0}^{N-1} \int_{I_i(u)\setminus I_{i+1}(u)} M_i^p E_{i+1}(|\psi_{k,i}|^p) \mathrm{d}\mu(y) \\
&\leq C|\hat{a}(k)|^p \sum_{i=0}^{N-1} \int_{I_i(u)\setminus I_{i+1}(u)} M_i^p (E_{i+1}(|\psi_{k,i}|))^p \mathrm{d}\mu(y) \\
&= C|\hat{a}(k)|^p \sum_{i=0}^{N-1} \int_{I_i(u)\setminus I_{i+1}(u)} M_i^p (E_{i+1}(\prod_{s=i}^{|k|} |r_s^{k^{(s)}}|))^p \mathrm{d}\mu(y) \\
&= C|\hat{a}(k)|^p \sum_{i=0}^{N-1} \int_{I_i(u)\setminus I_{i+1}(u)} M_i^p (E_{i+1}(\prod_{s=i}^{|k|-1} |r_s^{k^{(s)}}| E_{|k|}(|r_{|k|}^{k^{(|k|)}}|)))^p \mathrm{d}\mu(y) \\
&= C|\hat{a}(k)|^p \sum_{i=0}^{N-1} \int_{I_i(u)\setminus I_{i+1}(u)} M_i^p (E_{i+1}(\prod_{s=i}^{|k|-1} |r_s^{k^{(s)}}| (E_{|k|}(|r_{|k|}^{k^{(|k|)}}|^2)^{1/2}))^p \mathrm{d}\mu(y) \\
&\leq C|\hat{a}(k)|^p \sum_{i=0}^{N-1} \int_{I_i(u)\setminus I_{i+1}(u)} M_i^p \mathrm{d}\mu(y)
\end{aligned}
$$

$$\leq C|\hat{a}(k)|^p \sum_{i=0}^{N-1} \frac{M_i^p}{M_{i+1}} \leq C|\hat{a}(k)|^p,$$

since the series $\sum_{i=0}^{\infty} \frac{M_i^p}{M_{i+1}}$ converges for $0 < p < 1$.

By Theorem 1, the following inequality is true:

$$\begin{aligned}
&\sum_{n=N}^{\infty} \sum_{j=1}^{m_n-1} \frac{1}{(jM_n)^{2-p}} \sum_{k=jM_n}^{(j+1)M_n-1} \|S_k a\|_p^p \\
&= \sum_{n=N}^{\infty} \sum_{j=1}^{m_n-1} \frac{1}{(jM_n)^{2-p}} \sum_{k=jM_n}^{(j+1)M_n-1} \Big(\int_{I_N(u)} |S_k a(y)|^p \mathrm{d}\mu(y) + \\
&\quad \int_{G_m \setminus I_N(u)} |S_k a(y)|^p \mathrm{d}\mu(y)\Big) \\
&\leq \sum_{n=N}^{\infty} \sum_{j=1}^{m_n-1} \frac{1}{(jM_n)^{2-p}} \sum_{k=jM_n}^{(j+1)M_n-1} 1 + C \sum_{k=M_N}^{\infty} k^{p-2} |\hat{a}(k)|^p \\
&\leq C \sum_{n=N}^{\infty} \frac{1}{M_n^{1-p}} \sum_{j=1}^{\infty} \frac{1}{j^{2-p}} + C \leq C.
\end{aligned}$$

Thus we complete the proof of Theorem 2. □

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