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Generalized Semi-π-Regular Rings

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Abstract In this paper, the concept of right generalized semi-π-regular rings is defined. We prove that these rings are non-trival generalizations of both right GP-injective rings and semi-π-regular rings. Some properties of these rings are studied and some results about generalized semiregular rings and GP-injective rings are extended.

Keywords GP-injective rings; semi-π-regular rings; generalized semiregular rings; generalized semi-π-regular rings.

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1. Introduction

Throughout this paper, the ring R is always associative with identity and all modules are unitary. For any non-empty subset X of R, the right (resp. left) annihilator of X in R will be denoted by $r(X)$ (resp. $l(X)$). The symbols $J(R), Z(R_R), Z({}_RR)$ will denote the Jacobson radical, the right singular ideal and the left singular ideal of R, respectively. See [1–3] for the other undefined concepts and notations.

A module M is said to be AP-injective[4], if for any $a\in R$, there exists an S-submodule X_a of M such that $l_M(r_R(a)) = M_a \oplus X_a$, where $S = \mathrm{End}(M)$. We call R a right AP-injective ring if R_R is an AP-injective module. A ring R is called semiregular[5], if for any $a\in R$, there exists an idempotent $g\in Ra$ such that $a(1-g) \in J(R)$. A ring R is called semi-π-regular[6], if for any $a \in R$, there exist a positive integer n and $e^2 = e \in Ra^n$ such that $a^n(1-e) \in J(R)$. We call a ring R right generalized semiregular[6], if for any $a\in R$, there exist two left ideals P, L of R such that $lr(a) = P \oplus L$, where $P \subseteq Ra$ and $Ra \bigcap L$ is small in R. A ring R is called GP-injective[7], if for any $0 \neq a\in R$, there exists a positive integer n such that $a^n \neq 0$ and $l(r(a^n)) = Ra^n$. From [8, Example 1], we know that GP-injective rings need not be AP-injective, also AP-injective rings need not be GP-injective (See [4, Example 1.5]). Following [6], AP-injective rings and semiregular rings are generalized semiregular, but the converse is not true.

In this paper, we call a ring R right generalized semi-π-regular, if for any $a\in R$, there exists a positive integer n such that $a^n \neq 0$ and $lr(a^n) = P \oplus L$, where P, L are left ideals of R, $P \subseteq Ra^n$ and $Ra^n \bigcap L$ is small in R. The notion of semi-π-regular rings is left-right symmetric, but we do

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not know whether this is true for generalized semi-π-regular rings. In this paper, we mainly prove that generalized semi-π-regular rings are non-trivial generalizations of both semi-π-regular rings and GP-injective rings. Also some properties of generalized semi-π-regular rings are studied and some results about GP-injective rings and generalized semiregular rings are extended.

2. Generalized semi-ϕ-regular rings

Let R be a ring and M a left R-module. A submodule K of M is said to be small in M[1], if $K + N \neq M$ for every submodule $N \neq M$.

Definition 2.1 *An element $0 \neq a$ of a ring R is called right generalized semi-π-regular, if there exists a positive integer n such that $a^n \neq 0$ and $lr(a^n) = P \oplus L$, where P, L are left ideals of R, $P \subseteq Ra^n$ and $Ra^n \bigcap L$ is small in R. A ring R is called right generalized semi-π-regular if each element is right generalized semi-π-regular. Similarly, we may define left generalized semi-π-regular elements and left generalized semi-π-regular rings.*

Remark 1 By definition, AP-injective rings, GP-injective rings and generalized semiregular rings are generalized semi-π-regular. From [9], we know that a ring R is called right P-injective if, for any $a \in R$, $lr(a) = Ra$. Thus every right P-injective ring is right generalized semiregular, so it is right generalized semi-π-regular.

Proposition 2.2 *If R is a semi-π-regular ring, then R is right generalized semi-π-regular.*

Proof Let $0 \neq a \in R$. Since R is semi-π-regular, there exists $e^2 = e \in Ra^n$ for some positive integer n such that $a^n \neq 0$ and $a^n(1-e) \in J(R)$. Thus $R = Re \oplus R(1-e)$, where $Re \subseteq Ra^n$ and $Ra^n(1-e) \subseteq J(R)$ is small in R. Note that $Ra^n \subseteq lr(a^n)$, so by the modular law we have $lr(a^n) = lr(a^n)\bigcap R = lr(a^n)\bigcap(Re \bigoplus R(1-e)) = Re \bigoplus(lr(a^n)\bigcap R(1-e))$ and $Ra^n \bigcap(lr(a^n)\bigcap R(1-e)) = Ra^n \bigcap R(1-e) \subseteq Ra^n(1-e) \subseteq J(R)$ is small in R. Hence R is right generalized semi-π-regular.

The following two examples show that right generalized semi-π-regular rings are non-trivial generalizations of both GP-injective rings and semi-π-regular rings.

Example 2.3 Let ${}_RM_R$ be a bimodule over a ring R. The trivial extension of R by M is $T(R, M) = R \bigoplus M$ with pointwise addition and multiplication: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. It is shown in [4] that the trivial extension $R = T(Z_4, Z_4 \bigoplus Z_4)$ is an AP-injective ring, but R is not GP-injective. By Remark 1, R is a generalized semi-π-regular ring.

Lemma 2.4[6] *If R is a semi-π-regular ring, then R is an exchange ring.*

Example 2.5 Let $R = T(Z, Q/Z)$. By [10], R is a commutative P-injective ring, so R is a commutative generalized semi-π-regular ring. But $R/J(R) \cong Z$ is not an exchange ring. So R is not an exchange ring. Thus R is not semi-π-regular ring.

A module M is called AGP-injective[4], if for any $0 \neq a \in R$, there exist a positive integer

n and a S-submodule X_a of M such that $a^n \neq 0$ and $l_M(r_R(a^n)) = Ma^n \bigoplus X_a$, where $S = \mathrm{End}(M)$. In each case the module X_a may not be unique, but we take one such X_a for each a and form the S-module $b(M) = \sum_a X_a$. We call $b(M)$ an index bound of M and the set of those X_a an index set of M. We call R a right AGP-injective ring if R_R is an AGP-injective module. By definition, AGP-injective rings are generalized semi-π-regular, but the converse is not true.

Lemma 2.6[4] *If R is a right AGP-injective ring, then $J(R) = Z(R_R)$.*

Example 2.7 Let $R = \begin{pmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{pmatrix}$. Then $J(R) = \begin{pmatrix} 0 & Z_2 \\ 0 & 0 \end{pmatrix}$ and $Z(R_R) = Z({}_RR) = 0$. Thus R is not AGP-injective. But by [6], R is a semiregular ring, so R is a generalized semi-π-regular ring.

Although generalized semi-π-regular rings need not be GP-injective or semi-π-regular, but in the following we will show that right generalized semi-π-regular rings are GP-injective or semi-π-regular under some sufficient conditions.

Proposition 2.8 *Let R be a semiprimitive ring. Then R is right generalized semi-π-regular if and only if R is right AGP-injective.*

Proof One direction is obvious. Conversely, let $0 \neq a \in R$. Since R is right generalized semi-π-regular, there exists a positive integer n such that $a^n \neq 0$ and $lr(a^n) = P \bigoplus L$, where $P \subseteq Ra^n$ and $Ra^n \bigcap L$ is small in R. By assumption, $Ra^n \bigcap L \subseteq J(R) = 0$. Clearly, $lr(a^n) = Ra^n + L$. Thus $lr(a^n) = Ra^n \bigoplus L$, which shows that R is right AGP-injective.

Lemma 2.9[4] *A module M_R is GP-injective if and only if M_R is AGP-injective with an index bound $b(M) = 0$.*

Corollary 2.10 *Let R be a semiprimitive ring with index bound $b(R_R) = (0)$. Then R is right generalized semi-π-regular if and only if R is right GP-injective.*

Proposition 2.11 *If R is a right generalized semi-π-regular ring and for every $0 \neq a^n \in R$ there exists $e^2 = e \in R$ such that $r(a^n) = r(e)$, then R is semi-π-regular.*

Proof Let $0 \neq a \in R$. Since R is right generalized semi-π-regular, there exists a positive integer n such that $a^n \neq 0$ and $lr(a^n) = P \bigoplus L$, where $P \subseteq Ra^n$ and $Ra^n \bigcap L$ is small in R. Since $r(a^n) = r(e)$, we have $P \bigoplus L = lr(a^n) = lr(e) = Re$. So $a^n = a^n e$. Let $e = g + t$, where $g = ra^n \in P \subseteq Ra^n, t \in L$. Then $a^n = a^n e = a^n ra^n + a^n t$ and $ra^n = ra^n e = ra^n(ra^n + t) = ra^n ra^n + ra^n t$. So $ra^n - ra^n ra^n = ra^n t \in P \bigcap L = 0$, which gives $g^2 = g$ and $a^n - a^n ra^n = a^n t \in Ra^n \bigcap L \subseteq J(R)$. This shows that for any $0 \neq a \in R$, there exists $g^2 = g \in Ra^n$ such that $a^n(1-g) \in J(R)$, a is semi-π-regular. Hence R is semi-π-regular.

Lemma 2.12 *Let $a \in R$ such that $a^n R \cong eR$, where $e^2 = e \in R$. Then there exists an idempotent $f^2 = f \in R$ such that $a^n f = a^n$ and $r(a^n) = r(f)$.*

Proof Let $\sigma : a^n R \longrightarrow eR$ be the isomorphism. Let $\sigma(a^n) = ed, d \in R$ and $\sigma^{-1}(e) = a^n c, c \in R$.

Then $edc = \sigma(a^n c) = e$. Take $f = ced$. Then $f^2 = f$ and $a^n f = \sigma^{-1}(ed) = a^n$. Clearly, $r(f) \subseteq r(a^n)$. If $r \in r(a^n)$, then $a^n r = 0$. So $fr = c\sigma(a^n r) = c\sigma(0) = 0$, $r \in r(f)$. This shows that $r(a^n) = r(f)$.

Proposition 2.13 *Let a be a right generalized semi-π-regular element. If $a^n R \cong eR$, where $e^2 = e \in R$, then a is semi-π-regular element.*

A ring R is called right generalized $P.P$-ring if, for any $x \in R$ there exists $e^2 = e \in R$ such that $r(a^n) = eR$.

Corollary 2.14 *Let R be a right generalized $P.P$-ring. If R is a right generalized semi-π-regular ring, then R is semi-π-regular.*

Proof By assumption, for any $a \in R, r(a^n) = eR$, where $e^2 = e \in R$. So $lr(a^n) = l(eR) = l(e) = R(1-e)$. Let $f = 1-e$. Then $f^2 = f$ and $lr(a^n) = Rf$. Thus $r(a^n) = rlr(a^n) = r(Rf) = r(f)$. So R is semi-π-regular by Proposition 2.11.

A module M is said to satisfy C_2 if for any two submodules X and Y of M with $X \cong Y \mid M$, we have $X \mid M$. From [6], we know that if R is a right generalized semiregular ring with $J(R) \subseteq Z(R_R)$, then R_R satisfies C_2. But for right generalized semi-π-regular ring, we only have the following proposition.

Proposition 2.15 *Let R be a right generalized semi-π-regular ring with $J(R) \subseteq Z(R_R)$. If $e^2 = e \in R$ such that $a^n R \cong eR$, then there exists $g^2 = g \in R$ such that $a^n R = gR$.*

Proof Let $0 \neq a \in R$ such that $a^n R \cong eR$, where $e^2 = e \in R$. By Lemma 2.12, there exists $f^2 = f \in R$ such that $a^n = a^n f$ and $r(a^n) = r(f)$. By Proposition 2.11, R is semi-π-regular. So there exists $g^2 = g \in a^n R$ such that $(1-g)a^n \in J(R)$. Thus $a^n R = gR \bigoplus S$, where $S = (1-g)a^n R \subseteq J(R)$. By assumption, $S \subseteq Z(R_R)$ is a singular right R-module. Let φ be the epimorphism of fR to $a^n fR$ given by $\varphi(fr) = a^n fr$ for any $r \in R$. If $a^n fr = 0$, then $fr \in r(a^n) \bigcap fR = r(f) \bigcap fR = 0$. So φ is isomorphism. This shows that $a^n R = a^n fR \cong fR$ is a projective right R-module. Thus S is a projective and singular right R-module, and so $S = 0$ by [6, Lemma 2.2]. Hence $a^n R = gR$.

By Lemma 2.6, we know that if R is a right AGP-injective ring, then $J(R) = Z(R_R)$.

Proposition 2.16 *If R is a right generalized semi-π-regular ring, then $Z(R_R) \subseteq J(R)$.*

Proof Let $0 \neq a \in Z(R_R)$. Then for any $b \in R, ba \in Z(R_R)$. Let $u = 1 - ba$. Then $u \neq 0$. Since R is right generalized semi-π-regular, there exists a positive integer n such that $u^n \neq 0$ and $lr(u^n) = P \bigoplus L$, where $P \subseteq Ru^n$ and $Ru^n \bigcap L$ is small in R. Since $r(ba) \bigcap r(u^n) = 0$, we have $r(u^n) = 0$ and $R = lr(u^n) = P \bigoplus L$. So there exists $e^2 = e \in R$ such that $P = Re$. We claim that $e = 1$. If not, then $(1-e)R \neq 0$. Since $ba \in Z(R_R), u^n = (1-ba)^n$, there exists $v \in Z(R_R)$ such that $u^n = 1 - v$. Thus $(1-e)R \bigcap r(v) \neq 0$. Let $0 \neq (1-e)r \in (1-e)R \bigcap r(v)$. Then $v(1-e)r = 0$. So $(1-e)r = u^n(1-e)r$. Since $u^n \in lr(u^n) = R = Re \bigoplus L$, we take $u^n = se + t$, where $s \in R, t \in L$. Then $(1-t)(1-e)r = 0$. Note that $t = u^n - se \in Ru^n \bigcap L \subseteq J(R)$, so

$1-t$ is a unit, which implies $(1-e)r=0$, a contradiction. So $e=1$ and $P=R=Ru^n$. Thus $a \in J(R)$.

Remark 2 Example 2.7 shows that there exists a generalized semi-π-regular ring with $J(R) \neq Z({}_RR) = Z(R_R)$.

3. Corner subrings of generalized semi-π-regular rings

An idempotent element $e \in R$ is left (resp. right) semicentral in R[13] if $Re = eRe$ (resp. $eR = eRe$). In general we have

Theorem 3.1 *Let R be a right generalized semi-π-regular ring. If $e^2 = e \in R$ is right semicentral, then eRe is right generalized semi-π-regular.*

Proof Let $0 \neq a \in eRe$. By assumption, there exists a positive integer n such that $a^n \neq 0$ and $lr(a^n) = P \bigoplus L$, where $P \subseteq Ra^n$ and $Ra^n \bigcap L$ is small in R. We claim that $l_{eRe}r_{eRe}(a^n) = eP \bigoplus eL$.

In fact, $eP \bigcap eL \subseteq P \bigcap L = 0$. Take any $y \in eP \subseteq ePe$, where $y = ey_1, y_1 \in P \subseteq lr(a^n)$. Then for any $x \in r_{eRe}(a^n) \subseteq r(a^n), y_1x = 0$. So $yx = ey_1x = 0$. Hence $y \in l_{eRe}r_{eRe}(a^n), eP \subseteq l_{eRe}r_{eRe}(a^n)$. Similarly, $eL \subseteq l_{eRe}r_{eRe}(a^n)$. Thus $eP \bigoplus eL \subseteq l_{eRe}r_{eRe}(a^n)$. On the other hand, take $x \in l_{eRe}r_{eRe}(a^n)$. Then for any $y \in r(a^n), a^neye = a^nye = 0$ and $eye \in r_{eRe}(a^n)$. So $xeye = 0$. Since e is right semicentral, $ey = eye$. Thus $xy = xey = xeye = 0$, $x \in l(y)$. This shows that $l_{eRe}r_{eRe}(a^n) \subseteq lr(a^n)$. Let $x = s+t, s \in P, t \in L$. Then $x = ex = e(s+t) \in eP + eL$. Thus $l_{eRe}r_{eRe}(a^n) = eP \bigoplus eL$. It remains to prove that $eRea^n \bigcap eL$ is small in eRe since $eP \subseteq eRa^n = eRea^n$. Since e is right semicentral, we have $eRea^n \bigcap eL \subseteq e(eRea^n \bigcap eL)e$. But $eRea^n \bigcap eL \subseteq Ra^n \bigcap L \subseteq J(R)$, so $eRea^n \bigcap eL \subseteq eJ(R)e = J(eRe)$. Since $J(eRe)$ is small in eRe, $eRea^n \bigcap eL$ is small in eRe. Thus eRe is right generalized semi-π-regular.

Theorem 3.2 *Let $e^2 = e \in R$ such that $ReR = R$. If R is a right generalized semi-π-regular ring, then eRe is right generalized semi-π-regular.*

Proof Let $0 \neq a \in eRe$. By assumption, there exists a positive integer n such that $a^n \neq 0$ and $lr(a^n) = P \bigoplus L$, where $P \subseteq Ra^n$ and $Ra^n \bigcap L$ is small in R. We claim that $l_{eRe}r_{eRe}(a^n) = ePe \bigoplus eLe$.

Since $1-e \in r(e) \subseteq r(a) \subseteq r(a^n)$, we have $t(1-e) = 0$ for any $t \in L \subseteq lr(a^n)$. So $L = Le$. Similarly, $P = Pe$. Thus $ePe \bigcap eLe = eP \bigcap eL \subseteq P \bigcap L = 0$. Clearly, $ePe \subseteq l_{eRe}r_{eRe}(a^n), eLe \subseteq l_{eRe}r_{eRe}(a^n)$. Thus $ePe \bigoplus eLe \subseteq l_{eRe}r_{eRe}(a^n)$. On the other hand, take $x \in l_{eRe}r_{eRe}(a^n)$ and write $1 = \sum_{i=1}^n a_ieb_i$ since $R = eRe$, where $a_i, b_i \in R$. Then for any $y \in r(a^n)$, we have $a^neya_ie = a^nya_ie = 0$, $eya_ie \in r_{eRe}(a^n)$. So $xeya_ie = 0$. Thus $xy = xey = xey\sum_{i=1}^n a_ieb_i = \sum_{i=1}^n xeya_ieb_i = 0$, $x \in l(y)$. So $l_{eRe}r_{eRe}(a^n) \subseteq lr(a^n)$. Let $x = s+t, s \in P, t \in L$. Then $x = exe = ese + ete \in ePe + eLe$. Hence $l_{eRe}r_{eRe}(a^n) = ePe \bigoplus eLe$. It remains to prove that $eRea^n \bigcap eLe$ is small in eRe since $ePe \subseteq eRea^n$. Since $L = Le$, we have $eRea^n \bigcap eLe \subseteq Ra^n \bigcap L \subseteq J(R)$. So $eRea^n \bigcap eLe \subseteq eJ(R)e = J(eRe)$. Thus $eRea^n \bigcap eLe$ is small in eRe and eRe is right generalized semi-π-regular.

Proposition 3.3 *Let e and f be orthogonal central idempotents of R. If eR and fR are right generalized semi-π-regular, then $gR = eR\bigoplus fR$ is right generalized semi-π-regular.*

Proof Let $0 \neq a \in gR$. Then $ea \in eR, fa \in fR$. By assumption, there exists a positive integer n such that $a^n \neq 0$. Take $x \in l_{gR}r_{gR}(a^n)$. Then for any $y \in r_{eR}[(ea)^n], e^n a^n y = 0$. Hence $a^n y = a^n ey = a^n e^n y = e^n a^n y = 0$, this implies $a^n gy = a^n yg = 0$, $gy \in r_{gR}(a^n)$. Thus $xy = xgy = 0$ and $exy = xye = 0$. So $ex \in l_{eR}r_{eR}[(ea)^n]$. By assumption, $ex \in l_{eR}r_{eR}[(ea)^n] = P_e \bigoplus L_e$, where $P_e \subseteq eRea^n = eRa^n, eRa^n \bigcap L_e \subseteq J(eRe)$. Similarly, $fx \in l_{fR}r_{fR}[(fa)^n] = P_f \bigoplus L_f$, where $P_f \subseteq fRa^n, fRa^n \bigcap L_f \subseteq J(fRf)$. Then $x = gx = ex + fx \in P_e \bigoplus P_f \bigoplus L_e \bigoplus L_f$. For any $x \in L_e$ and any $y \in r_{gR}(a^n)$, $a^n y = 0$, so $a^n ey = a^n ye = 0$ and $xey = 0$ since $L_e \subseteq l_{eR}r_{eR}[(ea)^n]$. Note that $L_e \subseteq eR \subseteq gR$ and $x = xe$, so $xy = 0$ and $L_e \subseteq l_{gR}r_{gR}(a^n)$. Similarly, $L_f \subseteq l_{gR}r_{gR}(a^n)$. On the other hand, $P_e \bigoplus P_f \subseteq eRa^n \bigoplus fRa^n = gRa^n \subseteq l_{gR}r_{gR}(a^n)$. Thus $l_{gR}r_{gR}(a^n) = P_e \bigoplus P_f \bigoplus L_e \bigoplus L_f$. Since gR is a ring with identity, $J(gR)$ is small in gR. We have $gRa^n \bigcap(L_e \bigoplus L_f) \subseteq J(eR) \bigoplus J(fR) = J(gR)$ is small in gR. Hence gR is right generalized semi-π-regular.

Corollary 3.4 *Let $0 \neq e^2 = e \in R$ be a central idempotent. Then eRe and $(1-e)R(1-e)$ are right generalized semi-π-regular if and only if so is R.*

Theorem 3.5 *Let $1 = e_1 + e_2 + \cdots + e_n \in R$, where $e_1, e_2, \ldots, e_n$ are orthogonal central idempotents. Then R is right generalized semi-π-regular if and only if each e_iR is right generalized semi-π-regular.*

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