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A Note on the Monotone Product of Nuclear C^*-Algebras

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Abstract Given two nuclear C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ with states φ_1 and φ_2, we show that the monotone product C^*-algebra $\mathcal{A}_1 \rhd \mathcal{A}_2$ is still nuclear. Furthermore, if both the states φ_1 and φ_2 are faithful, then the monotone product $\mathcal{A}_1 \rhd \mathcal{A}_2$ is nuclear if and only if the C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ both are nuclear.

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1. Introduction and preliminaries

Let $\mathcal{H}$ be a Hilbert space. C^*-algebras are self-adjoint normed-closed subalgebras of $\mathcal{B}(\mathcal{H})$ which are the algebras of all bounded linear operators on $\mathcal{H}$. In this paper, if without special remark, we always assume that C^*-algebras are unital and the Hilbert spaces $\mathcal{H}$ are separable. As a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, von Neumann algebras are closed with respect to the weak operator topology. von Neumann algebras with trivial center are called factors.

In a series of papers, in order to solve the isomorphism problem of free group factors on different number of generators, Voiculescu[7] introduced a noncommutative probability theory (free probability theory) and free entropy in the framework of operator algebras. This new and powerful tool is crucial in solving some longstanding open problems in the filed on von Neumann algebras. In [8], Voiculescu defined a new concept which is called free entropy dimension. Using the new concept as basic tool, Voiculescu, Ge and other persons have solved several longstanding open problems in II_1 factors. Parrelling to the free probability theory, Muraki also introduced a monotone probability theory and the monotone product of C^*-algebras with respect to the states[4,5].

Muraki has also shown that the monotone product parrels to the tensor product and the free product of algebras. It is well known that the tensor product of two nuclear C^*-algebras is still

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nuclear but the free product of two nuclear C^*-algebras is not nuclear in general. How about the monotone product of two nuclear C^*-algebras?

In [9], Wu and Wang have studied the structures of the monotone product of C^*-algebras and von Neumann algebras. In this paper, by use of the results in [9] and the properties of nuclear C^*-algebras, we show that the monotone product of two nuclear C^*-algebras is still nuclear. Furthermore, if both given states φ_1 and φ_2 are faithful, we show that the monotone product C^*-algebra $\mathcal{A}_1 \rhd \mathcal{A}_2$ is nuclear if and only if both $\mathcal{A}_1$ and $\mathcal{A}_2$ are nuclear.

Now let us recall the construction of the monotone product of two C^*-algebras (for general construction[4]).

Given two C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ with states φ and φ_2, respectively. Let $(\pi_1, \mathcal{H}_1, \xi_1)$ and $(\pi_2, \mathcal{H}_2, \xi_2)$ be the GNS representations of $(\mathcal{A}_1, \varphi_1)$ and $(\mathcal{A}_2, \varphi_2)$ respectively. Then ξ_i is a unit vector and $\pi_i(\mathcal{A}_i)\xi_i$ is dense in the Hilbert space $\mathcal{H}_i$ $(i = 1, 2)$. We denote the orthogonal complements of $\mathbb{C}\xi_i$ in $\mathcal{H}_i$ as $\mathcal{H}_i^\circ$ $(i = 1, 2)$. The monotone product $(\mathcal{H}, \xi)$ of $(\mathcal{H}_1, \xi_1)$ and $(\mathcal{H}_2, \xi_2)$ is defined as follows

$$\mathcal{H} = \mathbb{C}\xi \oplus \mathcal{H}_1^\circ \oplus \mathcal{H}_2^\circ \oplus (\mathcal{H}_2^\circ \otimes \mathcal{H}_1^\circ)$$

and denoted by $(\mathcal{H}, \xi) = (\mathcal{H}_1, \xi_1) \rhd (\mathcal{H}_2, \xi_2)$, where ξ is a fixed unit vector in $\mathcal{H}$. It is easy to see that $\mathcal{H}$ is isomorphic to the Hilbert space $\mathcal{H}_2 \otimes \mathcal{H}_1$ which maps ξ to $\xi_2 \otimes \xi_1$. In fact, if we define the bounded linear operator U from $\mathcal{H}$ onto $\mathcal{H}_2 \otimes \mathcal{H}_1$ as follows:

$$U: \quad \xi \to \xi_2 \otimes \xi_1, \quad \mathcal{H}_1^\circ \ni \eta \to \xi_2 \otimes \eta,$$

$$\mathcal{H}_2^\circ \ni \eta \to \eta \otimes \xi_1, \quad U|_{\mathcal{H}_2^\circ \otimes \mathcal{H}_1^\circ} = I|_{\mathcal{H}_2^\circ \otimes \mathcal{H}_1^\circ},$$

where I is the identity operator on the tensor product Hilbert space $\mathcal{H}_2 \otimes \mathcal{H}_1$, then U is a unitary operator. From now on, we will identify the Hilbert space monotone product $(\mathcal{H}, \xi) = (\mathcal{H}_1 \rhd \mathcal{H}_2, \xi_1 \rhd \xi_2)$ with the tensor product $(\mathcal{H}_2 \otimes \mathcal{H}_1, \xi_2 \otimes \xi_1)$ through the operator U.

Let P_2 be the one-dimensional orthogonal projection from $\mathcal{H}_2$ onto the closed subspace $[\mathbb{C}\xi_2]$. Define a representation π of $\mathcal{A}_1$ and $\mathcal{A}_2$ on the Hilbert space $\mathcal{H}$ by

$$\pi(A) = \begin{cases} P_2 \otimes \pi_1(A) & A \in \mathcal{A}_1 \\ \pi_2(A) \otimes I_1 & A \in \mathcal{A}_2 \end{cases}.$$

Where I_1 is the identity operator on $\mathcal{H}_1$.

Let $\mathcal{A}$ be the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ which is generated by $\pi(\mathcal{A}_1)$ and $\pi(\mathcal{A}_2)$. Let φ be the vector state of $\mathcal{A}$ which is defined as $\varphi(A) = \langle A\xi, \xi \rangle$ for any $A \in \mathcal{A}$. Then φ is called the monotone product state of the states φ_1 and φ_2 and is denoted by $\varphi_1 \rhd \varphi_2$. We call $(\mathcal{A}, \varphi)$ as the C^*-algebra monotone product of $(\mathcal{A}_1, \varphi_1)$ and $(\mathcal{A}_2, \varphi_2)$ and denote it by $(\mathcal{A}, \varphi) = (\mathcal{A}_1, \varphi_1) \rhd (\mathcal{A}_2, \varphi_2)$.

2. Main results

The main result of this paper is the following theorem.

Theorem 1 *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are C^*-algebras with states φ_1 and φ_2, respectively. Then the monotone product C^*-algebra $(\mathcal{A}, \varphi) = (\mathcal{A}_1, \varphi_1) \rhd (\mathcal{A}_2, \varphi_2)$ is still nuclear.*

Before we prove the above-mentioned result, firstly let us recall the concept of nuclear C^*-algebra and some basic properties of it. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two C^*-algebras and that $\mathcal{A} \odot \mathcal{B}$ is their algebraic tensor product. $\mathcal{A} \odot \mathcal{B}$ forms a $*$-algebra under the $*$-operation defined as $(A \otimes B)^* = A^* \otimes B^*$ for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$. A norm $\|\cdot\|$ is called a C^*-norm if $(\mathcal{A} \odot \mathcal{B}, \|\cdot\|)$ is a normed $*$-algebra and the norm satisfies $\|S^*S\| = \|S\|^2$ for any $S \in \mathcal{A} \odot \mathcal{B}$. Maybe there are many C^*-norms on $\mathcal{A} \odot \mathcal{B}$.

Definition 2 *A C^*-algebra $\mathcal{A}$ is called a nuclear C^*-algebra if, for any C^*-algebra $\mathcal{B}$, there is only one C^*-norm on the algebraic tensor product $\mathcal{A} \odot \mathcal{B}$.*

There are lots of naturally occurring nuclear C^*-algebras. We list some classes and some basic properties of nuclear C^*-algebras as follows.

Proposition 3[5] *(1) All abelian C^*-algebras are nuclear;*

(2) All finite dimensional C^*-algebras are nuclear;

(3) Either of two stably isomorphic C^*-algebras is nuclear, then so is the other;

(4) The tensor product of two nuclear C^*-algebras is still nuclear;

(5) Given a short exact sequence of C^*-algebras

$$0 \to \mathcal{A}_1 \to \mathcal{A}_2 \to \mathcal{A}_3 \to 0.$$

If any two of the three C^*-algebras are nuclear, then so is the third.

It is a nontrivial result to show that the free group factors are not nuclear. In fact, Connes[1] showed that all separable amenable C^*-algebras are nuclear and Haagerup[3] has proved the inverse implication. In a series of papers, Lance and other mathematicians have shown that a C^*-algebra $\mathcal{A}$ is nuclear if and only if its bidual $\mathcal{A}^{**}$ is an injective von Neumann algebra (W^*-algebra). The following result is well known. Here we just give a short proof of the first result.

Proposition 4 *(1) If the dimension of the Hilbert space $\mathcal{H}$ is at most countably, then the ideal $\mathcal{K}(\mathcal{H})$ of all compact operators, as a C^*-algebra (maybe non-unital), is nuclear.*

(2)[8] *Suppose that $\mathcal{I}$ is a closed two-sided ideal of a C^*-algebra $\mathcal{A}$. Then $\mathcal{A}$ is nuclear if and only if both $\mathcal{I}$ and the quotient algebras $\mathcal{A}/\mathcal{I}$ are nuclear.*

Proof (1) If $\dim(\mathcal{H}) = n < +\infty$, then $\mathcal{K}(\mathcal{H}) = \mathcal{B}(\mathcal{H}) \cong \mathcal{M}_n(\mathbb{C})$ is finite dimensional. Therefore $\mathcal{K}(\mathcal{H})$ is nuclear by Proposition 3(2).

If $\dim(\mathcal{H})$ is countable, since the bidual $\mathcal{K}(\mathcal{H})^{**}$ is isomorphic to $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ is an injective W^*-algebra, $\mathcal{K}(\mathcal{H})$ is nuclear by above-mentioned results. □

According to the above proposition, we have the following result.

Corollary 5 *Suppose that $\mathcal{A}$ is a nuclear C^*-algebra and $\mathcal{B}$ is also a C^*-algebra. If $\pi : \mathcal{A} \to \mathcal{B}$ is a *-homomorphism, then the image of π is a nuclear C^*-algebra.*

Proof It is known that $\pi(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{B}$. The kernel $\ker(\pi)$ of π is a closed two-sided ideal of $\mathcal{A}$ since π is a $*$-homomorphism. The quotient algebra $\mathcal{A}/\ker(\pi)$ is a nuclear C^*-algebra by Proposition 4.(2) since $\mathcal{A}$ is nuclear. Thus the range $\pi(\mathcal{A})$ is also nuclear since it is $*$-isomorphic to $\mathcal{A}/\ker(\pi)$. $\square$

Now let us recall the description of the structure of the monotone product of C^*-algebras. Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are C^*-algebras with states φ_1 and φ_2. The GNS representations of them are $(\pi_1, \mathcal{H}_1, \xi_1)$ and $(\pi_2, \mathcal{H}_2, \xi_2)$, respectively. In [9], we have shown the following important results.

Theorem 6 *With notations as the above, the C^*-algebra monotone product $(\mathcal{A}, \varphi) = (\mathcal{A}_1, \varphi_1) \rhd (\mathcal{A}_2, \varphi_2)$ is generated by $\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1)$ and $\pi_2(\mathcal{A}_2) \otimes \mathbb{C}I_1$.*

Here we do not give the complete argument and just list the brief sketch. In fact, the monotone product $\mathcal{A}$ is generated by $P_2 \otimes \pi_1(A)$ for any $A \in \mathcal{A}_1$ and $\pi_2(A) \otimes I_1$ for any $A \in \mathcal{A}_2$. But P_2 is the one-dimensional orthogonal projection from $\mathcal{H}_2$ onto the closed subspace $[\mathbb{C}\xi_2]$ and ξ_2 is a cyclic vector in $\mathcal{H}_2$ for the C^*-algebra $\pi_2(\mathcal{A}_2)$. Thus the C^*-subalgebra of $\mathcal{B}(\mathcal{H}_2)$ generated by P_2 and $\pi_2(\mathcal{A}_2)$ contains the set $\mathcal{K}(\mathcal{H}_2)$ of all compact operators. The rest is obvious.

From the above result, we have the following result.

Corollary 7[9] *With notations as the above, $\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1)$ is a closed two-sided ideal of the monotone product C^*-algebra $\mathcal{A} = \mathcal{A}_1 \rhd \mathcal{A}_2$.*

Now we can prove the main result of this article.

Proof of Theorem 1 According to Corollary 7, $\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1)$ is a closed two-sided ideal of the monotone product $\mathcal{A}$. Hence by Proposition 4(2), we just need to show that both the ideal $\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1)$ and the quotient algebra $\mathcal{A}/(\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1))$ are nuclear.

The ideal $\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1)$ is nuclear since $\mathcal{K}(\mathcal{H}_2)$ is nuclear by Proposition 4(1) and $\pi_1(\mathcal{A}_1)$ is nuclear by Corollary 5 and the fact that $\mathcal{A}_1$ is nuclear. To prove that the quotient C^*-algebra $\mathcal{A}/(\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1))$ is nuclear, we just need to show that the following claim holds.

Claim The quotient C^*-algebra $\mathcal{A}/(\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1))$ is $*$-isomorphic to the C^*-algebra $\pi_2(\mathcal{A}_2)/(\pi_2(\mathcal{A}_2) \cap \mathcal{K}(\mathcal{H}_2))$.

Firstly, $\pi_2(\mathcal{A}_2) \cap \mathcal{K}(\mathcal{H}_2)$ is a closed and self-adjoint algebra since it is the intersection of two closed and self-adjoint algebras. For any $A \in \pi_2(\mathcal{A}_2)$ and $T \in \pi_2(\mathcal{A}_2) \cap \mathcal{K}(\mathcal{H}_2)$, then $AT, TA \in \pi_2(\mathcal{A}_2)$ since $\pi_2(\mathcal{A}_2)$ is an algebra and $AT, TA \in \mathcal{K}(\mathcal{H}_2)$ since $\mathcal{K}(\mathcal{H}_2)$ is an ideal of $\mathcal{B}(\mathcal{H}_2)$. Hence $AT, TA \in \pi_2(\mathcal{A}_2) \cap \mathcal{K}(\mathcal{H}_2)$. Therefore, $\pi_2(\mathcal{A}_2) \cap \mathcal{K}(\mathcal{H}_2)$ is a closed two-sided ideal of $\pi_2(\mathcal{A}_2)$.

Suppose that σ is the canonical quotient mapping from $\mathcal{A}$ into the quotient $\mathcal{A}/(\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1))$. For any $A \in \mathcal{A}$, there are sequences of operators $\{A_n : n \in \mathbb{N}\}$ in $\pi_2(\mathcal{A}_2) \otimes \mathbb{C}I_1$ and $\{B_n : n \in \mathbb{N}\}$ in $\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1)$ such that $A_n + B_n \to A$ in norm as $n \to +\infty$. Thus by the continuity of the mapping σ and since B_n is in the kernel of σ, we have

$$\sigma(A_n + B_n) = \sigma(A_n) \to \sigma(A), \quad n \to +\infty.$$

Hence for any $B \in \mathcal{A}/(\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1))$, there is a sequence of $A_n \in \pi_2(\mathcal{A}_2) \otimes \mathbb{C}I_1$ such that $\sigma(A_n) \to B$ as $n \to +\infty$. Now the restriction $\sigma|_{\pi_2(\mathcal{A}_2)\otimes\mathbb{C}I_1}$ of σ on the C^*-subalgebra $\pi_2(\mathcal{A}_2)\otimes\mathbb{C}I_1$ of $\mathcal{A}$ is still a $*$-homomorphism between C^*-algebras. Thus the range $\sigma(\pi_2(\mathcal{A}_2) \otimes \mathbb{C}I_1)$ is closed in norm. Then we have

$$\sigma(\mathcal{A}) = \mathcal{A}/(\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1)) = \sigma(\pi_2(\mathcal{A}_2) \otimes \mathbb{C}I_1).$$

Therefore by the basic rules in C^*-algebras,

$$(\pi_2(\mathcal{A}_2) \otimes \mathbb{C}I_1)/(\ker(\sigma|_{\pi_2(\mathcal{A}_2)\otimes\mathbb{C}I_1})) \cong \sigma(\pi_2(\mathcal{A}_2) \otimes \mathbb{C}I_1).$$

According to the definition of the quotient mapping σ, we have

$$\begin{aligned}\ker(\sigma|_{\pi_2(\mathcal{A}_2)\otimes\mathbb{C}I_1}) &= (\pi_2(\mathcal{A}_2) \otimes \mathbb{C}I_1) \cap (\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1)) \\ &= (\pi_2(\mathcal{A}_2) \cap \mathcal{K}(\mathcal{H}_2)) \otimes \mathbb{C}I_1.\end{aligned}$$

Therefore the claim holds, i.e.,

$$\mathcal{A}/(\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1)) \cong \pi_2(\mathcal{A}_2)/(\pi_2(\mathcal{A}_2) \cap \mathcal{K}(\mathcal{H}_2)).$$

Summarizing the above-mentioned results, we see that the monotone product C^*-algebra $\mathcal{A}$ is nuclear since the quotient $\mathcal{A}/(\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1))$ is also nuclear by the claim. □

As a natural corollary of Theorem 1, under the condition that the states are faithful, we can show that the monotone product C^*-algebra is nuclear if and only if both components C^*-algebras are nuclear.

Theorem 8 *Suppose that φ_1 and φ_2 are faithful states of the separable C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Then the monotone product C^*-algebra*

$$(\mathcal{A}, \varphi) = (\mathcal{A}_1, \varphi_1) \triangleright (\mathcal{A}_2, \varphi_2)$$

is nuclear if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ are nuclear.

Proof Firstly, according to Theorem 1, the result holds if both $\mathcal{A}_1$ and $\mathcal{A}_2$ are nuclear C^*-algebras.

Conversely, suppose that both states φ_1 and φ_2 are faithful and the monotone product C^*-algebra $\mathcal{A}_1 \triangleright \mathcal{A}_2$ is nuclear. Suppose that $(\mathcal{H}_i, \xi_i, \pi_i)$ is the GNS representation of $\mathcal{A}_i$ with respect to the state φ_i $(i = 1, 2)$. The representations π_1 and π_2 are faithful.

Since $\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1)$ is a closed two-sided ideal of $\mathcal{A} = \mathcal{A}_1 \triangleright \mathcal{A}_2$ (Corollary 7), $\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1)$ is nuclear and the quotient C^*-algebra $\mathcal{A}/(\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1))$ is also nuclear. Then $\pi_1(\mathcal{A}_1)$ is nuclear since it is stably isomorphic to $\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1)$. In the proof of Theorem 1, we have shown that $\mathcal{A}/(\mathcal{K}(\mathcal{H}_2) \otimes \pi_1(\mathcal{A}_1))$ is $*$-isomorphic to the C^*-algebra $\pi_2(\mathcal{A}_2)/(\pi_2(\mathcal{A}_2) \cap \mathcal{K}(\mathcal{H}_2))$. Hence $\pi_2(\mathcal{A}_2)/(\pi_2(\mathcal{A}_2) \cap \mathcal{K}(\mathcal{H}_2))$ is nuclear. At the same time, $\pi_2(\mathcal{A}_2) \cap \mathcal{K}(\mathcal{H}_2)$ is a non-unital C^*-algebra generated by some compact operators. Thus by the structure theorem of the C^*-algebras which are generated by compact operators[2], $\pi_2(\mathcal{A}_2) \cap \mathcal{K}(\mathcal{H}_2)$ is nuclear. Thus $\pi_2(\mathcal{A}_2)$ is nuclear.

By the faithfulness of the states φ_1 and φ_2 and the faithfulness of π_1 and π_2, $\mathcal{A}_1$ and $\mathcal{A}_2$ are nuclear. □

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