Journal of Mathematical Research & Exposition
May, 2009, Vol. 29, No. 3, pp. 507–516
DOI:10.3770/j.issn:1000-341X.2009.03.016
Http://jmre.dlut.edu.cn

The Existence of Three Positive Solutions for a Class of Nonlinear Three-Point Boundary Value Problem with p-Laplacian

LI Xiang Feng, XU Hong Wu
(Department of Mathematics, Longdong University, Gansu 745000, China)
(E-mail: lxf66006@sina.com)

Abstract This paper deals with the existence of three positive solutions for a class of nonlinear singular three-point boundary value problem with p-Laplacian. By means of a fixed point theorem duo to Leggett and Williams, sufficient condition for the existence of at least three positive solutions to the nonlinear singular three-point boundary value problem is established.

Keywords p-Laplacian operator; positive solution; three-point singular boundary value problem; fixed-point theorem.

Document code A
MR(2000) Subject Classification 34B16; 34B18
Chinese Library Classification O175.8

1. Introduction

In this paper, we shall consider the existence of multiple positive solutions for the following p-Laplacian differential equation

$$(\varphi_p(u'))' + a(t)f(u(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

subject to one of the following boundary conditions

$$\alpha\varphi_p(u(0)) - \beta\varphi_p(u'(\xi)) = 0, \quad \gamma\varphi_p(u(1)) + \delta\varphi_p(u'(1)) = 0, \tag{1.2}$$

or

$$\alpha\varphi_p(u(0)) - \beta\varphi_p(u'(0)) = 0, \quad \gamma\varphi_p(u(1)) + \delta\varphi_p(u'(\eta)) = 0, \tag{1.3}$$

where $\varphi_p(x) = |x|^{p-2}x$, $p > 1$, $\varphi_q(x) = |x|^{q-2}x$ is the inverse function to φ_p, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 0$, $\beta \geq 0$, $\gamma > 0$, $\delta \geq 0$, ξ, $\eta \in (0,1)$ are prescribed.

In recent years, the study of the existence of multiple positive solutions for three-point boundary value problem with p-Laplacian operator has been an interesting topic, and much attention is currently focused on the study of nonlinear three-point boundary value problem[1−12]. Liu et al.[1] and Li et al.[3] studied the existence of at least two or three positive solutions of Eq.(1.1) subject to one of the following two pairs of three-point boundary value conditions:

$$u(0) - B_0(u'(\eta)) = 0, \quad u'(1) = 0, \tag{1.4}$$

Received date: 2007-04-03; **Accepted date**: 2007-10-30
Foundation item: the Tutorial Scientific Research Program Foundation of Education Department of Gansu Province (Nos. 0710-04; 0810-03).

$$u'(0)=0, \quad u(1)+B_1(u'(\eta))=0, \tag{1.5}$$

by applying the three functionals fixed point theorem and the five functionals fixed point theorem duo to Avery and Henderson[10,11], respectively. In a recent paper[2], by using two new existence principles, Ma et al. considered the existence of positive solution of Eq.(1.1) subject to the following two pairs of nonlinear three-point boundary value conditions

$$u(0)-g(u'(0))=0, \quad u(1)-\beta u(\eta)=0, \tag{1.6}$$

$$u(0)-\alpha u(\eta)=0, \quad u(1)-g(u'(1))=0. \tag{1.7}$$

Liu[4,6] studied the existence of one and two positive solutions and infinitely many positive solutions of Eq.(1.1) subject to the nonlinear three-point boundary value conditions

$$u'(0)=0, \quad u(1)=\beta u(\eta). \tag{1.8}$$

The main tool is the fixed point index theory. He and Ge[7,8] discussed the existence of twin positive solutions Eq.(1.1) subject to the following two-, three-point boundary value conditions

$$u(0)-B_0(u'(0))=0, \quad u(1)+B_1(u'(1))=0, \tag{1.9}$$

$$u(0)-B_0(u'(\eta))=0, \quad u(1)+B_1(u'(1))=0, \tag{1.10}$$

$$u(0)-B_0(u'(0))=0, \quad u(1)+B_1(u'(\eta))=0, \tag{1.11}$$

by using the three functionals fixed point theorem duo to Avery and Henderson. Ma and Ge[5] applied the monotone iterative technique to prove the existence of positive pseudo-symmetric solutions for the following three-point second order p-Laplacian BVP

$$\begin{gathered}(\varphi_p(u'(t)))'+q(t)f(t,u(t))=0, \quad 0<t<1,\\ u(0)=0, \quad u(\eta)=u(1).\end{gathered} \tag{1.12}$$

Avery and Henderson[10] considered the existence of at least three positive solutions Eq.(1.1)–(1.12) by means of the five functionals fixed point theorem duo to Avery and Henderson. Su et al.[12] studied the existence of positive solutions Eq.(1.1) subject to the following boundary value conditions

$$\alpha\varphi_p(u(0))-\beta\varphi_p(u'(0))=0, \quad \gamma\varphi_p(u(1))+\delta\varphi_p(u'(1))=0, \tag{1.13}$$

by using the fixed-point theorem duo to Guo and Krasnoselskii.

However, for Eq.(1.1) and (1.2) or (1.3), there are currently few papers dealing with the existence of three positive solutions. Motivated by the papers mentioned above, we shall consider in this paper that nonlinear singular boundary value problem (1.1) and (1.2) or (1.3) has at least three positive solutions, by using the Leggett-Williams fixed point theorem. The results obtained in this paper essentially improve and generalize many well-known results.

In the rest of the paper, we make the following assumption:

(H_1) $f:[0,\infty)\to[0,\infty)$ is continuous;

(H_2) $a(t)\in C((0,1),[0,\infty))$, and $0<\int_0^1 a(t)\mathrm{d}t<\infty$. Furthermore, $a(t)$ is not identically zero on any subinterval of $(0,1)$.

2. Preliminaries and lemmas

In this section, we provide some background materials from the theory of cones in Banach spaces and some lemmas, and we state a three fixed point theorem duo to Leggett-Williams[13] for multiple fixed-point of a cone preserving operator.

Definition 2.1 *Let $(E, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is called a cone if it satisfies the following two conditions:*

(i) $u \in P, \lambda \geq 0$, implies $\lambda u \in P$;

(ii) $u \in P, -u \in P$, implies $u = 0$.

Definition 2.2 *A map α is said to be a nonnegative continuous concave functional on cone P of real Banach space E if $\alpha : P \to [0, \infty)$ is continuous and $\alpha(tu+(1-t)v) \geq t\alpha(u)+(1-t)\alpha(v)$ for all $u, v \in P$ and $t \in [0, 1]$.*

Let $0 < a < b, r > 0$ be real numbers, and $\alpha(u)$ be a nonnegative continuous concave functional on P. We define the following convex sets

$$P_r = \{u \in P : \|u\| < r\}, \quad \overline{P}_r = \{u \in P : \|u\| \leq r\},$$

$$\partial P_r = \{u \in P : \|u\| = r\}, \quad P(\alpha, a, b) = \{u \in P : a \leq \alpha(u), \|u\| \leq b\}.$$

Lemma 2.1 *Suppose condition (H_2) holds. Then there exists a constant $\omega \in (0, \frac{1}{2})$ that satisfies*

$$0 < \int_\omega^{1-\omega} a(t)\mathrm{d}t < \infty.$$

Furthermore, the function

$$A(t) = \varphi_q\Big(\int_\omega^t a(t)\,\mathrm{d}t\Big) + \varphi_q\Big(\int_t^{1-\omega} a(t)\,\mathrm{d}t\Big), \quad \omega \leq t \leq 1-\omega$$

is positive continuous function on $[\omega, 1-\omega]$. Therefore, $A(t)$ has a minimum on $[\omega, 1-\omega]$. Hence we suppose that there exists $L > 0$ such that $L = \min_{t\in[\omega,1-\omega]} A(t)$.

Proof By condition (H_2), it is easily seen that

$$0 < \int_\omega^{1-\omega} a(t)\mathrm{d}t < \infty$$

is satisfied. And it is easy to prove that $A(t)$ is continuous on $[\omega, 1-\omega]$. Next, let

$$A_1(t) = \varphi_q\Big(\int_\omega^t a(t)\,\mathrm{d}t\Big), A_2(t) = \varphi_q\Big(\int_t^{1-\omega} a(t)\,\mathrm{d}t\Big).$$

Then, it follows from condition (H_2) that the function $A_1(t)$ is strictly monotone increasing on $[\omega, 1-\omega]$ and $A_1(\omega) = 0$, and $A_2(t)$ is strictly monotone decreasing on $[\omega, 1-\omega]$ and $A_2(1-\omega) = 0$, which shows $L = \min_{t\in[\omega,1-\omega]} A(t) > 0$. The proof is completed. □

For convenience, we write

$$\lambda_1 = \varphi_q\Big(\frac{\beta}{\alpha}\int_\xi^1 a(r)\,\mathrm{d}r\Big) + \int_0^1 \varphi_q\Big(\int_s^1 a(r)\,\mathrm{d}r\Big)\,\mathrm{d}s,$$

$$\lambda_2=\varphi_q\Big(\frac{\delta}{\gamma}\int_0^{\eta} a(r)\,\mathrm{d}r\Big)+\int_0^1 \varphi_q\Big(\int_0^s a(r)\,\mathrm{d}r\Big)\,\mathrm{d}s.$$

In order to obtain multiple positive solutions of (1.1)–(1.2), the following fixed point theorem of Leggett and Williams will be fundamental.

Theorem A[13] *Let $c>0$ and $T:\overline{P}_c\to\overline{P}_c$ be completely continuous and $\phi(u)$ be a nonnegative continuous concave functional on P such that $\phi(u)\le\|u\|$, for arbitrary $u\in\overline{P}_c$. Suppose there exists $0<a<b<d\le c$, such that*

(C_1) $\{u\in P(\phi,b,d):\phi(u)>b\}\neq\emptyset$, and $\phi(Tu)>b$, for arbitrary $u\in P(\phi,b,d)$;

(C_2) $\|Tu\|<a$, for $\|u\|\le a$;

(C_3) $\phi(Tu)>b$ for $u\in P(\phi,b,c)$ with $\|Tu\|>d$.

Then T has at least three fixed points u_1,u_2 and u_3 on $\overline{P}_c$ satisfying

$$\|u_1\|<a, b<\phi(u_2),\ \text{and } \|u_3\|>a \text{ with } \phi(u_3)<b.$$

3. Existence of three positive solutions of problem (1.1)–(1.2)

In this section, we firstly define an appropriate Banach space and a cone, then give our main result and proof of the result.

Let $E=C[0,1]$. Then E is a Banach space with the norm $\|u\|=\sup_{t\in[0,1]}|u(t)|$. Let $P=\{u\in E: u(t)\ge 0, \alpha\varphi_p(u(0))-\beta\varphi_p(u'(\xi))=0, \gamma\varphi_p(u(1))+\delta\varphi_p(u'(1))=0,\ u(t)$ is concave function on $[0,1]\}$. It is easy to prove that P is a cone of E.

In the proof of our main results, we will make use of the following lemmas:

Lemma 3.1[4,12, Lemma 2.2] *Let $u\in P$ and ω be as in Lemma 2.1. Then*

$$u(t)\ge\omega\|u\|,\quad t\in[\omega,1-\omega].$$

Lemma 3.2 *Suppose that conditions (H_1),(H_2) hold. Then $u(t)\in E\bigcap C^2(0,1)$ is a solution of boundary value problem (1.1)–(1.2), if and only if $u(t)\in E$ is a solution of the following integral equation:*

$$u(t)=\begin{cases}\varphi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(u(r))\,\mathrm{d}r\right)+\int_0^t\varphi_q\left(\int_s^\sigma a(r)f(u(r))\,\mathrm{d}r\right)\,\mathrm{d}s, 0\le t\le\sigma,\\ \varphi_q\left(\frac{\delta}{\gamma}\int_\sigma^1 a(r)f(u(r))\,\mathrm{d}r\right)+\int_t^1\varphi_q\left(\int_\sigma^s a(r)f(u(r))\,\mathrm{d}r\right)\,\mathrm{d}s, \sigma\le t\le 1,\end{cases}\tag{3.1}$$

where $\sigma\in[\xi,1]$ and $u'(\sigma)=0$.

Proof Firstly, assume that(3.1) holds. We have

$$u'(t)=\begin{cases}\ \ \varphi_q\left(\int_t^\sigma a(r)f(u(r))\,\mathrm{d}r\right)\ge 0,\quad 0\le t\le\sigma,\\ -\varphi_q\left(\int_\sigma^t a(r)f(u(r))\,\mathrm{d}r\right)\le 0,\quad \sigma\le t\le 1.\end{cases}\tag{3.2}$$

Hence, thanks to (3.2), we have $(\varphi_p(u'(t)))'+a(t)f(u(t))=0,\ 0<t<1$, i.e., equation (1.1) holds. Furthermore, by letting $t=0$ and $t=1$ on (3.1) and letting $t=\xi$ and $t=1$(3.2), we can show the boundary conditions (1.2) are satisfied. Consequently, sufficiency is proved.

Next, by the boundary conditions (1.2), we have $u'(\xi) \geq 0$, $u'(1) \leq 0$. Then there exists a constant $\sigma \in [\xi, 1]$ such that $u'(\sigma) = 0$. Thus, for $t \in (0, \sigma)$, by integrating the equation (1.1) on $(0, \sigma)$, and note that $u'(\sigma) = 0$, we have

$$\varphi_p(u'(\sigma)) - \varphi_p(u'(t)) = -\int_t^\sigma a(s)f(u(s))\,\mathrm{d}s. \tag{3.3}$$

Then $u'(t) = \varphi_q\left(\int_t^\sigma a(s)f(u(s))\,\mathrm{d}s\right)$. Hence,

$$u(\sigma) - u(t) = \int_t^\sigma \varphi_q\Big(\int_s^\sigma a(r)f(u(r))\,\mathrm{d}r\Big)\,\mathrm{d}s. \tag{3.4}$$

Let $t = \xi$ in (3.3) and note that $u'(\sigma) = 0$. We have

$$\varphi_p(u'(\xi)) = \int_\xi^\sigma a(s)f(u(s))\,\mathrm{d}s.$$

With the boundary conditions (1.2), we have $\varphi_p(u(0)) = \frac{\beta}{\alpha}\varphi_p(u'(\xi))$. Then

$$u(0) = \varphi_q\Big(\frac{\beta}{\alpha}\int_\xi^\sigma a(s)f(u(s))\,\mathrm{d}s\Big). \tag{3.5}$$

Let $t = 0$ in (3.4). With (3.4) and (3.5), we can obtain

$$u(t) = \varphi_q\Big(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(u(r))\,\mathrm{d}r\Big) + \int_0^t \varphi_q\Big(\int_s^\sigma a(r)f(u(r))\,\mathrm{d}r\Big)\,\mathrm{d}s, 0 \leq t \leq \sigma.$$

Similarly, for $t \in (\sigma, 1)$, by integrating the equation (1.1) on $(\sigma, 1)$, we can obtain

$$u(t) = \varphi_q\Big(\frac{\delta}{\gamma}\int_\sigma^1 a(r)f(u(r))\,\mathrm{d}r\Big) + \int_t^1 \varphi_q\Big(\int_\sigma^s a(r)f(u(r))\,\mathrm{d}r\Big)\,\mathrm{d}s, \sigma \leq t \leq 1.$$

Therefore, the necessity holds. The proof of Lemma 3.2 is completed. □

We are now ready to apply the Leggett-Williams fixed point theorem to give sufficient conditions for the existence of at least three positive solutions to problem (1.1)–(1.2).

Theorem 3.1 *Assume that* (H_1), (H_2) *hold, and suppose that there exist positive constants* a, b, d *such that* $0 < a < \omega b < b < d$. *Also assume that* f *satisfies*

(H_3) $f(u) < \varphi_p(\frac{a}{\lambda_1})$, *for* $0 \leq u \leq a$;

(H_4) *One of the following conditions holds:*

(i) There exists constant $e > d$ *such that* $f(u) \leq \varphi_p(\frac{e}{\lambda_1})$ *for* $0 \leq u \leq e$ *or*

(ii) $\limsup_{u\to\infty} \frac{f(u)}{u^{p-1}} < \varphi_p(\frac{1}{\lambda_1})$.

(H_5) $f(u) > \varphi_p(\frac{2b}{\omega L})$, *for* $\omega b \leq u \leq d$.

Then the boundary value problem (1.1)–(1.2) has at least three positive solutions u_1, u_2, *and* u_3 *satisfying*

$$\|u_1\| < a, \quad \phi(u_2) > b, \quad \|u_3\| > a, \quad \text{and} \quad \phi(u_3) < b.$$

Proof We define the operator $T : P \to E$ given by

$$(Tu)(t) = \begin{cases} \varphi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(u(r))\,\mathrm{d}r\right) + \int_0^t \varphi_q\left(\int_s^\sigma a(r)f(u(r))\,\mathrm{d}r\right)\,\mathrm{ds}, 0 \leq \mathrm{t} \leq \sigma, \\ \varphi_q\left(\frac{\delta}{\gamma}\int_\sigma^1 a(r)f(u(r))\,\mathrm{d}r\right) + \int_t^1 \varphi_q\left(\int_\sigma^s a(r)f(u(r))\,\mathrm{d}r\right)\,\mathrm{ds}, \sigma \leq \mathrm{t} \leq 1. \end{cases} \tag{3.6}$$

Thus, it follows from Lemma 3.2 that operator T is well-defined. Since

$$(Tu)'(t)=\begin{cases}\varphi_q\left(\int_t^\sigma a(r)f(u(r))\,\mathrm{d}r\right)\ge 0, & 0\le t\le\sigma,\\ -\varphi_q\left(\int_\sigma^t a(r)f(u(r))\,\mathrm{d}r\right)\le 0, & \sigma\le t\le 1,\end{cases}\tag{3.7}$$

it is easy to see that the operator $(Tu)'$ is continuous monotone decreasing on $[0,1]$, and $(Tu)'(\sigma)=0$. Meanwhile, it follows from the definition of operator T that for each $u\in P, Tu\in E$ is nonnegative continuous. And with (3.6), (3.7), we can obtain

$$\alpha\varphi_p((Tu)(0))-\beta\varphi_p((Tu)'(\xi))=0,\quad \gamma\varphi_p((Tu)(1))+\delta\varphi_p((Tu)'(1))=0.$$

These show $Tu\in P$, i.e., $T(P)\subset P$, and $(Tu)(\sigma)=\|Tu\|$ with

$$\begin{aligned}&\varphi_q\Big(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(u(r))\,\mathrm{d}r\Big)+\int_0^\sigma\varphi_q\Big(\int_s^\sigma a(r)f(u(r))\,\mathrm{d}r\Big)\,\mathrm{d}s\\ &=\varphi_q\Big(\frac{\delta}{\gamma}\int_\sigma^1 a(r)f(u(r))\,\mathrm{d}r\Big)+\int_\sigma^1\varphi_q\Big(\int_\sigma^s a(r)f(u(r))\,\mathrm{d}r\Big)\,\mathrm{d}s.\end{aligned}$$

Furthermore, we can also obtain that operator $T:P\to P$ is completely continuous[11, Lemma 2.4], and it follows from Lemma 3.2 that each fixed point of T in P is a positive solution of problem (1.1)–(1.2).

We now show that the conditions of Theorem A are satisfied. Firstly, we define

$$\phi(u)=\frac{1}{2}(u(\omega)+u(1-\omega)),\quad \text{for all}\quad u\in P.\tag{3.8}$$

Obviously, $\phi(u)$ is concave functional, and we have $\phi(u)\le\|u\|$, for arbitrary $u\in P$. Now we choose $u\in\overline{P}_a$. Then $\|u\|\le a$ and assumption (H_3) yields $f(u(t))<\varphi_p(\frac{a}{\lambda_1})$, for $0\le t\le 1$. Thus, by Lemmas 2.1 and 3.2, we have

$$\begin{aligned}\|Tu\|=(Tu)(\sigma)&=\varphi_q\Big(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(u(r))\,\mathrm{d}r\Big)+\int_0^\sigma\varphi_q\Big(\int_s^\sigma a(r)f(u(r))\,\mathrm{d}r\Big)\,\mathrm{d}s\\ &\le\varphi_q\Big(\frac{\beta}{\alpha}\int_\xi^1 a(r)f(u(r))\,\mathrm{d}r\Big)+\int_0^1\varphi_q\Big(\int_s^1 a(r)f(u(r))\,\mathrm{d}r\Big)\,\mathrm{d}s\\ &<\frac{a}{\lambda_1}\Big[\varphi_q\Big(\frac{\beta}{\alpha}\int_\xi^1 a(r)\,\mathrm{d}r\Big)+\int_0^1\varphi_q\Big(\int_s^1 a(r)\,\mathrm{d}r\Big)\,\mathrm{d}s\Big]\\ &=\frac{a}{\lambda_1}\cdot\lambda_1=a.\end{aligned}$$

Hence, condition (C_2) of Theorem A is satisfied.

Secondly, we claim that condition (H_4) guarantees the existence of a number c with $c>d$ and

$$T:\overline{P}_c\to\overline{P}_c.\tag{3.9}$$

It is clear that if (i) holds, then as the proof above we immediately have (3.9) with $e=c$. Suppose now that (ii) is satisfied. Then there exist $l>0$ and $\varepsilon<\varphi_p(\frac{1}{\lambda_1})$ such that

$$\frac{f(u)}{\varphi_p(u)}\le\varepsilon,\quad\text{for}\quad u\ge l.\tag{3.10}$$

Let $M = \max_{u\in[0,l]} f(u)$. In view of (3.10), it is easy to see that

$$f(u) \le M + \varepsilon\varphi_p(u), \quad \text{for} \quad u \ge 0. \tag{3.11}$$

Now, assume that there exists a constant c such that

$$\varphi_p(c) > \max\{\varphi_p(d), M(\varphi_p(\frac{1}{\lambda_1}) - \varepsilon)^{-1}\}. \tag{3.12}$$

Then, for arbitrary $u \in \overline{P}_c$ and $t \in [0,1]$, from (3.11), (3.12) and Lemma 3.2, we can obtain

$$\begin{aligned}
\|Tu\| = (Tu)(\sigma) &= \varphi_q\Big(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(u(r))\,\mathrm{d}r\Big) + \int_0^\sigma \varphi_q\Big(\int_s^\sigma a(r)f(u(r))\,\mathrm{d}r\Big)\,\mathrm{d}s \\
&\le \varphi_q\Big(\frac{\beta}{\alpha}\int_\xi^1 a(r)f(u(r))\,\mathrm{d}r\Big) + \int_0^1 \varphi_q\Big(\int_s^1 a(r)f(u(r))\,\mathrm{d}r\Big)\,\mathrm{d}s \\
&\le \varphi_q\Big(\frac{\beta}{\alpha}\int_\xi^1 a(r)(M+\varepsilon\varphi_p(u(r)))\,\mathrm{d}r\Big) + \int_0^1 \varphi_q\Big(\int_s^1 a(r)(M+\varepsilon\varphi_p(u(r)))\,\mathrm{d}r\Big)\,\mathrm{d}s \\
&< \Big[\varphi_q\Big(\frac{\beta}{\alpha}\int_\xi^1 a(r)\,\mathrm{d}r\Big) + \int_0^1 \varphi_q\Big(\int_s^1 a(r)\,\mathrm{d}r\Big)\,\mathrm{d}s\Big]\varphi_q\Big(\varphi_p(c)(\varphi_p(\frac{1}{\lambda_1}) - \varepsilon) + \varepsilon\varphi_p(c)\Big) \\
&= \lambda_1 \cdot \frac{c}{\lambda_1} = c.
\end{aligned}$$

Hence, (3.9) holds.

We shall now show that condition (C_1) of theorem A is satisfied. For this, we suppose that

$$u(t) \equiv \frac{b+d}{2}.$$

Note that $\phi(u(t)) = \phi(\frac{b+d}{2}) = \frac{b+d}{2} > b$. Thus $\{u \in P(\phi,b,d) : \phi(u) > b\} \ne \emptyset$. Next, let $u \in P(\phi,b,d)$. Then $\phi(u) \ge b$, and so $b \le \|u\| \le d$. It follows from Lemma 3.1 that $\omega b \le \omega\|u\| \le u(t) \le d, t \in [\omega, 1-\omega]$, and condition ($H_5$) yields $f(u(t)) > \varphi_p(\frac{2b}{\omega L})$, for $\omega b \le u(t) \le d, t \in [\omega, 1-\omega]$. In the following, we shall discuss it from three respects.

(i) If $\sigma \in [\xi, \omega]$, by (H_5), (3.6)–(3.8) and Lemma 2.1, we have

$$\begin{aligned}
\phi(Tu) &= \frac{1}{2}\left(Tu(\omega) + Tu(1-\omega)\right) \ge Tu(1-\omega) \\
&= \varphi_q\Big(\frac{\delta}{\gamma}\int_\sigma^1 a(r)f(u(r))\,\mathrm{d}r\Big) + \int_{1-\omega}^1 \varphi_q\Big(\int_\sigma^s a(r)f(u(r))\,\mathrm{d}r\Big)\,\mathrm{d}s \\
&\ge \int_{1-\omega}^1 \varphi_q\Big(\int_\sigma^s a(r)f(u(r))\,\mathrm{d}r\Big)\,\mathrm{d}s \\
&> \int_{1-\omega}^1 \varphi_q\Big(\int_\omega^{1-\omega} a(r)\varphi_p(\frac{2b}{\omega L})\,\mathrm{d}r\Big)\,\mathrm{d}s \\
&> \omega\varphi_q\Big(\int_\omega^{1-\omega} a(r)\,\mathrm{d}r\Big)\frac{2b}{\omega L} \ge 2b > b.
\end{aligned}$$

(ii) If $\sigma \in [\omega, 1-\omega]$, then for arbitrary $u \in P(\phi,b,d)$, similarly we have

$$\begin{aligned}
2\phi(Tu) &= (Tu(\omega) + Tu(1-\omega)) \\
&\ge \int_0^\omega \varphi_q\Big(\int_s^\sigma a(r)f(u(r))\,\mathrm{d}r\Big)\,\mathrm{d}s + \int_{1-\omega}^1 \varphi_q\Big(\int_\sigma^s a(r)f(u(r))\,\mathrm{d}r\Big)\,\mathrm{d}s
\end{aligned}$$

$$\geq \int_0^{\omega} \varphi_q\Big(\int_{\omega}^{\sigma} a(r)f(u(r))\,\mathrm{d}r\Big)\,\mathrm{d}s + \int_{1-\omega}^{1} \varphi_q\Big(\int_{\sigma}^{1-\omega} a(r)f(u(r))\,\mathrm{d}r\Big)\,\mathrm{d}s$$
$$> \omega\Big(\varphi_q\Big(\int_{\omega}^{\sigma} a(r)\varphi_p(\frac{2b}{\omega L})\,\mathrm{d}r\Big) + \varphi_q\Big(\int_{\sigma}^{1-\omega} a(r)\varphi_p(\frac{2b}{\omega L})\,\mathrm{d}r\Big)\Big)$$
$$> \omega\Big(\varphi_q\Big(\int_{\omega}^{\sigma} a(r)\,\mathrm{d}r\Big) + \varphi_q\Big(\int_{\sigma}^{1-\omega} a(r)\,\mathrm{d}r\Big)\Big)\frac{2b}{\omega L} \geq 2b.$$

(iii) If $\sigma \in [1-\omega, 1]$, then for arbitrary $u \in P(\phi, b, d)$, similarly we have

$$\begin{aligned}\phi(Tu) &= \frac{1}{2}\,(Tu(\omega) + Tu(1-\omega)) \geq Tu(\omega)\\ &\geq \int_0^{\omega} \varphi_q\Big(\int_{s}^{\sigma} a(r)f(u(r))\,\mathrm{d}r\Big)\,\mathrm{d}s\\ &> \int_0^{\omega} \varphi_q\Big(\int_{\omega}^{1-\omega} a(r)\varphi_p(\frac{2b}{\omega L})\,\mathrm{d}r\Big)\,\mathrm{d}s\\ &> \omega\varphi_q\Big(\int_{\omega}^{1-\omega} a(r)\,\mathrm{d}r\Big)\frac{2b}{\omega L} \geq 2b > b.\end{aligned}$$

Hence, for arbitrary $u \in P(\phi, b, d), \phi(Tu) > b$. Thus, condition ($\mathrm{C}_1$) of Theorem A is satisfied.

Finally, we show that condition (C_3) of Theorem A is also satisfied. For this, let $d \geq \frac{b}{\omega}$, and let $u \in P(\phi, b, c)$ be such that $\|Tu\| > d$. Then, it follows from Lemma 3.1 that

$$\phi(Tu) = \frac{1}{2}\,(Tu(\omega) + Tu(1-\omega)) \geq \omega\|Tu\| > \omega d \geq b.$$

Consequently, condition (C_3) of Theorem A is satisfied.

Thus, it now follows from Theorem A that the boundary value problem (1.1)–(1.2) has at least three positive solutions u_1, u_2 and u_3 such that

$$\|u_1\| < a, \quad \phi(u_2) > b, \quad \|u_3\| > a, \quad \text{and} \quad \phi(u_3) < b.$$

The proof of Theorem 3.1 is completed. □

4. Existence of three positive solutions of problem (1.1)–(1.3)

Now we discuss problem (1.1)–(1.3). The method is just similar to that we have done in Section 3, so we omit the proof of main result of this section. We define the cone $P = \{u \in E : u(t) \geq 0, \alpha\varphi_p(u(0)) - \beta\varphi_p(u'(0)) = 0, \gamma\varphi_p(u(1)) + \delta\varphi_p(u'(\eta)) = 0,\ u(t)$ is concave function on $[0,1]\}$.

Lemma 4.1 *Suppose that conditions (H_1), (H_2) hold. Then $u(t) \in E\bigcap C^2(0,1)$ is a solution of boundary value problem (1.1)–(1.3), if and only if $u(t) \in E$ is a solution of the following integral equation:*

$$u(t) = \begin{cases} \varphi_q\left(\frac{\beta}{\alpha}\int_0^{\sigma} a(r)f(u(r))\,\mathrm{d}r\right) + \int_0^t \varphi_q\left(\int_s^{\sigma} a(r)f(u(r))\,\mathrm{d}r\right)\,\mathrm{d}s, 0 \leq t \leq \sigma,\\ \varphi_q\left(\frac{\delta}{\gamma}\int_{\sigma}^{\eta} a(r)f(u(r))\,\mathrm{d}r\right) + \int_t^1 \varphi_q\left(\int_{\sigma}^{s} a(r)f(u(r))\,\mathrm{d}r\right)\,\mathrm{d}s, \sigma \leq t \leq 1,\end{cases}$$

where $\sigma \in [0, \eta]$ and $u'(\sigma) = 0$.

Proof The proof is similar to Lemma 3.2, and we omit it here. □

Theorem 4.1 *Assume that* (H_1),(H_2) *hold, and suppose that there exist positive constants* a, b, d *such that* $0 < a < \omega b < b < d$. *Also assume that* f *satisfies*

(H_3) $f(u) < \varphi_p(\frac{a}{\lambda_2})$, *for* $0 \le u \le a$;

(H_4) *One of the following conditions holds:*

(i) *There exists constant* $e > d$ *such that* $f(u) \le \varphi_p(\frac{e}{\lambda_2})$ *for* $0 \le u \le e$ *or*

(ii) $\limsup_{u\to\infty} \frac{f(u)}{u^{p-1}} < \varphi_p(\frac{1}{\lambda_2})$.

(H_5) $f(u) > \varphi_p(\frac{2b}{\omega L})$, *for* $\omega b \le u \le d$.

Then the boundary value problem (1.1)–(1.3) has at least three positive solutions u_1, u_2 *and* u_3 *satisfying*

$$\|u_1\| < a, \quad \phi(u_2) > b, \quad \|u_3\| > a, \quad \text{and} \quad \phi(u_3) < b.$$

Proof The proof is similar to Theorem 3.1 and we omit it here. □

5. Application

In order to illustrate our results, we present an example as follows.

Example Consider the the following p-Laplacian differential equation

$$(\varphi_p(u'))' + t^{-\frac{1}{2}} f(u) = 0, \quad 0 < t < 1, \tag{5.1}$$

subject to one of the following boundary conditions

$$\alpha\varphi_p(u(0)) - \beta\varphi_p(u'(\xi) = 0, \quad \gamma\varphi_p(u(1)) + \delta\varphi_p(u'(1)) = 0, \tag{5.2}$$

or

$$\alpha\varphi_p(u(0)) - \beta\varphi_p(u'(0)) = 0, \quad \gamma\varphi_p(u(1)) + \delta\varphi_p(u'(\eta) = 0, \tag{5.3}$$

where

$$p = \frac{3}{2}, \alpha = 2, \beta = \gamma = 1, \delta = 2, \xi = \frac{1}{4}, \eta = \frac{1}{2}, \omega = \frac{1}{4}, a(t) = t^{-\frac{1}{2}}$$

and

$$f(u) = \begin{cases} \frac{u^2}{2}, & 0 \le u \le 40, \\ 792 + \frac{4}{5}(50 - u), & 40 < u \le 50, \\ \frac{u + 15790}{2\sqrt{2u}}, & u > 50. \end{cases}$$

Then (5.1)–(5.2) or (5.1)–(5.3) has at least three positive solutions.

Proof In this example we have

$$L = \min_{\frac{1}{4} \le t \le \frac{3}{4}} A(t) = \min_{\frac{1}{4} \le t \le \frac{3}{4}} \Big(\varphi_q\Big(\int_{\frac{1}{4}}^{t} t^{-\frac{1}{2}}\,\mathrm{d}t\Big) + \varphi_q\Big(\int_{t}^{\frac{3}{4}} t^{-\frac{1}{2}}\,\mathrm{d}t\Big)\Big) = 2 - \sqrt{3},$$

$$\lambda_1 = \varphi_q\Big(\frac{1}{2}\int_{\frac{1}{4}}^{1} a(r)\,\mathrm{d}r\Big) + \int_0^1 \varphi_q\Big(\int_s^1 a(r)\,\mathrm{d}r\Big)\,\mathrm{d}s = \frac{11}{12},$$

$$\lambda_2 = \varphi_q\Big(2\int_0^{\frac{1}{2}} a(r)\,\mathrm{d}r\Big) + \int_0^1 \varphi_q\Big(\int_0^s a(r)\,\mathrm{d}r\Big)\,\mathrm{d}s = 10.$$

And obviously, condition (H_1), (H_2) are satisfied. Let $a=1, b=40, d=50$. Then we have

$$\begin{aligned} f(u) &\le \max_{0\le u\le 1} f(u)=\frac{1}{2}<\varphi_p(\frac{a}{\lambda_1})=\varphi_p(\frac{1}{\lambda_1}) \\ &=\varphi_p(\frac{12}{11})=(\frac{12}{11})^{\frac{1}{2}}, \quad \text{for} \quad 0\le u\le 1 \\ f(u) &> \min_{10\le u\le 50} f(u)=50>\varphi_p(\frac{2b}{\omega L}) \\ &=\varphi_p(320(2+\sqrt{3}))=(320(2+\sqrt{3}))^{\frac{1}{2}}, \quad \text{for} \quad 10\le u\le 50 \end{aligned}$$

and

$$\limsup_{u\to\infty}\frac{f(u)}{u^{\frac{1}{2}}}=\frac{\sqrt{2}}{4}<\varphi_p(\frac{1}{\lambda_1}).$$

Consequently, conditions (H_3), (H_4)(ii) and (H_5) of Theorem 3.1 are satisfied. Then by Theorem 3.1, the singular boundary value problem with p-Laplacian (5.1)–(5.2) has at least three positive solutions such that

$$\|u_1\|<a, \quad \phi(u_2)>b, \quad \|u_3\|>a, \quad \text{and} \quad \phi(u_3)<b.$$

Similarly, we can also show that problem (4.1)–(4.3) has at least three positive solutions.

Remark We note that in the above example, $a(t)$ has singularity at $t=0$.

References

[1] LIU Yuji, GE Weigao. *Multiple positive solutions to a three-point boundary value problem with p-Laplacian* [J]. J. Math. Anal. Appl., 2003, **277**(1): 293–302.

[2] MA Dexiang, HAN Jianxin, CHEN Xuegang. *Positive solution of three-point boundary value problem for the one-dimensional p-Laplacian with singularities* [J]. J. Math. Anal. Appl., 2006, **324**(1): 118–133.

[3] LI Jianli, SHEN Jianhua. *Existence of three positive solutions for boundary value problems with p-Laplacian* [J]. J. Math. Anal. Appl., 2005, **311**(2): 457–465.

[4] LIU Bing. *Positive solutions of three-point boundary value problems for the one-dimensional p-Laplacian with infinitely many singularities* [J]. Appl. Math. Lett., 2004, **17**(6): 655–661.

[5] MA Dexiang, GE Weigao. *Existence and iteration of positive pseudo-symmetric solutions for a three-point second-order p-Laplacian BVP* [J]. Appl. Math. Lett., 2007, **20**(12): 1244–1249.

[6] LIU Bing. *Positive solutions of singular three-point boundary value problems for the one-dimensional p-Laplacian* [J]. Comput. Math. Appl., 2004, **48**(5-6): 913–925.

[7] HE Xiaoming, GE Weigao. *Twin positive solutions for the one-dimensional p-Laplacian boundary value problems* [J]. Nonlinear Anal., 2004, **56**(7): 975–984.

[8] HE Xiaoming. *Double positive solutions of a three-point boundary value problem for the one-dimensional p-Laplacian* [J]. Appl. Math. Lett., 2004, **17**(8): 867–873.

[9] LI Zhiyan, YAN Shulin, GE Weigao. *Multiple positive solutions to a nonlinear two-point boundary value problem with p-Laplacian* [J]. J. Math. Res. Exposition, 2006, **26**(3): 480–488.

[10] AVERY R, HENDERSON J. *Existence of three positive pseudo-symmetric solutions for a one-dimensional p-Laplacian* [J]. J. Math. Anal. Appl., 2003, **277**(2): 395–404.

[11] AVERY R I, CHYAN C J, Henderson J. *Twin solutions of boundary value problems for ordinary differential equations and finite difference equations* [J]. Comput. Math. Appl., 2001, **42**(3-5): 695–704.

[12] SU Hua, WEI Zhongli, XU Fuyi. *The existence of positive solutions for nonlinear singular boundary value system with p-Laplacian* [J]. Appl. Math. Comput., 2006, **181**(2): 826–836.

[13] LEGGETT R W, WILLIAMS L R. *Multiple positive fixed points of nonlinear operators on ordered Banach spaces* [J]. Indiana Univ. Math. J., 1979, **28**(4): 673–688.