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The Crossing Numbers of Cartesian Products of Stars with a 5-Vertex Graph

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Abstract In this paper, we compute the crossing number of a specific graph H_n, and then by contraction, we obtain the conclusion that $\mathrm{cr}(G_{13} \times S_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$. The result fills up the blank of the crossing numbers of Cartesian products of stars with all 5-vertex graphs presented by Marián Klešč.

Keywords graph; drawing; crossing number; Cartesian products; star.

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1. Introduction

For definitions not explained in this paper, readers are referred to [1]. Let G be a simple graph with vertex set V and edge set E. A drawing of an (undirected) graph $G = (V, E)$ is a mapping f that assigns to each vertex in V a distinct point in the plane and to each edge uv in E a continuous arc (i.e, a homeomorphic image of a closed interval) connecting $f(u)$ and $f(v)$, not passing through the image of any other vertex. As for the drawing we need the additional assumptions: (i) No three edges have an interior point in common; (ii) If two edges share an interior point p, then they cross at p; (iii) Any two edges of a drawing have only a finite number of crossings (common interior points).

The crossing number, $\mathrm{cr}(G)$, of a graph G is the minimum number of edge crossings in any drawing of G in the plane. Let ϕ be a drawing of graph G. We denote the number of crossings in ϕ by $\mathrm{cr}_\phi(G)$. The Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \times G_2) = \{(u_i, v_j)(u_h, v_k) | u_i = u_h$ and $u_j v_k \in E(G_2)$ or $v_j = v_k$ and $u_i u_h \in E(G_1)\}$.

Generally, computing the crossing number of graphs is an NP-complete problem[2,3]. At present, the classes of graphs whose crossing numbers have been determined are very scarce, and there are only some classes of special graphs whose crossing numbers are known. For example, these include the complete graph K_n[4] and the complete bipartite graph $K_{m,n}$ for small m and n[5], certain generalized Peterson graphs[6] and cyclic graphs[7] and so on.

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Recently, the crossing numbers of Cartesian products graphs become more interested. Let C_n and P_n be the cycle and the path with n edges, and S_n the star $K_{1,n}$. The crossing numbers of the Cartesian products of all 4-vertex graphs with C_n, P_n and S_n are determined[8−11]. There are several known exact results on the crossing number of Cartesian products of C_n, P_n and S_n with 5-vertex graphs. In [12], Marián gave a description of the Cartesian products of all the 5-vertex graphs with P_n, C_n, S_n, where some results have been given, but in the blank of this table, the crossing numbers have not been determined. In this paper, we obtain the crossing number of $G_{13} \times S_n$ in Marián's table which was in blank (the graph G_{13} is shown in Figure 1).

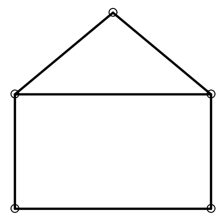

Figure 1 G_{13}

2. Some lemmas

First, we give some useful results.

Lemma 1 *Let A, B, C be mutually disjoint subsets of E. Then*

$$\mathrm{cr}_\phi(A \bigcup B) = \mathrm{cr}_\phi(A) + \mathrm{cr}_\phi(B) + \mathrm{cr}_\phi(A, B);$$

$$\mathrm{cr}_\phi(A, B \bigcup C) = \mathrm{cr}_\phi(A, B) + \mathrm{cr}_\phi(A, C), \tag{1}$$

where ϕ is a good drawing of E.

Proof By the definition, it is easy to obtain the conclusion.

On the crossing numbers of the complete bipartite graphs $K_{m,n}$, Kleitmain obtained the following result in [4].

Lemma 2 If $K_{5,n}$ *is a complete bipartite graphs, then*

$$\mathrm{cr}(K_{5,n}) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor. \tag{2}$$

Let H be a graph isomorphic to G_{13}. Consider a graph G_H obtained by joining all vertices of H to five vertices of a connected graph G such that every vertex of H will only be adjacent to exactly one vertex of G. Let G_H^* be the graph obtained from G_H by contracting the edges of H.

Lemma 3 *The crossing number of G_H^* is no more than the crossing number of G_H, i.e.,*

$$\mathrm{cr}(G_H^*) \leq \mathrm{cr}(G_H).$$

Proof Let ϕ be an optimal drawing of G_H. Since the plane is a normal space, for an edge e

of the drawing ϕ there is an open set M_e homeomorphic to the open disk such that M_e contains e, together with ends of edges incident with endpoints of e, and open arcs of edges which are crossing e (see Figure 2(a)). All remaining edges of ϕ are disjoint with M_e.

Let x denote the number of crossings of e in ϕ. If we draw in M_e two edges e_1 and e_2 instead of e, these two edges have $2x$ crossings (see Figure 2(b)).

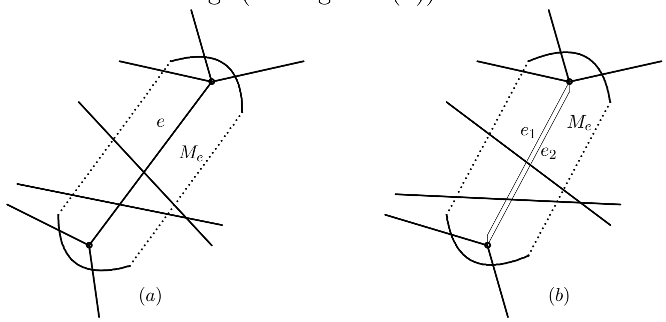

Figure 2 Open disks which contain edge

The subgraph H has six edges and let x_1, x_2, x_3, x_4, x_5 and x_6 denote the numbers of crossing on the edges of H.

Let $x_1 + x_2 \leq x_4 + x_6$. Figure 3 shows that H can be contracted to the vertex b without

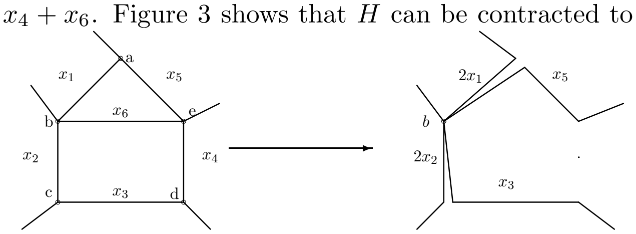

Figure 3 H is contracted to the vertex b

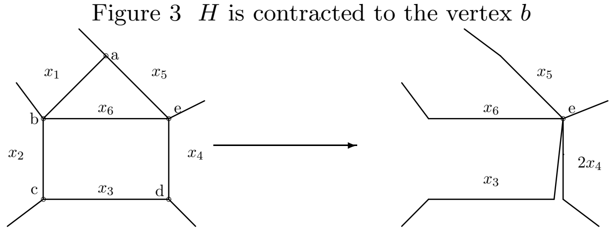

Figure 4 H is contracted to the vertex e

increasing the number of crossings.

Let $x_4 \leq x_1 + x_2$. Figure 4 shows that H can be contracted to the vertex e without increasing the number of crossing. This completes the proof, because for nonnegative integers the system of inequalities

$$x_1 + x_2 > x_4 + x_6,$$
$$x_4 > x_1 + x_2$$

holds only for $x_6 < 0$. This is impossible because x_6 is a nonnegative integer. The Lemma 3 has been proved. $\square$

3. The main theorem and proof

Let us denote by H_n the graph obtained by $G_{13} \cup K_{5,n}$, where the five vertices of degree n in $K_{5,n}$, and the vertices of G_{13} are the same. Let, for $i = 1, 2, \ldots, n$, t_i denote the vertex which belongs to one of the n of $K_{5,n}$, and T^i denote the subgraph of $K_{5,n}$ which consists of the five edges incident with t_i. Thus, we have

$$H_n = G_{13} \cup K_{5,n} = G_{13} \cup (\bigcup_{i=1}^{n} T^i). \tag{3}$$

Theorem 1 *For $n \geq 1$, we have* $\mathrm{cr}(H_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$.

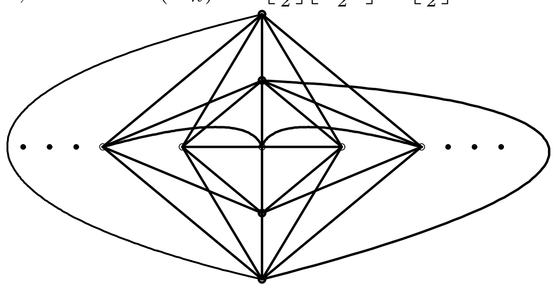

Figure 5 A good drawing of H_n

Proof The drawing in Figure 5 shows that

$$\mathrm{cr}(H_n) \leq \mathrm{cr}(K_{5,n}) + \lfloor \frac{n}{2} \rfloor = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$$

and that the theorem is true if equality holds. We prove the reverse inequality by induction on n. The case $n = 1$ and 2 are trivial. Suppose now that for $n \geq 3$

$$\mathrm{cr}(H_{n-2}) \geq 4\lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor + \lfloor \frac{n-2}{2} \rfloor \tag{4}$$

and consider such a drawing ϕ that

$$\mathrm{cr}_{\phi}(H_n) < 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor. \tag{5}$$

Our next analysis depends on whether or not there are different subgraphs T^i and T^j that do not cross each other in each good drawing ϕ.

Case 1 We suppose that every pair of T^i cross each other. Using (1), (2) and (3), we have

$$\begin{aligned}\mathrm{cr}_{\phi}(H_n) &= \mathrm{cr}_{\phi}(K_{5,n}) + \mathrm{cr}_{\phi}(G_{13}) + \mathrm{cr}_{\phi}(K_{5,n}, G_{13}) \\ &\geq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + cr_{\phi}(G_{13}) + cr_{\phi}(\bigcup_{i=1}^{n} T^i, G_{13}).\end{aligned} \tag{6}$$

This, together with our assumption (5), implies that

$$\mathrm{cr}_{\phi}(\bigcup_{i=1}^{n} T^i, G_{13}) < \lfloor \frac{n}{2} \rfloor.$$

So, we can see that in ϕ, there are no more than $\lfloor \frac{n}{2} \rfloor$ subgraphs T^i which cross G_{13}, and at least $\lfloor \frac{n}{2} \rfloor$ subgraphs T^i which do not cross G_{13}. Now, we consider T^i, which satisfy $\mathrm{cr}_{\phi}(G_{13}, T^i) = 0$.

Without loss of generality, we denote T^1, $\mathrm{cr}_\phi(G_{13}, T^1) = 0$.

Consider the subgraphs ϕ^* and ϕ^{**} of G_{13} and $G_{13} \cup T^1$, respectively, induced by ϕ. Since $\mathrm{cr}_\phi(G_{13}, T^1) = 0$, the subdrawing ϕ^* divides the plane in such a way that all vertices are on the boundary of one region. Thus, G_{13} have the following cases (see Figure 6).

(i) G_{13} do not intersect, as in case (a).

(ii) The quadrangle intersect, but the triangle do not cross with quadrangle, as in case (b).

(iii) The quadrangle do not intersect, but the triangle cross with quadrangle, as in cases (c) and (d).

(iv) The quadrangle intersect, and the triangle cross with quadrangle, as in cases (e),(f), and (g). So $G_{13} \cup T^1$ have the following cases (see Figure 7).

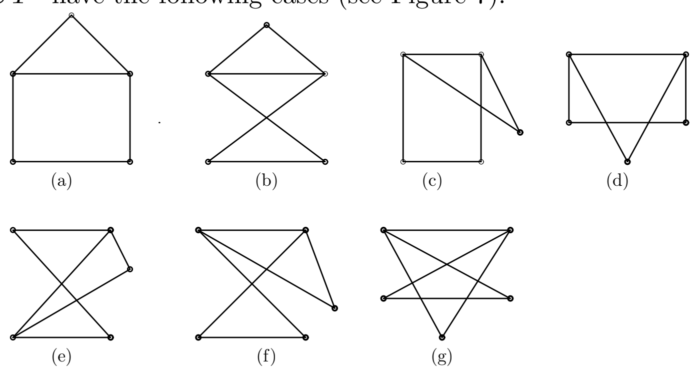

Figure 6 All possibilities of the subdrawing ϕ^*

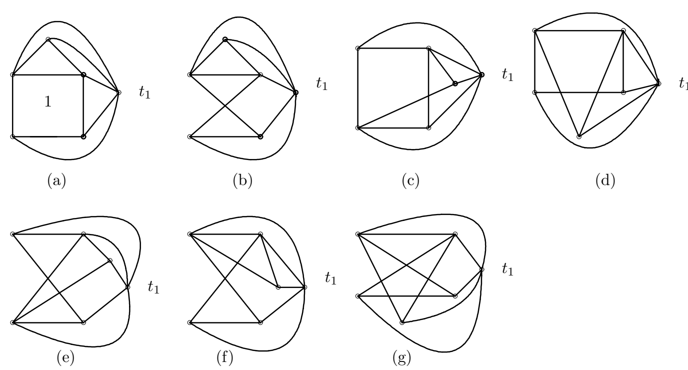

Figure 7 All possibilities of the subdrawing ϕ^{**}

Now consider the subdrawing of $T^i \cup G \cup T^1$ for some $i \in \{2, 3, \ldots, n\}$.

In case (a), except the region 1, no matter which region t_i lies in, using $\mathrm{cr}_\phi(T^i, T^1) \geq 1$, we have $\mathrm{cr}_\phi(T^i, G_{13} \cup T^1) \geq 3$. When t_i lies in the region 1, we have $\mathrm{cr}_\phi(T^i, G_{13} \cup T^1) \geq 2$, where $\mathrm{cr}_\phi(T^i, G_{13}) \geq 1$. We suppose the sum of vertex t_i lying in the region 1 is x. So we have

$$\mathrm{cr}_\phi\Big(\bigcup_{i=2}^{n} T_i, G_{13} \cup T^1\Big) \geq 2x + 3(n - 1 - x). \tag{7}$$

Using (1), (2), (3), (7), and the fact that x is no more than $\lfloor \frac{n}{2} \rfloor$, we have

$$\mathrm{cr}_\phi(H_n) = \mathrm{cr}_\phi(K_{5,n-1}) + \mathrm{cr}_\phi(G_{13} \cup T^1) + \mathrm{cr}_\phi(G_{13} \cup T^1, K_{5,n-1})$$

$$= \mathrm{cr}_\phi(K_{5,n-1}) + \mathrm{cr}_\phi(G_{13} \cup T^1) + \mathrm{cr}_\phi(G_{13} \cup T^1, \bigcup_{i=2}^{n} T_i)$$
$$\geq 4\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor + 2x + 3(n-1-x) \geq 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + \lfloor\frac{n}{2}\rfloor.$$

In cases (b), (c), (d), (e), (f), (g), no matter what region t_i lies in, using $cr_\phi(T^i, T^1) \geq 1$, we have $\mathrm{cr}_\phi(T^i, G_{13} \cup T^1) \geq 3$. So using (1) (2) (3) gives

$$\begin{aligned}
\mathrm{cr}_\varphi(H_n) &= \mathrm{cr}_\phi(K_{5,n-1}) + \mathrm{cr}_\phi(G_{13} \cup T^1) + \mathrm{cr}_\phi(G_{13} \cup T^1, K_{5,n-1}) \\
&= \mathrm{cr}_\phi(K_{5,n-1}) + \mathrm{cr}_\phi(G_{13} \cup T^1) + \mathrm{cr}_\phi(G_{13} \cup T^1, \bigcup_{i=2}^{n} T^i) \\
&\geq 4\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor + 3(n-1) \\
&\geq 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + \lfloor\frac{n}{2}\rfloor
\end{aligned}$$

which contradicts our assumption of (5).

Case 2 There are at least two different subgraphs T^i and T^j that do not cross each other in ϕ. Without loss of generality, we may assume that $\mathrm{cr}_\phi(T^1, T^2) = 0$. The subgraph $G_{13} \cup T^1 \cup T^2$ of H_n contains a subgraphs $K_{3,3}$, whose crossing number is 1. As $\mathrm{cr}(K_{3,5}) = 4$, for all $i, i = 3, 4, \ldots, n$, $\mathrm{cr}_\phi(T^i, T^1 \cup T^2) \geq 4$. This implies that

$$\mathrm{cr}_\phi(H_{n-2}, T^1 \cup T^2) \geq 4(n-2) + 1 = 4n - 7. \tag{8}$$

Since $H_n = H_{n-2} \cup (T^1 \cup T^2)$, using (1),(8) and (4), we have

$$\begin{aligned}
\mathrm{cr}_\phi(H_n) &= \mathrm{cr}_\phi(H_{n-2}) + \mathrm{cr}_\phi(T^1 \cup T^2) + \mathrm{cr}_\phi(H_{n-2}) + cr_\phi(H_{n-2}, (T^1 \cup T^2)) \\
&\geq 4\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor + \lfloor\frac{n-2}{2}\rfloor + 4n - 7 \\
&\geq 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + \lfloor\frac{n}{2}\rfloor.
\end{aligned}$$

This contradicts (5) and the proof is completed. □

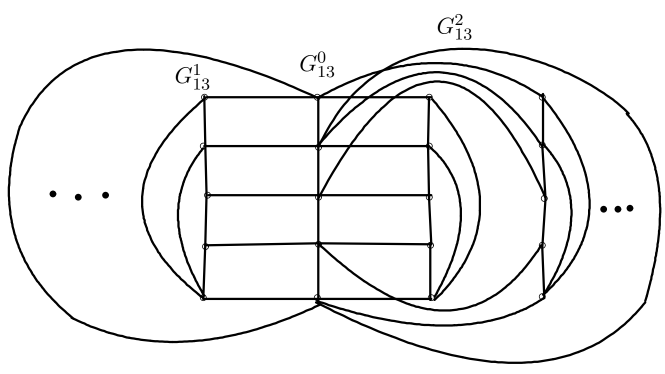

Figure 8 An optimal drawing of $G_{13} \times S_n$

Theorem 2 *For $n \geq 1$, we have* $\mathrm{cr}(G_{13} \times S_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$.

Proof The drawing in Figure 8 shows that $\mathrm{cr}(G_{13} \times S_n) \leq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$. Then, we only prove the reverse inequality for each good drawing ϕ. Use Lemma 3, and contract the copy of $G_{13}^i (i = 1, 2, \ldots, n)$ to the vertex t_i. Then, we obtain a graph isomorphic to H_n. So, for each good drawing ϕ, we have $\mathrm{cr}_\phi(G_{13} \times S_n) \geq \mathrm{cr}(H_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$. This completes the proof. □

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