Journal of Mathematical Research & Exposition
July, 2009, Vol. 29, No. 4, pp. 599–606
DOI:10.3770/j.issn:1000-341X.2009.04.004
Http://jmre.dlut.edu.cn

Locally Inverse Semigroups with Inverse Transversals

SHAO Yong[1], *ZHAO Xian Zhong*[1,2]
(1. Department of Mathematics, Northwest University, Shaanxi 710127, China;
2. College of Mathematics and Information Science, Jiangxi Normal University, Jiangxi 330022, China)
(E-mail: yongshaomath@126.com)

Abstract Let S be a locally inverse semigroup with an inverse transversal S°. In this paper, we construct an amenable partial order on S by an R-cone. Conversely, every amenable partial order on S can be constructed in this way. We give some properties of a locally inverse semigroup with a Clifford transversal. In particular, if S is a locally inverse semigroup with a Clifford transversal, then there is an order-preserving bijection from the set of all amenable partial orders on S to the set of all R-cones of S.

Keywords locally inverse semigroup; inverse transversal; amenable partial order; order-preserving bijection.

Document code A
MR(2000) Subject Classification 31B05; 47B35
Chinese Library Classification O174.56

1. Introduction and preliminary

A semigroup S is said to be a p.o. semigroup, or to be partially ordered, if it admits a compatible ordering $\leq$; that is, $\leq$ is a partial order on S such that

$$(\forall a,\, b \in S,\, x \in S^1)\ a \leq b \Longrightarrow xa \leq xb \ \text{ and } \ ax \leq bx.$$

Let S be a regular semigroup with set $E(S)$ of idempotent elements. As usual, we use $\preceq$ to denote the natural partial order on S. That is, for any $a,\, b \in S$,

$$a \preceq b \Longleftrightarrow (\exists e,\, f \in E(S))\, a = eb = bf.$$

By Corollary II.4.2 in [1], the natural partial order $\preceq$ on S is compatible with the multiplication if and only if S is a locally inverse semigroup. Thus a locally inverse semigroup equipped with the natural partial order is a p.o. semigroup. Particularly, an inverse semigroup is a p.o. semigroup under the natural partial order.

The amenable partial orders on an inverse semigroup were introduced and studied by McAlister in [3]. We now give this concept in the following definition.

Definition 1.1[3] *Let $(S, \cdot, \leq)$ be a p.o. inverse semigroup. The partial order $\leq$ is said to be*

Received date: 2007-09-08; **Accepted date**: 2008-01-04
Foundation item: the National Natural Science Foundation of China (No. 10471112); the Natural Science Foundation of Education Committee of Shaanxi Province (No. 07JK413).

left(right) amenable if it coincides with $\preceq$ on idempotents and for every a, $b \in S$, $a \leq b$ implies $a^{-1}a \preceq b^{-1}b\,(aa^{-1} \preceq bb^{-1})$. If $\leq$ is both left amenable and right amenable on S, then $\leq$ is said to be amenable and S is called an amenable p.o. inverse semigroup.

It is easy to see that the natural partial order on inverse semigroup is amenable. For an inverse semigroup S, McAlister in [3] introduced the notion of a cone as the full subsemigroup Q of the subsemigroup $R(E(S)) = \{x \in S|\,(\forall e \in E(S)),\, exe = xe\}$ with the properties that $Q \bigcap Q^{-1} = E(S)$ and $xQx^{-1} \subseteq Q$ for all $x \in S$. He showed that there exists an order-preserving bijection between the set of cones in S and the set of left amenable partial orders on S.

Blyth and Almeida generalized (left) amenable partial orders on an inverse semigroup to a regular semigroup with an inverse transversal in [4]. Let S be a regular semigroup, for any $a \in S$, and let $V(a)$ denote the set of all inverses of a. An inverse transversal of a regular semigroup S is an inverse subsemigroup S° with the property that $|S^\circ \bigcap V(a)| = 1$ for every a in S. The unique inverse of a in $S^\circ \bigcap V(a)$ is written by a° and $(a^\circ)^\circ$ is written by $a^{\circ\circ}$. The set of idempotents in S° is denoted by $E(S^\circ)$. We formulate the following definition.

Definition 1.2[4] *Let $(S, \cdot, \leq)$ be a p.o. regular semigroup with an inverse transversal S°. If $\leq$ coincides with $\preceq$ on idempotents and $\leq$ has the following property*

$$(\forall a,\, b \in S)\;\; a \leq b \Longrightarrow a^\circ a \preceq b^\circ b,$$

then $\leq$ is said to be left amenable on S. Dually, if $a \leq b$ implies $aa^\circ \preceq bb^\circ$, then $\leq$ is said to be right amenable on S. If $\leq$ is both left amenable and right amenable on S, then $\leq$ is said to be amenable and S is called an amenable p.o. regular semigroup with an inverse transversal S°.

Let $(S, \cdot)$ be a regular semigroup with an inverse transversal S°. Then Blyth and Almeida pointed out in [5] that S satisfies the following equalities.

$$\begin{aligned}(\forall a,\, b \in S)\,(ab)^\circ = (a^\circ ab)a^\circ = b^\circ(ab^\circ b)^\circ = b^\circ(a^\circ abb^\circ)^\circ,\\ (a^\circ b)^\circ = b^\circ a^{\circ\circ},\, (ab^\circ)^\circ = b^{\circ\circ}a^\circ.\end{aligned} \tag{1}$$

According to Blyth and Almeida in [5], if S is locally inverse, then

$$(\forall a,\, b,\, c \in S) \quad a^\circ bc^\circ = a^\circ b^{\circ\circ}c^\circ. \tag{2}$$

Suppose that S is a locally inverse semigroup with an inverse transversal S°. Then Blyth and Almeida stated in [5] that the two subsets of $E(S)$

$$\Lambda = \{x^\circ x|\, x \in S\}, \quad I = \{xx^\circ|\, x \in S\}$$

are right normal subband and left normal subband of $E(S)$, respectively. Consider the following two subsets of S

$$L = \{xx^\circ x^{\circ\circ}|x \in S\},\, R = \{x^{\circ\circ}x^\circ x|x \in S\}.$$

Then, the following statements were given in [5].

(α) L is a left normal orthodox subsemigroup of S and $I = \{xx^\circ|\, x \in S\}$ is the set of all idempotents of L;

(β) R is a right normal orthodox subsemigroup of S and $\Lambda = \{x^\circ x | x \in S\}$ is the set of all idempotents of R;

(γ) $L \bigcap R = S^\circ$, $\Lambda \bigcap I = E(S^\circ)$;

(δ) $S^\circ S \subseteq R$, $SS^\circ \subseteq L$.

Let

$$\Lambda^* = \{x \in S | (\forall l \in \Lambda)\, lxl = xl\},\ I^* = \{x \in S | (\forall r \in I)\, rxr = rx\}.$$

It is easy to see that $\Lambda^*(I^*)$ is a subsemigroup of S and $\Lambda = E(\Lambda^*)(I = E(I^*))$, Furthermore, Blyth and Alemida proved that Λ^* is a subsemigrup of R containing Λ and I^* is a subsemigroup of L containing I (see Theorem 3 in [5]), and after that they introduced the concepts of R-cone and L-cone of S, which generalized the notion of cone in an inverse semigroup. We have the following definition.

Definition 1.3[5] *Let S be a locally inverse semigroup with an inverse transversal S° and Q be a non-empty subset of S. If*

(C1) Q is a full subsemigroup of $\Lambda^*(\Lambda \subseteq Q)$;

(C2) $Q \bigcap Q^\circ = E(S^\circ)$;

(C3) $(\forall x \in R)\, xQx^\circ \subseteq Q$,

then Q is called an R-cone. Dually, we can consider the subset $I^* = \{x \in S | (\forall r \in I)\, rxr = rx\}$ and formulate the L-cone.

It is known that an amenable partial order on S can be constructed by an R-cone P and an L-cone Q. Conversely, every amenable partial order on S can be obtained in this way (see Theorems 8 and 10 in [5]).

In this paper, we give a new characterization of amenable partial order on S by constructing amenable partial order on S by using any R-cone. Conversely, we also show that every amenable partial order on a locally inverse semigroup can be obtained in this way. Thus, a new characterization theorem of the natural partial order on S is obtained. Also, we characterize the locally inverse semigroup with a Clifford transversal. In particular, if S is a locally inverse semigroup with a Clifford transversal, then there is an order-preserving bijection from the set of all amenable partial orders on S to the set of all R-cones of S.

2. Constructing amenable partial orders

We first give a new characterization of amenable partial order on a locally inverse semigroup with an inverse transversal.

The following lemma follows from Theorem 5 in [5].

Lemma 2.1 *Let S be a locally inverse semigroup with an inverse transversal S°. If C is an R-cone of S, then*

$$(\forall x \in S)\ x \in C \Longrightarrow x^{\circ\circ} \in C.$$

We now prove the following theorem.

Theorem 2.2 *Suppose that S is a locally inverse semigroup with an inverse transversal S° and C is an R-cone of S. Then the relation $\leq_C$ defined on S by*

$$x \leq_C y \Longleftrightarrow xx^\circ \preceq yy^\circ,\ x^\circ x \preceq y^\circ y,\ x^\circ y^{\circ\circ},\ y^{\circ\circ}x^\circ \in C \tag{*}$$

is an amenable partial order on S.

Proof It can be easily seen that $\leq_C$ is reflexive. Suppose that $x \leq_C y$ and $y \leq_C x$. Then, by $\leq_C$, we have $xx^\circ = yy^\circ$, $x^\circ x = y^\circ y$, $x^\circ y^{\circ\circ}$, $y^\circ x^{\circ\circ} \in C$. Thus $y^\circ x^{\circ\circ} = (x^\circ y^{\circ\circ})^\circ \in C \bigcap C^\circ = E(S^\circ)$ since C is an R-cone. It follows from $xx^\circ = yy^\circ$ and (1) that $x^{\circ\circ}x^\circ = (xx^\circ)^\circ = (yy^\circ)^\circ = y^{\circ\circ}y^\circ$ and so $y^\circ = y^\circ y^{\circ\circ} y^\circ = y^\circ x^{\circ\circ} x^\circ$, which gives $y^\circ \preceq x^\circ$. Likewisely, $x^\circ \preceq y^\circ$ and so $x^\circ = y^\circ$, furthermore, we have $x^{\circ\circ} = y^{\circ\circ}$. Hence, $x = xx^\circ \cdot x^{\circ\circ} \cdot x^\circ x = yy^\circ \cdot y^{\circ\circ} \cdot y^\circ y = y$. This shows that $\leq_C$ is anti-symmetric. If $x \leq_C y$ and $y \leq_C z$, then $x^\circ x \preceq y^\circ y \preceq z^\circ z$, $xx^\circ \preceq yy^\circ \preceq zz^\circ$ and $x^\circ y^{\circ\circ}$, $y^\circ z^{\circ\circ} \in C$. We obtain from $xx^\circ \preceq yy^\circ$ that $xx^\circ yy^\circ = xx^\circ$. Since C is a subsemigroup of S, $x^\circ y^{\circ\circ} y^\circ z^{\circ\circ} \in C$. Thus, we obtain

$$\begin{aligned} x^\circ y^{\circ\circ} y^\circ z^{\circ\circ} &= x^\circ x x^\circ y^{\circ\circ} y^\circ z^{\circ\circ} = x^\circ (xx^\circ yy^\circ) z^{\circ\circ} \quad \text{(by (2))} \\ &= x^\circ z^{\circ\circ}. \end{aligned}$$

Consequently $x^\circ z^{\circ\circ} \in C$. Similarly, we have $z^{\circ\circ} x^\circ \in C$ and so $x \leq_C z$. Thereby, $\leq_C$ is transitive and $\leq_C$ is a partial order on S.

Suppose that $x \leq_C y$. For any $z \in S$, we have

$$\begin{aligned} (zx)^\circ (zy)^{\circ\circ} &= x^\circ (zxx^\circ)^\circ (zy)^{\circ\circ} && \text{(by (1))} \\ &= x^\circ (zyy^\circ xx^\circ)^\circ (zy)^{\circ\circ} \\ &= x^\circ x^{\circ\circ} x^\circ y^{\circ\circ} (zy)^\circ (zy)^{\circ\circ} \\ &= x^\circ y^{\circ\circ} (zy)^\circ (zy)^{\circ\circ} \\ &\in CE(S^\circ) \\ &\subseteq C && (E(S^\circ) \subseteq C) \end{aligned}$$

and

$$\begin{aligned} (zy)^{\circ\circ} (zx)^\circ &= (zy)^{\circ\circ} x^\circ (zxx^\circ)^\circ && \text{(by (1))} \\ &= (zyy^\circ y)^{\circ\circ} x^\circ (zxx^\circ yy^\circ)^\circ \\ &= (zyy^\circ)^{\circ\circ} y^{\circ\circ} x^\circ (zyy^\circ xx^\circ)^\circ \\ &= (zyy^\circ)^{\circ\circ} y^{\circ\circ} x^\circ (zyy^\circ x^{\circ\circ} x^\circ)^\circ && \text{(by (2))} \\ &= (zyy^\circ)^{\circ\circ} y^{\circ\circ} x^\circ x^{\circ\circ} x^\circ (zyy^\circ)^\circ && \text{(by (1))} \\ &= (zyy^\circ)^{\circ\circ} y^{\circ\circ} x^\circ (zyy^\circ)^\circ \\ &\in C. && (y^{\circ\circ} x^\circ \in C) \end{aligned}$$

Hence, $(zx)^\circ (zy)^{\circ\circ}$, $(zy)^{\circ\circ} (zx)^\circ \in C$. It follows from Theorem 8 in [5] that $zx(zx)^\circ \preceq zy(zy)^\circ$ and $(zx)^\circ zx \preceq (zy)^\circ zy$. Thus, $zx \leq_C zy$, and therefore $\leq_C$ is compatible on the left. Dually, $\leq_C$ is also compatible on the right and so $(S, \cdot, \leq_C)$ is a p.o. semigroup.

In the following, we will show that the partial order $\leq_C$ coincides with the natural partial order on $E(S)$. Suppose that e, $f \in E(S)$ and $e \leq_C f$. Then $e^\circ e \preceq f^\circ f$, $ee^\circ \preceq ff^\circ$. From Theorem 2 in [5], we obtain $e \preceq f$. Conversely, if $e \preceq f$, by using Theorem 2 in [5], then $e^\circ e \preceq$

$f^\circ f$, $ee^\circ \preceq ff^\circ$ and $e^\circ f \in \Lambda$, $fe^\circ \in I$. Furthermore, we have $(e^\circ f)^{\circ\circ} = e^\circ f^{\circ\circ} \in E(S^\circ) \subseteq C$, and likewise, $f^{\circ\circ}e^\circ \in C$. Thus $e \leq_C f$, and consequently $\leq_C$ coincides with $\preceq$ on idempotents. This shows that $\leq_C$ is an amenable partial order. □

Theorem 2.2 shows that an amenable partial order can be constructed by using an R-cone. The following result shows that an R-cone can be obtained from an amenable partial order.

We first prove the following lemma.

Lemma 2.3 *Suppose that S is a locally inverse semigroup with an inverse transversal S°. If the partial order $\leq$ is an amenable partial order on S, then*

$$(\forall a,\, b \in S)\ a \leq b \Longrightarrow a^{\circ\circ} \leq b^{\circ\circ}.$$

Proof Suppose that $a \leq b$. Then $aa^\circ \preceq bb^\circ$ and $a^\circ a \preceq b^\circ b$, and whence $aa^\circ bb^\circ = aa^\circ$ and $a^\circ ab^\circ b = a^\circ a$. From (1) and (2), we have $(aa^\circ bb^\circ)^\circ = (aa^\circ b^{\circ\circ}b^\circ)^\circ = b^{\circ\circ}b^\circ a^{\circ\circ}a^\circ = (aa^\circ)^\circ = a^{\circ\circ}a^\circ$. This shows that $a^{\circ\circ}a^\circ \preceq b^{\circ\circ}b^\circ$. Likewisely, $a^\circ a^{\circ\circ} \preceq b^\circ b^{\circ\circ}$. Since $\leq$ is an amenable partial order, $a^{\circ\circ}a^\circ \leq b^{\circ\circ}b^\circ$ and $a^\circ a^{\circ\circ} \leq b^\circ b^{\circ\circ}$, and consequently $a^{\circ\circ} = a^{\circ\circ}a^\circ aa^\circ a^{\circ\circ} \leq b^{\circ\circ}b^\circ bb^\circ b^{\circ\circ} = b^{\circ\circ}$, as required. □

Lemma 2.4 *Suppose that S is a locally inverse semigroup with an inverse transversal S°. If the partial order $\leq$ is an amenable partial order on S, then the set $C = \{x \in \Lambda^* | x^\circ x \leq x\}$ is an R-cone of S and $\leq_C = \leq$.*

Proof It follows from Theorem 10 in [5] that C is an R-cone of S. Consider the corresponding partial order $\leq_C$ given by

$$x \leq_C y \Longleftrightarrow xx^\circ \preceq yy^\circ,\ x^\circ x \preceq y^\circ y, x^\circ y^{\circ\circ},\, y^{\circ\circ}x^\circ \in C.$$

We can obtain from Theorem 2.2 that $\leq_C$ is an amenable partial order on S.

In the following, we will show that $\leq_C = \leq$.

Suppose that $x \leq_C y$. Then $xx^\circ \preceq yy^\circ$ and $x^\circ y^{\circ\circ} \in C$, from the definition of C, we have $(x^\circ y^{\circ\circ})^\circ x^\circ y^{\circ\circ} = y^\circ x^{\circ\circ}x^\circ y^{\circ\circ} \leq x^\circ y^{\circ\circ}$. Since $x^\circ x \preceq y^\circ y$, $x^\circ x^{\circ\circ} = (x^\circ x)^\circ = (x^\circ xy^\circ y)^\circ = y^\circ y^{\circ\circ}x^\circ x^{\circ\circ}$, similarly, $x^{\circ\circ}x^\circ = x^{\circ\circ}x^\circ y^{\circ\circ}y^\circ$. Hence, we deduce that

$$\begin{aligned}
x^{\circ\circ}x^\circ &= y^{\circ\circ}y^\circ x^{\circ\circ}x^\circ \\
&= y^{\circ\circ}y^\circ x^{\circ\circ}x^\circ y^{\circ\circ}y^\circ \qquad (E(S^\circ) \text{ is a semilattice}) \\
&= y^{\circ\circ}(y^\circ x^{\circ\circ}x^\circ y^{\circ\circ})y^\circ \\
&\leq y^{\circ\circ}(y^\circ y^{\circ\circ}x^\circ x^{\circ\circ}x^\circ y^{\circ\circ})y^\circ \\
&= y^{\circ\circ}x^\circ(x^{\circ\circ}x^\circ y^{\circ\circ}y^\circ) \\
&= y^{\circ\circ}x^\circ x^{\circ\circ}x^\circ \\
&= y^{\circ\circ}x^\circ.
\end{aligned}$$

Since $\leq$ is an amenable partial order, $x = xx^\circ x^{\circ\circ}x^\circ x \leq xx^\circ y^{\circ\circ}x^\circ x \leq yy^\circ y^{\circ\circ}y^\circ y = y$. Consequently $\leq_C \subseteq \leq$.

Suppose that $a, b \in S$ and $a \leq b$. It follows from $a \leq b$ that $aa^\circ \preceq bb^\circ$ and $a^\circ a \preceq b^\circ b$. Furthermore, we have $b^\circ ba^\circ = b^\circ ba^\circ aa^\circ = a^\circ$. By Lemma 2.3, we have $a^{\circ\circ} \leq b^{\circ\circ}$. Hence,

$(a^\circ b)^\circ(a^\circ b) = b^\circ a^{\circ\circ} a^\circ b \leq b^\circ b^{\circ\circ} a^\circ b = b^\circ b a^\circ b = a^\circ b$. Since

$$\begin{aligned}(a^\circ b)^{\circ\circ}(a^\circ b)^\circ a^\circ b &= a^\circ b^{\circ\circ} b^\circ a^{\circ\circ} a^\circ b = a^\circ (bb^\circ aa^\circ)^{\circ\circ} b \\ &= a^\circ a^{\circ\circ} a^\circ b = a^\circ b, \quad a^\circ b \in R.\end{aligned}$$

To prove $a^\circ b \in \Lambda^*$, we first denote the Green's $\mathcal{L}$ relation on R by $\mathcal{L}$. For any $e \in \Lambda$, it can be easily seen that $(a^\circ b)e\mathcal{L}(a^\circ b)^\circ a^\circ be$ since $\mathcal{L}$ is a right congruence on R. Since every $\mathcal{L}$-class contains a single idempotent, $(a^\circ be)^\circ(a^\circ be) = (a^\circ b)^\circ a^\circ be$. Post-multiplying $(a^\circ b)^\circ(a^\circ b) \leq a^\circ b$ by e and pre-multiplying $(a^\circ b)^\circ(a^\circ b) \leq a^\circ b$ by $(a^\circ b)^\circ a^\circ be$, we have $(a^\circ b)^\circ a^\circ be \leq (a^\circ b)^\circ a^\circ be \cdot a^\circ be$. By Theorem 7 in [4], we have $a^\circ be = (a^\circ b)^\circ a^\circ be \cdot a^\circ be$. Pre-multiplying this by e, we have $ea^\circ be = e(a^\circ b)^\circ a^\circ be \cdot a^\circ be$. Since Λ is a right normal band, $ea^\circ be = e(a^\circ b)^\circ a^\circ be \cdot a^\circ be = (a^\circ b)^\circ a^\circ be \cdot a^\circ be = a^\circ be$. This shows that $a^\circ b \in \Lambda^*$. Thus $(a^\circ b)^\circ(a^\circ b) \leq a^\circ b$ and $a^\circ b \in \Lambda^*$. It follows from the definition of C that $a^\circ b \in C$. By Lemma 2.1, we have $(a^\circ b)^{\circ\circ} = a^\circ b^{\circ\circ} \in C$. Likewisely, $b^{\circ\circ} a^\circ \in C$. Hence, $a \leq_C b$ and so $\leq \subseteq \leq_C$. Therefore $\leq_C = \leq$. □

Suppose that S is a locally inverse semigroup with an inverse transversal S°. It can be easily seen that Λ is the smallest R-cone of S. By Lemma 2.4 and Theorem 11 in [5], we obtain a new characterization of the natural partial order on S.

Theorem 2.5 *Suppose that S is a locally inverse semigroup with an inverse transversal S°. Then $\leq_\Lambda$ defined by $(*)$ is the smallest amenable partial order on S and $\leq_\Lambda = \preceq$.*

3. Clifford transversals

Suppose that S is a locally inverse semigroup with an inverse transversal S°. Consider the following subset of S

$$T = \{x \in S | x = x^\circ x^2\}.$$

Then, by Theorem 9 in [5], $T \subseteq R$ and $E(T) = \Lambda$, where $E(T)$ is the set of all idempotents of T.

Suppose that S is a locally inverse semigroup with an inverse transversal S°. If S° is a Clifford semigroup, i.e., for any $a \in S^\circ$, $a^\circ a = aa^\circ$, then S° is called a Clifford transversal.

We have the following lemma.

Lemma 3.1 *Suppose that S is a locally inverse semigroup with an inverse transversal S°. If $T = R$, then S° is a Clifford transversal.*

Proof For any $x \in S^\circ$, by (β), we have $S^\circ \subseteq R = T$ and so $x = x^\circ x^2$. It can be easily seen that $x = x^{\circ\circ}$ since $x \in S^\circ$. Thus $x^{\circ\circ} = x^\circ (x^{\circ\circ})^2$. Post-multiplying this by x°, we obtain $x^{\circ\circ} x^\circ = x^\circ x^{\circ\circ} \cdot x^{\circ\circ} x^\circ$. Since $x^\circ \in S^\circ \subseteq R = T$, $x^\circ = x^{\circ\circ}(x^\circ)^2$. Post-multiplying this by $x^{\circ\circ}$, we have $x^\circ x^{\circ\circ} = x^{\circ\circ} x^\circ \cdot x^\circ x^{\circ\circ}$. Consequently $x^{\circ\circ} x^\circ = x^\circ x^{\circ\circ}$ since $E(S^\circ)$ is a semilattice. Furthermore, we have $xx^\circ = x^\circ x$. Hence, S° is a Clifford transversal. □

Suppose that S is a completely regular semigroup and $E(S)$ is the set of all idempotents of S. If $E(S)$ is a right normal band, then S is called a right normal orthogroup. In the following, we will give a characterization of locally inverse semigroup with a Clifford transversal.

Theorem 3.2 *Suppose that S is a locally inverse semigroup with an inverse transversal S°. Then the following conditions are equivalent:*

(i) $T = R$;

(ii) S° *is a Clifford transversal;*

(iii) R *is a right normal orthogroup.*

Proof (i)$\Longrightarrow$ (iii). Assume that $x \in R$. Then $x = x^{\circ\circ}x^\circ x$. Since $T = R$, $x = x^\circ x^2$. Post-multiplying this by x°, we obtain $xx^\circ = x^\circ x \cdot xx^\circ$. It can be easily seen that $xx^\circ \cdot x^\circ x = (x^{\circ\circ}x^\circ x)x^\circ \cdot x^\circ x = x^{\circ\circ}(x^\circ)^2 x$. From $x^\circ \in S^\circ \subseteq R = T$, we have $x^\circ = x^{\circ\circ}(x^\circ)^2$. Thus, $xx^\circ \cdot x^\circ x = x^\circ x$. We denote by $\mathcal{R}$ and $\mathcal{H}$ the Green's $\mathcal{R}$ relation and Green's $\mathcal{H}$ relation on R, respectively. Then $xx^\circ \mathcal{R} x^\circ x$. Since $x\mathcal{R}xx^\circ$ and $x\mathcal{L}x^\circ x$, $x\mathcal{R}x^\circ x$ and so $x(\mathcal{L}\bigcap\mathcal{R})x^\circ x$, i.e., $x\mathcal{H}x^\circ x$. This shows that every $\mathcal{H}$-class of R is a group. Consequently R is a completely regular semigroup. From the fact that $E(R) = \Lambda$ is a right normal band, R is a right normal orthogroup.

(iii)$\Longrightarrow$ (i). For any $x \in R$, we denote x^{-1} the inverse of x in $\mathcal{H}$-class containing x. Then it is clear that $x = xx^{-1}x = x^{-1}x \cdot x$. From the fact that $x\mathcal{L}x^{-1}x\mathcal{L}x^\circ x$ and every $\mathcal{L}$-class of R contains a single idempotent, we have $x^{-1}x = x^\circ x$. Consequently $x = x^{-1}x \cdot x = x^\circ x \cdot x = x^\circ x^2$, by the definition of T, we have $x \in T$, moreover, $R \subseteq T$. Now, by Theorem 9 in [5], $T \subseteq R$. Hence, $T = R$.

(ii)$\Longrightarrow$ (i). For any $x \in R$, we have $x = x^{\circ\circ}x^\circ x$. From (1) and (2), we deduce that

$$\begin{aligned}(x^2)^{\circ\circ} &= (x^{\circ\circ}x^\circ x x^{\circ\circ}x^\circ x)^{\circ\circ} = (x^{\circ\circ}x^\circ x^{\circ\circ}x^{\circ\circ}x^\circ x)^{\circ\circ}\\ &= (x^{\circ\circ}x^{\circ\circ}x^\circ x)^{\circ\circ} = x^{\circ\circ}\cdot x^{\circ\circ}.\end{aligned}$$

Since S° is a Clifford transversal, we deduce that

$$\begin{aligned}x &= x^{\circ\circ}x^\circ x = x^\circ x^{\circ\circ}(x^{\circ\circ}x^\circ x)\\ &= x^\circ(x^2)^{\circ\circ}x^\circ x = x^\circ x^2 x^\circ x \quad \text{(by (2))}\\ &= x^\circ x^2.\end{aligned}$$

This shows that $R \subseteq T$. By Theorem 9 in [5], we have $T \subseteq R$, and whence $T = R$.

(i)$\Longrightarrow$ (ii). By Lemma 3.1, it can be easily seen that S° is a Clifford transversal. $\square$

Suppose that S is a locally inverse semigroup with an inverse transversal S° and C is an R-cone. It follows by Theorem 2.2 that $\leq_C$ is an amenable partial order on S. From Lemma 2.4, the set $\{x \in \Lambda^* | x^\circ x \leq_C x\}$ is an R-cone.

Lemma 3.3 *Suppose that S is a locally inverse semigroup with a Clifford transversal S° and C is an R-cone. Then $C = \{x \in \Lambda^* | x^\circ x \leq_C x\}$, in which $\leq_C$ is an amenable partial order defined by $(*)$.*

Proof It follows by Theorem 2.2 that $\leq_C$ is an amenable partial order on S. By Lemma 2.4, the set $D = \{x \in \Lambda^* | x^\circ x \leq_C x\}$ is an R-cone. We now show that $C = D$.

If $x \in C$, then $x \in \Lambda^*$ from $(C1)$. Since Λ^* is a subsemigroup of R, $x = x^{\circ\circ}x^\circ x$. From that

S° is a Clifford transversal, we obtain

$$\begin{aligned} x^\circ x(x^\circ x)^\circ &= x^\circ x x^\circ x^{\circ\circ} \quad (\text{by } (1)) \\ &= x^\circ x^{\circ\circ} = x^{\circ\circ} x^\circ \\ &= (x^{\circ\circ} x^\circ x) x^\circ = x x^\circ. \end{aligned}$$

It is easy to see that $(x^\circ x)^\circ x^{\circ\circ} = x^\circ x^{\circ\circ} x^{\circ\circ}$. Since S° is a Clifford transversal, $x^\circ x^{\circ\circ} x^{\circ\circ} = x^{\circ\circ} x^\circ x^{\circ\circ} = x^{\circ\circ}$. It follows from Lemma 2.1 and $x \in C$ that $x^{\circ\circ} \in C$, i.e., $(x^\circ x)^\circ x^{\circ\circ} = x^{\circ\circ} \in C$. From $x^{\circ\circ} \in C$, we have $x^{\circ\circ}(x^\circ x)^\circ = x^{\circ\circ} \in C$. Consequently $x^\circ x \leq_C x$ by Theorem 2.2. This shows that $x \in D$ and so $C \subseteq D$.

If $x \in D$, then $x \in R$. Furthermore, we have $x = x^{\circ\circ} x^\circ x$. From $x^\circ x \leq_C x$ and Theorem 2.2, we have $x^{\circ\circ} = x^{\circ\circ}(x^\circ x)^\circ \in C$. It is obvious that $x^\circ x \in \Lambda$ and Λ is the smallest R-cone, so we have $x^\circ x \in C$. Consequently, $x = x^{\circ\circ} x^\circ x \in CC \subseteq C$. This shows that $D \subseteq C$. Hence, $C = D$. $\square$

The following theorem is the main theorem of this paper.

Theorem 3.4 *Let S be a locally inverse semigroup with a Clifford transversal S°. Then there is an order-preserving bijection from the set $\mathbf{RC}(S)$ of all R-cones of S to the set $\mathbf{AO}(S)$ of all amenable partial orders on S.*

Proof For any $C \in \mathbf{RC}(S)$, it follows by Theorem 2.2 that $\leq_C$ is an amenable partial order on S. Hence, we can define a mapping ψ from the set $\mathbf{RC}(S)$ of all R-cones on S to the set $\mathbf{AO}(S)$ of all amenable partial orders on S by:

$$\psi : \mathbf{RC}(S) \longrightarrow \mathbf{AO}(S),\ \psi(C) = \leq_C.$$

Suppose that $C_1, C_2 \in \mathbf{RC}(S)$ but $C_1 \neq C_2$. Then, by Lemma 3.3, we have $C_1 = \{x \in \Lambda^* | x^\circ x \leq_{C_1} x\}$, $C_2 = \{x \in \Lambda^* | x^\circ x \leq_{C_2} x\}$. If $\leq_{C_1} = \leq_{C_2}$, then it can be easily seen that $C_1 = C_2$. This contradicts $C_1 \neq C_2$. Hence, ψ is injective..

If $\leq$ is an amenable partial order on S. It follows by Lemma 2.4 that $C = \{x \in \Lambda^* | x^\circ x \leq x\}$ is an R-cone and $\leq = \leq_C$. Thus ψ is surjective. If $C_1, C_2 \in \mathbf{RC}(S)$ and $C_1 \subseteq C_2$, then it is easy to see that $\leq_{C_1} \subseteq \leq_{C_2}$. Hence, ψ is an order-preserving bijection. The proof is completed. $\square$

References

[1] PETRICH M, REILLY N R. *Completely Regular Semigroups* [M]. John Wiley & Sons, Inc., New York, 1999.
[2] HOWIE J M. *Fundamentals of Semigroup Theory* [M]. Oxford University Press, New York, 1995.
[3] MCALISTER D B. *Amenably ordered inverse semigroups* [J]. J. Algebra, 1980, **65**(1): 118–146.
[4] BLYTH T S, ALMEIDA-SANTOS M H. *Amenable orders on orthodox semigroups* [J]. J. Algebra, 1994, **169**(1): 49–70.
[5] BLYTH T S, ALEMIDA-SANTOS M H. *Amenable orders associated with inverse transversals* [J]. J. Algebra, 2001, **240**(1): 143–164.