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Two Kinds of Weak Berwald Metrics of Scalar Flag Curvature

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Abstract In this paper, we study the (α,β)-metrics of scalar flag curvature in the form of $F=\alpha+\varepsilon\beta+k\frac{\beta^2}{\alpha}$ (ε and $k\neq 0$ are constants) and $F=\frac{\alpha^2}{\alpha-\beta}$. We prove that these two kinds of metrics are weak Berwaldian if and only if they are Berwaldian and their flag curvatures vanish. In this case, the metrics are locally Minkowskian.

Keywords Finsler metric; (α,β)-metric; weak Berwald metric; Berwald metric; flag curvature.

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1. Introduction

In the past several years, Finsler geometry has achieved rapid and great progress. Various Riemannian curvatures and non-Riemannian curvatures in Finsler geometry have been studied deeply and their geometric meanings are better understood. Finsler geometry has been applied extensively in phisics, biology (ecology) and other fields in natural science. These are partially duo to the study of (α,β)-metrics[1].

In this paper we discuss two important kinds of (α,β)-metrics in the form of $F=\alpha+\varepsilon\beta+k\beta^2/\alpha$ (ε and $k\neq 0$ are constants) and $F=\alpha^2/(\alpha-\beta)$. The (α,β)-metric in the form of $F=\alpha^2/(\alpha-\beta)$ is called Matsumoto metric. In [2] and [7], Bácsó and Matsumoto proved that Matsumoto metric F is Douglas metric if and only if β is parallel with respect to α. In this case F is a Berwald metric. In [10], Shen and Yildirim obtained the necessary and sufficient conditions for the metric $F=(\alpha+\beta)^2/\alpha$ to be projectively flat. They also obtained the necessary and sufficient conditions for the metric $F=(\alpha+\beta)^2/\alpha$ to be projectively flat Finsler metric of constant flag curvature and proved that, in this case, the flag curvature vanishes. In particular, for these two kinds of (α,β)-metrics, Cui proved the following result.

Theorem 1.1[5] *For Matsumoto metric $F=\alpha^2/(\alpha-\beta)$ and (α,β)-metric in the form of $F=\alpha+\varepsilon\beta+k\beta^2/\alpha$ on an n-dimensional manifold M, where ε and $k\neq 0$ are constants, the following are equivalent.*

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(a) *F is of isotropic S-curvature, $\mathbf{S}=(n+1)c(x)F$;*

(b) *F is of isotropic mean Berwald curvature, $\mathbf{E}=\frac{n+1}{2}c(x)F^{-1}h$;*

(c) *β is a Killing 1-form with constant length with respect to α, that is, $r_{00}=0$ and $s_0=0$;*

(d) *S-curvature vanishes, $\mathbf{S}=0$;*

(e) *F is a weak Berwald metric, $\mathbf{E}=0$,*

where $c=c(x)$ is a scalar function on the manifold M (for the related difinitions, see the next section).

We note that, the discussion in [5] does not involve whether or not F is Berwald metric. By the definitions (see the next section), Berwald metrics must be weak Berwald metrics but the reverse is not true. Inspired by this observation, we further study the metrics $F=\alpha+\varepsilon\beta+k\beta^2/\alpha$ (ε and $k\neq 0$ are constants) and $F=\alpha^2/(\alpha-\beta)$ and obtain the following

Theorem 1.2 *Let (M,F) be an n-dimensional Finsler manifold ($n\geq 3$). Assume that (α,β)-metric $F=\alpha+\varepsilon\beta+k\beta^2/\alpha$ (or $F=\alpha^2/(\alpha-\beta)$) is of scalar flag curvature $\mathbf{K}=K(x,y)$, where ε and $k\neq 0$ are constants. Then F is weak Berwald metric if and only if F is Berwald metric and $\mathbf{K}=0$. In this case, F must be locally Minkowskian.*

2. Preliminaries

Let M be an n-dimensional C^∞ manifold and $TM=\cup_{x\in M}T_xM$ denote the tangent bundle of M. A Finsler metric on M is a function $F:TM\to[0,\infty)$ with the following properties

(a) F is C^∞ on $TM\backslash\{0\}$;

(b) At each point $x\in M$, $F_x(y)=F(x,y)$ is a Minkowskian norm on T_xM.

We call the couple (M,F) Finsler manifold.

Let (M,F) be a Finsler manifold and

$$g_{ij}(x,y):=\frac{1}{2}[F^2(x,y)]_{y^iy^j}. \tag{1}$$

For a vector $y=y^i\frac{\partial}{\partial x^i}|_x\neq 0$, F induces an inner product g_y on T_xM as follows

$$g_y(u,v)=g_{ij}(x,y)u^iv^j,$$

where $u=u^i\frac{\partial}{\partial x^i}|_x$ and $v=v^i\frac{\partial}{\partial x^i}|_x$. Further, the Cartan torsion $\mathbf{C}$ and the mean Cartan torsion $\mathbf{I}$ are defined as follows[4]

$$\mathbf{C}_y(u,v,w):=C_{ijk}u^iv^jw^k,\quad \mathbf{I}_y(u):=I_i(x,y)u^i, \tag{2}$$

where

$$C_{ijk}(x,y):=\frac{1}{4}[F^2]_{y^iy^jy^k}(x,y), \tag{3}$$

$$I_i(x,y):=g^{jk}(x,y)C_{ijk}(x,y)=\frac{\partial}{\partial y^i}\Big[\ln\sqrt{\det(g_{jk})}\Big], \tag{4}$$

where $\left(g^{jk}\right):=\left(g_{jk}\right)^{-1}$.

Let $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric and $\beta=b_i(x)y^i$ be a 1-form on an n-

dimensional manifold M. The norm $\|\beta_x\|_\alpha$ of β with respect to α is defined by

$$\|\beta_x\|_\alpha := \sqrt{a^{ij}(x)b_i(x)b_j(x)}.$$

Let $\phi = \phi(s)$ be a C^∞ positive function on an open interval $(-b_0, b_0)$ satisfying the following condition:

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \le b < b_0.$$

Then the function $F = \alpha\phi(\beta/\alpha)$ is a Finsler metric if and only if $\|\beta_x\|_\alpha < b_0$. We call such Finsler metrics are (α, β)-metrics.

Throughout this paper, the Einstein summation convention is adopted. Let ";" and "|" denote the horizontal covariant derivative with respect to F and α, respectively. Let

$$s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \tag{5}$$

$$s^i{}_j := a^{ik}s_{kj}, \quad s_i := b^j s_{ji} = b_j s^j{}_i, \quad r_i := b^j r_{ji}. \tag{6}$$

We will denote $r_{00} := r_{ij}y^iy^j, s_{i0} := s_{ij}y^j, s_0 := s_iy^i$, etc.

Let

$$h_i := \frac{\alpha b_i - sy_i}{\alpha}, \quad Q := \frac{\phi'}{\phi - s\phi'},$$

$$\Delta := 1 + sQ + (b^2 - s^2)Q', \quad \Theta := \frac{Q - sQ'}{2\Delta},$$

$$\Phi := -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q'',$$

$$\Psi_1 := \sqrt{b^2 - s^2}\Delta^{\frac{1}{2}}\Big[\frac{\sqrt{b^2 - s^2}\Phi}{\Delta^{\frac{3}{2}}}\Big]', \quad \Psi_2 := 2(n+1)(Q - sQ') + 3\frac{\Phi}{\Delta},$$

where $y_i := a_{ij}y^j$.

Using a Maple program, the fundamental tensor of an (α, β)-metric $F = \alpha\varphi(s)$ is given by[4]

$$g_{ij}(x, y) = \rho a_{ij} + \rho_0 b_ib_j + \rho_1[b_i\alpha_j + b_j\alpha_i] - s\rho_1\alpha_i\alpha_j,$$

where

$$\alpha_i := \alpha_{y^i}, \quad \rho := \phi^2 - s\phi\phi', \quad \rho_0 := \phi\phi'' + \phi\phi',$$

$$\rho_1 := -s(\phi\phi'' + \phi'\phi') + \phi\phi', \quad s = \beta/\alpha.$$

Further, we can obtain[4]

$$\det(g_{ij}) = \phi^{n+1}(\phi - s\phi')^{n-2}[(\phi - s\phi') + (b^2 - s^2)\phi'']\det(a_{ij}). \tag{7}$$

From (4) and (7), by a direct computation , we obtain a formula for the mean Cartan torsion of (α, β)-metrics as follows

$$\begin{aligned} I_i =& \frac{\partial}{\partial y^i}[\ln\sqrt{\det(g_{jk})}] \\ =& \frac{\partial}{\partial y^i}\Big\{\frac{n+1}{2}\ln\phi + \frac{n-2}{2}\ln(\phi - s\phi') + \frac{1}{2}\ln\big[(\phi - s\phi') + (b^2 - s^2)\phi''\big] + \\ & \frac{1}{2}\ln\big[\det(a_{jk}(x))\big]\Big\} \end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{2}\frac{\partial s}{\partial y^i}\{(n+1)\frac{\phi'}{\phi}-(n-2)\frac{s\phi''}{\phi-s\phi'}-\frac{3s\phi''-(b^2-s^2)\phi'''}{(\phi-s\phi')+(b^2-s^2)\phi''}\}\\
&=\frac{\alpha b_i-sy_i}{2\alpha^2}\{(n+1)\frac{\phi'}{\phi}-(n-2)\frac{s\phi''}{\phi-s\phi'}-\frac{3s\phi''-(b^2-s^2)\phi'''}{(\phi-s\phi')+(b^2-s^2)\phi''}\}\\
&=-\frac{\Phi(\phi-s\phi')}{2\Delta\phi\alpha^2}(\alpha b_i-sy_i). \qquad (8)
\end{aligned}
$$

According to Deicke's theorem[9], a Finsler metric is Riemannian if and only if the mean Cartan torsion vanishes, $\mathbf{I}=0$. Clearly, (α,β)-metric $F=\alpha\phi(s), s=\beta/\alpha$, is Riemannian if and only if $\Phi=0$.

The geodesic $x=x(t)$ of a Finsler metric F is characterized by the following system of 2nd order ordinary differential equations:

$$\frac{d^2x^i(t)}{\mathrm{d}t^2}+2G^i\big(x(t),x'(t)\big)=0,$$

where

$$G^i:=\frac{1}{4}g^{ij}\{[F^2]_{x^my^j}y^m-[F^2]_{x^j}\}.$$

G^i are called the geodesic coefficients of F.

For an (α,β)-metric $F=\alpha\varphi(s)$, $s=\beta/\alpha$, using a Maple program, we can get the following

$$G^i=\bar{G}^i+\alpha Qs_0^i+\Theta[-2\alpha Qs_0+r_{00}]\{\frac{y^i}{\alpha}+\frac{Q'}{Q-sQ'}b^i\}, \tag{9}$$

where $\bar{G}^i$ denote the spray coefficients of α.

For a Finsler metric $F=F(x,y)$ on a manifold M, the Riemann curvature $\mathbf{R}_y=R^i{}_k\frac{\partial}{\partial x^i}\otimes\mathrm{d}x^k$ is defined by

$$R^i{}_k:=2\frac{\partial G^i}{\partial x^k}-\frac{\partial^2G^i}{\partial x^m\partial y^k}y^m+2G^m\frac{\partial^2G^i}{\partial y^m\partial y^k}-\frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}.$$

Let $R_{jk}:=g_{ji}R^i{}_k$. Then $R_{jk}y^j=0,\quad R_{jk}=R_{kj}$.

For a flag $\{P,y\}$, where $P:=\mathrm{span}\{y,u\}\subset T_xM$, the flag curvature $\mathbf{K}=\mathbf{K}(P,y)$ of F is defined by

$$\mathbf{K}(P,y):=\frac{R_{jk}(x,y)u^ju^k}{F^2(x,y)h_{jk}(x,y)u^ju^k}, \tag{10}$$

where $h_{jk}:=g_{jk}-F^{-2}g_{jp}y^pg_{kq}y^q$.

The flag curvature in Finsler geometry is the analogue of the sectional curvature in Riemann geometry. When Finsler metric F is Riemannian, the flag curvature is just the sectional curvature. We say that Finsler metric F is of scalar flag curvature if the flag curvature $\mathbf{K}=\mathbf{K}(x,y)$ is independent of the flag P. By the definition, F is of scalar flag curvature $\mathbf{K}=\mathbf{K}(x,y)$ if and only if in a standard local coordinate system,

$$R^i{}_k=\mathbf{K}F^2h^i{}_k, \tag{11}$$

where $h^i{}_k:=g^{ij}h_{jk}=g^{ij}FF_{y^jy^k}$[4,9].

The Schur Lemma in Finsler geometry tells us that, in dimension $n\geq 3$, if F is of isotropic flag curvature, $\mathbf{K}=\mathbf{K}(x)$, then it is of constant flag curvature, $\mathbf{K}=$ constant. One of the most important problems in Finsler geometry is to study and characterize Finsler metrics of scalar flag curvature $K = K(x, y)$.

The Berwald curvature $\mathbf{B}_y = B_j{}^i{}_{kl} \mathrm{d}x^j \otimes \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^k \otimes \mathrm{d}x^l$ and the mean Berwald curvature $\mathbf{E}_y = E_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$ are defined respectively by

$$B_j{}^i{}_{kl} := \frac{\partial^3 G^i(x,y)}{\partial y^j \partial y^k \partial y^l}, \quad E_{ij} := \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} (\frac{\partial G^m}{\partial y^m}) = \frac{1}{2} B_i{}^m{}_{jm}.$$

A Finsler metric F is called (weak) Berwald metric if the (mean) Berwald curvature vanishes, i.e., ($\mathbf{E} = 0$) $\mathbf{B} = 0$. A Finsler metric F is said to be of isotropic mean Berwald curvature if $\mathbf{E} = \frac{1}{2}(n+1)c(x)F^{-1}h$, where $c = c(x)$ is a scalar function on the manifold M.

For a Finsler metric $F = F(x, y)$ on an n-dimensional manifold M, the Busemann-Hausdorff volume form $\mathrm{d}V_F = \sigma_F(x) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ is given by

$$\sigma_F(x) := \frac{\mathrm{Vol}\big(B^n(1)\big)}{\mathrm{Vol}\big\{(y^i) \in R^n | F(x,y) < 1\big\}}.$$

Here Vol denotes the Euclidean volume in R^n. The well-known S-curvature is given by

$$\mathbf{S}(x,y) = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial (\ln \sigma_F)}{\partial x^m}. \tag{12}$$

Clearly, the mean Berwald curvature $\mathbf{E} = E_{ij} dx^i \otimes dx^j$ can be characterized by use of S-curvature as follows

$$E_{ij} = \frac{1}{2} \frac{\partial^2 \mathbf{S}}{\partial y^i \partial y^j}.$$

A Finsler metric F is said to be of isotropic S-curvature if $\mathbf{S} = (n+1)c(x)F$, where $c = c(x)$ is a scalar function on the manifold M. S-curvature is closely related to the flag curvature. The first author Mo and Shen proved the following important result.

Theorem 2.1[3] *Let (M, F) be an n-dimensional Finsler manifold of scalar flag curvature with flag curvature $\mathbf{K}(x, y)$. Suppose that the S-curvature is isotropic, $\mathbf{S} = (n+1)c(x)F(x, y)$, where $c = c(x)$ is a scalar function on M. Then there is a scalar function $\sigma(x)$ on M such that*

$$\mathbf{K} = \frac{3c_{x^m}(x) y^m}{F(x,y)} + \sigma(x).$$

The Landsberg curvature $\mathbf{L} = L_{ijk} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k$ and the mean Landsberg curvature $\mathbf{J} = J_k \mathrm{d}x^k$ are defined respectively by

$$L_{ijk} := -\frac{1}{2} F F_{y^m} \big[G^m\big]_{y^i y^j y^k}, \quad J_k := g^{ij} L_{ijk}.$$

A Finsler metric F is called (weak) Landsberg metric if the (mean) Landsberg curvature vanishes, i.e., ($\mathbf{J} = 0$) $\mathbf{L} = 0$.

For an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, Li and Shen[6] obtained the following formula of the mean Landsberg curvature

$$J_i = -\frac{1}{2\Delta\alpha^4} \Big\{ \frac{2\alpha^2}{b^2 - s^2} \big[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\big](s_0 + r_0) h_i +$$
$$\frac{\alpha}{b^2 - s^2} \big[\Psi_1 + s\frac{\Phi}{\Delta}\big](r_{00} - 2\alpha Q s_0) h_i + \alpha\big[-\alpha Q' s_0 h_i + \alpha Q(\alpha^2 s_i - y_i s_0) +$$

$$\alpha^2\Delta s_{i0}+\alpha^2(r_{i0}-2\alpha Qs_i)-(r_{00}-2\alpha Qs_0)y_i\big]\frac{\Phi}{\Delta}\Big\}. \tag{13}$$

Besides, they also obtained

$$\bar{J}:=J_ib^i=-\frac{1}{2\Delta\alpha^2}\big\{\Psi_1(r_{00}-2\alpha Qs_0)+\alpha\Psi_2(r_0+s_0)\big\}. \tag{14}$$

The horizontal covariant derivatives $J_{i;m}$ and $J_{i|m}$ of J_i with respect to F and α respectively are given by

$$J_{i;m}=\frac{\partial J_i}{\partial x^m}-J_l\Gamma^l_{im}-\frac{\partial J_i}{\partial y^l}N^l_m,\quad J_{i|m}=\frac{\partial J_i}{\partial x^m}-J_l\bar{\Gamma}^l_{im}-\frac{\partial J_i}{\partial y^l}\bar{N}^l_m,$$

where $\Gamma^l_{ij}:=\frac{\partial G^l}{\partial y^i\partial y^j},\ N^l_j:=\frac{\partial G^l}{\partial y^j}$ and $\bar{\Gamma}^l_{ij}:=\frac{\partial \bar{G}^l}{\partial y^i\partial y^j},\ \bar{N}^l_j:=\frac{\partial \bar{G}^l}{\partial y^j}$. Further, we have

$$\begin{aligned}J_{i;m}y^m&=\big\{J_{i|m}-J_l(\Gamma^l_{im}-\bar{\Gamma}^l_{im})-\frac{\partial J_i}{\partial y^l}(N^l_m-\bar{N}^l_m)\big\}y^m\\&=J_{i|m}y^m-J_l(N^l_i-\bar{N}^l_i)-2\frac{\partial J_i}{\partial y^l}(G^l-\bar{G}^l).\end{aligned}$$

Akbar-Zadeh proved that[9], if a Finsler metric F is of constant flag curvature $\mathbf{K}$, then

$$J_{i;m}y^m+KF^2I_i=0.$$

So, if an (α,β)-metric $F=\alpha\varphi(s), s=\beta/\alpha$, is of constant flag curvature $\mathbf{K}$, then

$$J_{i|m}y^m-J_l\frac{\partial(G^l-\bar{G}^l)}{\partial y^i}-2\frac{\partial J_i}{\partial y^l}(G^l-\bar{G}^l)+\mathbf{K}\alpha^2\phi^2I_i=0.$$

Contracting the above equation by b^i yields the following equation

$$\bar{J}_{|m}y^m-J_ia^{ik}b_{k|m}y^m-J_l\frac{\partial(G^l-\bar{G}^l)}{\partial y^i}b^i-2\frac{\partial\bar{J}}{\partial y^l}(G^l-\bar{G}^l)+\mathbf{K}\alpha^2\phi^2I_ib^i=0. \tag{15}$$

Lemma 2.2 *For (α,β)-metrics $F=\alpha+\varepsilon\beta+k\beta^2/\alpha$ (where ε and $k\neq 0$ are constants) and $F=\frac{\alpha^2}{\alpha-\beta}$, $\Phi\neq 0$.*

Proof We just give the proof for $F=\alpha+\varepsilon\beta+k\frac{\beta^2}{\alpha}$. The proof for $F=\frac{\alpha^2}{\alpha-\beta}$ is similar. So we omit it.

By a direct computation, we have

$$\Phi=-\frac{A\phi}{(1-ks^2)^4},$$

where

$$\begin{aligned}\phi=&1+\varepsilon s+ks^2,\\A:=&12nk^3s^5+3k^2\varepsilon(3n-1)s^4-4k^2(2nkb^2-kb^2+n+4)s^3+\\&6\varepsilon k(kb^2-nkb^2-n-1)s^2+12b^2k^2s+\varepsilon(n+1)(2kb^2+1).\end{aligned}$$

Assume that $\Phi=0$. Then $A=0$. Multiplying $A=0$ with α^5 yields

$$\begin{aligned}&[12b^2k^2\beta\alpha^4-4k^2(2nkb^2-kb^2+n+4)\beta^3\alpha^2+12nk^3\beta^5]+\\&\alpha[\varepsilon(12nkb^2+2kb^2+n+1)\alpha^4-6\varepsilon k\beta^2(nkb^2-kb^2+n+1)\alpha^2+\\&3k^2\varepsilon\beta^4(3n-1)]=0.\end{aligned}$$

Hence we have

$$12b^2k^2\beta\alpha^4 - 4k^2(2nkb^2 - kb^2 + n + 4)\beta^3\alpha^2 + 12nk^3\beta^5 = 0, \tag{16}$$

$$\varepsilon(12nkb^2 + 2kb^2 + n + 1)\alpha^4 - 6\varepsilon k\beta^2(nkb^2 - kb^2 + n + 1)\alpha^2 + 3k^2\varepsilon\beta^4(3n - 1)] = 0.$$

Note that β^5 is not divisible by α^2. Therefore, we have $k = 0$ by (16), which is a contradiction with $k \neq 0$. So $\Phi \neq 0$. □

In fact, by (8) and Lemma 2.2, we know that (α, β)-metrics $F = \alpha + \varepsilon\beta + k\frac{\beta^2}{\alpha}$ and $F = \frac{\alpha^2}{\alpha - \beta}$ cannot denote Riemannian metrics, where ε and $k \neq 0$ are constants and $\beta \neq 0$.

3. Proof of Theorem 1.2

The sufficiency is obvious. We just prove the necessity.

Assume that the metric F is weak Berwald, i.e., $\mathbf{E} = 0$. By Theorem 1.1, we know that $S = (n+1)c(x)F$ with $c(x) = 0$ and

$$r_{00} = 0, \quad s_0 = 0. \tag{17}$$

By Theorem 2.1, F must be of isotropic flag curvature $\mathbf{K} = \sigma(x)$. Further, F must be of constant flag curvature by Schur Lemma.

By (17), we may simplify (9), (13) and (14) as follows

$$G^i - \bar{G}^i = \alpha Q s^i{}_0, \quad J_i = -\frac{\Phi s_{i0}}{2\alpha\Delta}, \quad \bar{J} = 0.$$

From (8), we obtain

$$I_i b^i = \frac{-\Phi}{2\Delta F}(\phi - s\phi')(b^2 - s^2).$$

Thus (15) can be expressed as follows

$$\frac{\Phi s_{i0}}{2\alpha\Delta} a^{ik} s_{k0} + \frac{\Phi s_{l0}}{2\alpha\Delta}(\alpha Q s^l{}_0)_{\cdot i} b^i - \mathbf{K} F \frac{\Phi}{2\Delta}(\phi - s\phi')(b^2 - s^2) = 0.$$

By Lemma 2.2, we have

$$s_{i0} s^i{}_0 + s_{l0}(\alpha Q s^l{}_0)_{\cdot i} b^i - \mathbf{K} F \alpha(\phi - s\phi')(b^2 - s^2) = 0.$$

On the other hand, it is easy to show that

$$(\alpha Q s^l{}_0)_{\cdot i} b^i = s Q s^l{}_0 + Q' s^l{}_0 (b^2 - s^2).$$

Note that $F = \alpha\phi(s)$, $s = \beta/\alpha$. We have

$$s_{i0} s^i{}_0 \Delta - \mathbf{K}\alpha^2\phi(\phi - s\phi')(b^2 - s^2) = 0. \tag{18}$$

Case 1 $F = \alpha + \varepsilon\beta + k\frac{\beta^2}{\alpha}$ (ε and $k \neq 0$ are constants). In this case,

$$\Delta = \frac{\phi(1 + 2kb^2 - 3ks^2)}{(1 - ks^2)^2}.$$

Then (18) becomes

$$(1 + 2kb^2 - 3ks^2) s_{i0} s^i{}_0 - \mathbf{K}\alpha^2(b^2 - s^2)(1 - ks^2)^3 = 0.$$

Multiplying this equation with α^6 yields

$$\begin{aligned}&-\mathbf{K}b^2\alpha^8+\big\{k\beta^2(1+3kb^2)+s_{i0}s^i{}_0(1+2kb^2)\big\}\alpha^6-\\&3k\beta^2\big\{k\beta^2(1+kb^2)+s_{i0}s^i{}_0\big\}\alpha^4+\mathbf{K}\beta^6k^2(3+kb^2)\alpha^2=\mathbf{K}\beta^8k^3.\end{aligned}\tag{19}$$

Clearly, the left of (19) is divisible by α^2. So we can obtain that the flag curvature $\mathbf{K}=0$ because $k\neq 0$ and β^8 is not divisible by α^2.

Substituting $\mathbf{K}=0$ into (18), we have $s_{i0}s^i{}_0=a_{ij}(x)s^j{}_0s^i{}_0=0$. Because $(a_{ij}(x))$ is positive definite, we have $s^i_0=0$, i.e., β is closed. By (17), we know that β is parallel with respect to α. Then $F=\alpha+\varepsilon\beta+k\frac{\beta^2}{\alpha}$ is a Berwald metric, where ε and $k\neq 0$ are constants. Hence F must be locally Minkowskian.

Case 2 $F=\frac{\alpha^2}{\alpha-\beta}$. In this case,

$$\Delta=\frac{1+2b^2-3s}{(2s-1)^2}.$$

Then (18) becomes

$$(1+2b^2-3s)(s-1)^3s_{i0}s^i{}_0-\mathbf{K}\alpha^2(b^2-s^2)(2s-1)^3=0.$$

Multiplying this equation with α^4 yields $A+\alpha B=0$, where

$$\begin{aligned}A=&-\mathbf{K}b^2\alpha^6+[(1+2b^2)s_{i0}s^i{}_0+\mathbf{K}\beta^2(1-12b^2)]\alpha^4+\\&6\beta^2[2K\beta^2+(2+b^2)s_{i0}s^i{}_0]\alpha^2+3s_{i0}s^i{}_0\beta^4,\\B=&6\mathbf{K}\beta b^2\alpha^4+2\beta[K\beta^2(4b^2-3)-3(1+b^2)s_{i0}s^i{}_0]\alpha^2-\\&2\beta^3[4K\beta^2+(5+b^2)s_{i0}s^i{}_0].\end{aligned}$$

Obviously, we have $A=0$ and $B=0$.

By $A=0$ and the fact that β^4 is not divisible by α^2, we obtain $s_{i0}s^i{}_0=0$. Hence β is closed. By (17), we know that β is parallel with respect to α. Then F is a Berwald metric. From (18), we find that $\mathbf{K}=0$. Hence F is locally Minkowskian. □

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