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Pointwise Approximation by Szàsz-Mirakjan Quasi-Interpolants

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Abstract Recently some classical operator quasi-interpolants were introduced to obtain much faster convergence. A.T. Diallo investigated some approximation properties of Szàsz-Mirakjan Quasi–Interpolants, but he obtained only direct theorem with Ditzian-Totik modulus $\omega_\varphi^{2r}(f,t)$. In this paper, we extend Diallo's result and solve completely the characterization on the rate of approximation by the method of quasi-interpolants to functions $f\in C_B[0,\infty)$ by making use of the unified modulus $\omega_{\varphi^\lambda}^{2r}(f,t)$ $(0\le\lambda\le 1)$.

Keywords Quasi-interpolants; Szàsz-Mirakjan operator; equivalence theorem; Ditzian-Totik modulus; unified modulus.

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1. Introduction

For function $f\in C_B[0,\infty)$ ($C_B[0,\infty)$ denotes the set of continuous and bounded function), the n-th classical Szàsz-Mirakjan operator $S_n, n\in N$, is defined by

$$S_n(f,x)=:\sum_{k=0}^{\infty}s_{n,k}(x)f\left(\frac{k}{n}\right),\quad x\in[0,\infty), n\in N$$

where for $k=0,1,2,\ldots$,

$$s_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!}.$$

It is well known that for $f\in C_B[0,\infty)$, $\varphi(x)=\sqrt{x}$ and $0<\alpha<1$[4,8]

$$\|S_nf-f\|=O(n^{-\alpha})\Longleftrightarrow\omega_\varphi^2(f,t)=O(t^{2\alpha}),$$

where $\omega_\varphi^2(f,t)$ is Ditzian-Totik modulus. In [3], Ditzian introduced the unified modulus $\omega_{\varphi^\lambda}^2(f,t)=\sup_{0<h\le t}\|\Delta_{h\varphi^\lambda}^2f\|$, $0\le\lambda\le1$. With $\omega_{\varphi^\lambda}^2(f,t)$ the following pointwise approximation equivalence result is obtained in [5]

$$|S_n(f,x)-f(x)|=O\Big(\big(\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\big)^\alpha\Big)\Longleftrightarrow\omega_{\varphi^\lambda}^2(f,t)=O(t^\alpha)\quad(0<\alpha<2).$$

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In order to obtain faster convergence the so-called Szàsz-Mirakian quasi-interpolants were considered[1,2,7]. We first recall the construction of $S_n^{(k)}(f,x)$.

Let Π_n denote the space of algebraic polynomials of degree at most n. On Π_n the Szàsz-Mirakian operator S_n and its inverse S_n^{-1} can be expressed as linear differential operators with polynomials coefficients in the form $S_n=\sum_{j=0}^n\beta_j^nD^j$ and $S_n^{-1}=A_n=\sum_{j=0}^n\alpha_j^nD^j$ with $D=\frac{\mathrm{d}}{\mathrm{d}x}$ and $D^0=\mathrm{id}$.

Therefore Szàsz-Mirakian quasi-interpolant is defined by[2]

$$S_n^{(r)}=A_n^{(r)}\circ S_n=\sum_{j=0}^r\alpha_j^n(x)D^jS_n(f,x)=:\sum_{j=0}^r\alpha_j^n(x)S_{n,j}(f,x),\quad 0\le r\le n,$$

where $A_n^{(r)}=\sum_{j=0}^r\alpha_j^nD^j$. Of course, $S_n^{(0)}=S_n$, $S_n^{(n)}=\mathrm{id}$ on Π_n. Moreover, for $0\le r\le n$, $S_n^{(r)}p=p$ for all $p\in\Pi_r$. Diallo[2] estimates $\alpha_j^n(x)$ and obtains expression as follows:

$$\alpha_0^n(x)=1,\quad \alpha_1^n(x)=0\ \text{ and}$$

$$\alpha_j^n(x)=c_{j-1}^n\frac{x}{n^{j-1}}+c_{j-2}^n\frac{x^2}{n^{j-2}}+\cdots+c_{j'}^n\frac{x^{j-j'}}{n^{j'}},\quad j\ge 2 \tag{1.1}$$

where $j'=\left[\left(\frac{j+1}{2}\right)\right]$ and c_j^n are constants independent of n. Some approximation properties of $S_n^{(r)}$ have been investigated too.

Theorem 1.1[2] *Let $f\in C_B[0,\infty)$, $\varphi(x)=\sqrt{x}$, $n\ge 2r-1$, $r\in N$. Then there exists a constant $C>0$ independent of n and f such that*

$$\|S_n^{(2r-1)}f-f\|_\infty\le C\omega_\varphi^{2r}\Big(f,\frac{1}{\sqrt{n}}\Big)_\infty. \tag{1.2}$$

We note that there are not inverse and equivalent results in [2]. The intention of this paper is to extend this result and solve completely the characterization on the rate of approximation by the method of quasi-interpolants to functions $f\in C_B[0,\infty)$ by making use of the unified modulus $\omega_{\varphi^\lambda}^2(f,t)$. This leads to the following

Theorem 1.2 (Equivalence Result) *Let $f\in C_B[0,\infty)$, $\varphi(x)=\sqrt{x}$, $n\ge 4r$, $r\in N$, $0\le\lambda\le 1$. Then for $0<\alpha<2r$ the following two statements are equivalent:*

$$\begin{aligned}&(i)\quad |S_n^{(2r-1)}(f,x)-f(x)|=O\Big(\big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\big)^\alpha\Big)\\ &(ii)\quad \omega_{\varphi^\lambda}^{2r}(f,t)=O\big(t^{2\alpha}\big),\end{aligned} \tag{1.3}$$

where $\delta_n(x)=\max\Big\{\varphi(x),\frac{1}{\sqrt{n}}\Big\}\sim\varphi(x)+\frac{1}{\sqrt{n}}$.

In next section, we give a direct theorem which implies Theorem 1.1. Now we give the definitions of the unified modulus and K-functional

$$\omega_{\varphi^\lambda}^s(f,t)=\sup_{0<h\le t}\ \sup_{x\pm\frac{r}{2}h\varphi^\lambda(x)\in[0,\infty]}|\Delta_{h\varphi^\lambda(x)}^sf(x)|, \tag{1.4}$$

$$K_{\varphi^\lambda}^s(f,t^s)=\inf_{g\in w^s(\varphi,[0,\infty])}\{\|f-g\|_\infty+t^s\|\varphi^{s\lambda}g^{(s)}\|_\infty\}, \tag{1.5}$$

$$\overline{K}_{\varphi^\lambda}^s(f,t^s)=\inf_{g\in w^s(\varphi,[0,\infty])}\{\|f-g\|_\infty+t^s\|\varphi^{s\lambda}g^{(s)}\|_\infty+t^{\frac{s}{1-\lambda/2}}\|g^{(s)}\|_\infty\}, \tag{1.6}$$

where $w^s(\varphi,[0,\infty))=\{g\in C[0,\infty),\ g^{(s-1)}\in A.C._{loc}[0,\infty),\ \|\varphi^{s\lambda}g^{(s)}\|<\infty,\ \|g^{(s)}\|<\infty\}$.

It was proved in [4] that

$$\omega^s_{\varphi^\lambda}(f,t)\sim K^s_{\varphi^\lambda}(f,t^s)\sim \overline{K}^s_{\varphi^\lambda}(f,t^s). \tag{1.7}$$

Throughout this paper $\|\cdot\|$ denotes $\|\cdot\|_\infty$, C denotes a positive constant not necessarily the same at each occurrence.

2. Direct theorem

We will use the following results.

Lemma 2.1[2] *The following estimates hold:*

(1) For $x\in E_n^c=[0,\frac{1}{n})$ and $j\ge 2$, we have

$$|\alpha^n_j(x)|\le Cn^{-j}. \tag{2.1}$$

(2) For $m\ge 1$ and $x\in E_n=[\frac{1}{n},\infty)$, we get

$$|\alpha^n_{2m}(x)|\le Cn^{-m}\varphi^{2m}(x),\quad |\alpha^n_{2m+1}(x)|\le Cn^{-m-\frac12}\varphi^{2m+1}(x). \tag{2.2}$$

Lemma 2.2 *(1) For $x\in E_n^c$ and $j\ge 2$, we have*

$$\left|D^r(\alpha^n_j(x))\right|\le Cn^{-j+r}. \tag{2.3}$$

(2) For $m\ge 1$ and $x\in E_n$, we get

$$|D^r\alpha^n_{2m}(x)|\le Cn^{-m+\frac{r}{2}}\varphi^{2m-r}(x),\quad |D^r\alpha^n_{2m+1}(x)|\le Cn^{-m+\frac{r-1}{2}}\varphi^{2m-r+1}(x). \tag{2.4}$$

Proof Let us consider the r-th derivative of α^n_j in (1.1) for $j=2m$, $r\le m$ by

$$\begin{aligned} D^r(\alpha^n_{2m}) &= D^r\Big(c^n_{2m-1}\frac{x}{n^{2m-1}}+c^n_{2m-2}\frac{x^2}{n^{2m-2}}+\cdots+c^n_m\frac{x^m}{n^m}\Big)\\ &\le C\Big(c^n_{2m-r}\frac{1}{n^{2m-r}}+c^n_{2m-r-1}\frac{x}{n^{2m-r-1}}+\cdots+c^n_m\frac{x^{m-r}}{n^m}\Big). \end{aligned}\tag{2.5}$$

Firstly, we consider $x\in E_n^c=[0,\frac{1}{n})$, where $x<\frac{1}{n}$. So

$$\begin{aligned} D^r(\alpha^n_{2m}) &\le C\Big(c^n_{2m-r}\frac{1}{n^{2m-r}}+c^n_{2m-r-1}\frac{x}{n^{2m-r-1}}+\cdots+c^n_m\frac{x^{m-r}}{n^m}\Big)\\ &= \frac{C}{n^{2m-r}}\Big(c^n_{2m-r}+c^n_{2m-r-1}nx+\cdots+c^n_m(nx)^{m-r}\Big)\\ &\le Cn^{-2m+r}. \end{aligned}\tag{2.6}$$

Secondly, we consider $x\in E_n$, where $x\ge\frac{1}{n}$. So

$$\begin{aligned} D^r(\alpha^n_{2m}) &\le C\Big(c^n_{2m-r}\frac{1}{n^{2m-r}}+c^n_{2m-r-1}\frac{x}{n^{2m-r-1}}+\cdots+c^n_m\frac{x^{m-r}}{n^m}\Big)\\ &= \frac{Cx^{m-r}}{n^m}\Big(c^n_{2m-r}(nx)^{r-m}+c^n_{2m-r-1}(nx)^{r-m+1}+\cdots+c^n_m\Big)\\ &\le Cn^{-m}x^{m-r}\le Cn^{-m+\frac{r}{2}}x^{m-\frac{r}{2}}(nx)^{-\frac{r}{2}}\\ &\le Cn^{-m+\frac{r}{2}}\varphi^{2m-r}(x). \end{aligned}\tag{2.7}$$

When $r > m$, (2.4) is valid obviously.

So we proved the estimates for the derivatives of α_j^n for the polynomial of even degree $j = 2m$. For the polynomial of odd degree $j = 2m+1$ we can get the estimation with similar computations. Then the lemma is proved. □

Theorem 2.3 *If $\varphi(x) = \sqrt{x}$, $\delta_n(x) = \max\{\varphi(x), \frac{1}{\sqrt{n}}\}$, $0 \le \lambda \le 1$, $n \ge 2r-1$, then for $f \in C_B[0,\infty)$, we have*

$$\left|S_n^{(2r-1)}(f,x) - f(x)\right| \le C\omega_{\varphi^\lambda}^{2r}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{2.8}$$

Remark If $\lambda = 1$, then (2.8) is the result in [2].

Proof By the definition of $\overline{K}_{\varphi^\lambda}^{2r}(f, t^{2r})$, for fixed n, x, λ, we can choose $g(t) = g_{\lambda,n,x}(t)$ such that

$$\begin{aligned} &\|f-g\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{2r}\left\|\varphi^{2r\lambda}g^{(2r)}\right\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{2r}{1-\lambda/2}}\|g^{(2r)}\| \\ &\le 2\overline{K}_{\varphi^\lambda}^{2r}\left(f, \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{2r}\right). \end{aligned} \tag{2.9}$$

It is known that[2] $\|S_n^{(k)}\| \le M$, where M is a constant independent of n.

Since $S_n^{(k)}$ is exact on Π_k, i.e., $S_n^{(k)}p = p$ for $p \in \Pi_k$[2], we have

$$\begin{aligned} |S_n^{(2r-1)}(f,x) - f(x)| &\le C\left(\|f-g\| + |S_n^{(2r-1)}(g,x) - g(x)|\right) \\ &= C\left(\|f-g\| + |S_n^{(2r-1)}(R_{2r}(g,\cdot,x),x)|\right) \\ &=: C(\|f-g\| + I), \end{aligned} \tag{2.10}$$

where $R_{2r}(g,\cdot,x) = \frac{1}{(2r-1)!}\int_x^t (t-u)^{2r-1}g^{(2r)}(u)\mathrm{d}u$.

We only need estimate I. As $\alpha_0^n = 1, \alpha_1^n = 0$[2], we have

$$\begin{aligned} I &\le |S_n(R_{2r}(g,\cdot,x),x)| + \left|\sum_{j=2}^{2r-1}\alpha_j^n(x)D^jS_n(R_{2r}(g,\cdot,x),x)\right| \\ &=: \ I_0 + \left|\sum_{j=2}^{2r-1}\alpha_j^n(x)I_j\right|. \end{aligned} \tag{2.11}$$

The following estimate is known[5]

$$I_0 \le C\omega_{\varphi^\lambda}^{2r}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{2.12}$$

To estimate I_j we have to consider two cases

Case 1 For $x \in E_n^c$, $\delta_n(x) \sim \frac{1}{\sqrt{n}}$, using formula[4,(9.4.3)]

$$S_{n,j}(f,x) = n^j\sum_{k=0}^{\infty}s_{n,k}(x)\left(\overrightarrow{\Delta}_{\frac{1}{n}}^j f\right)\left(\frac{k}{n}\right) \tag{2.13}$$

where $\overrightarrow{\Delta}^j_{\frac{1}{n}} f\left(\frac{k}{n}\right)$ are j-th forward differences, and formula [2,(35)],[4,(9.6.1)]

$$|R_{2r}(g,t,x)| \leq \frac{\left|\frac{k}{n}-x\right|^{2r-1}}{\delta_n^{2r\lambda}(x)}\left|\int_x^{\frac{k}{n}} \delta_n^{2r\lambda}(x) g^{(2r)}(u)\mathrm{d}u\right|,$$

we have

$$\begin{aligned}
|I_j| &= |D^j S_n(R_{2r}(g,\cdot,x),x)| \\
&= \left| n^j \sum_{k=0}^{\infty} s_{n,k}(x) \overrightarrow{\Delta}^j_{\frac{1}{n}} \left(\frac{1}{(2r-1)!} \int_x^{\frac{k}{n}} \left(\frac{k}{n}-u\right)^{2r-1} g^{(2r)}(u)\mathrm{d}u \right) \right| \\
&\leq n^j \sum_{k=0}^{\infty} s_{n,k}(x) \sum_{i=0}^{j} \binom{j}{i} \left| \int_x^{\frac{k+i}{n}} \left(\frac{k+i}{n}-u\right)^{2r-1} g^{(2r)}(u)\mathrm{d}u \right| \\
&\leq Cn^j \sum_{k=0}^{\infty} s_{n,k}(x) \sum_{i=0}^{j} \|\delta_n^{2r\lambda} g^{(2r)}\| \frac{\left(\frac{k+i}{n}-x\right)^{2r}}{\delta_n^{2r\lambda}(x)} \\
&\leq Cn^j \delta_n^{-2r\lambda}(x) \|\delta_n^{2r\lambda} g^{(2r)}\| \sum_{i=0}^{j}\sum_{k=0}^{\infty} s_{n,k}(x) \left(\left(\frac{k}{n}-x\right)^{2r} + \left(\frac{i}{n}\right)^{2r} \right) \\
&\leq Cn^j \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}} \right)^{2r} \|\delta_n^{2r\lambda} g^{(2r)}\|. \qquad (2.14)
\end{aligned}$$

In the last step we have used that $S_n\big((t-x)^{2r},x\big) \leq cn^{-r}\delta_n^{2r}(x)$. Noting that $|\alpha_j^n(x)| \leq Cn^{-j}$ and $\delta_n(x) \sim \frac{1}{\sqrt{n}}$ for $x \in E_n^c$, we have by (2.11), (2.13), (2.14)

$$\begin{aligned}
\left|\sum_{j=2}^{2r-1} \alpha_j^n(x) I_j\right| &\leq C\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{2r} \|\delta_n^{2r\lambda} g^{(2r)}\| \\
&\leq C\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{2r} \|\varphi^{2r\lambda} g^{(2r)}\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{2r}{1-\lambda/2}} \|g^{(2r)}\|. \qquad (2.15)
\end{aligned}$$

Case 2 For $x \in E_n$, $\delta_n(x) \sim \varphi(x)$, by the formula[2,4]

$$D^j s_{n,k}(x) \leq C \sum_{i=0}^{j} \left(\frac{\sqrt{n}}{\varphi(x)}\right)^{j+i} \left|\frac{k}{n}-x\right|^i s_{n,k}(x), \qquad (2.16)$$

we have

$$\begin{aligned}
|I_i| &= \left| D^j \sum_{k=0}^{n} s_{n,k}(x) \frac{1}{(2r-1)!} \int_x^{\frac{k}{n}} \big(\frac{k}{n}-u\big)^{2r-1} g^{(2r)}(u)\mathrm{d}u \right| \\
&\leq C \sum_{k=0}^{n} \sum_{i=0}^{j} \Big(\frac{\sqrt{n}}{\varphi(x)}\Big)^{j+i} \big|\frac{k}{n}-x\big|^i s_{n,k}(x) \frac{(\frac{k}{n}-x)^{2r}}{\varphi^{2r\lambda}(x)} \|\varphi^{2r\lambda} g^{(2r)}\| \\
&\leq C \sum_{i=0}^{j} \Big(\frac{\sqrt{n}}{\varphi(x)}\Big)^{j+i} \varphi^{-2r\lambda}(x) \|\varphi^{2r\lambda} g^{(2r)}\| \frac{\varphi^{2r+i}(x)}{n^{r+\frac{i}{2}}}.
\end{aligned}$$

Hence with (2.2)

$$\Big|\sum_{j=2}^{2r-1} \alpha_j^n(x) I_j\Big| \leq C\left(\frac{\varphi^{(1-\lambda)}(x)}{\sqrt{n}}\right)^{2r} \|\varphi^{2r\lambda} g^{(2r)}\|$$

$$= C\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{2r}\|\varphi^{2r\lambda}g^{(2r)}\|. \tag{2.17}$$

From (2.10), (2.12), (2.15) and (2.16) it follows

$$\left|S_n^{(2r-1)}(f,x)-f(x)\right|\le C\omega_{\varphi^\lambda}^{2r}\left(f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right).$$

The proof is completed. □

3. Inverse theorem

To prove inverse theorem we need the following lemma.

Lemma 3.1 *For $n\ge 4r, r\in N, r\ge 2$, we have*

$$\left|\varphi^{2r\lambda}(x)D^{2r}S_n^{(2r-1)}(f,x)\right|\le Cn^r\delta_n^{2r(\lambda-1)}(x)\|f\|\quad (f\in C_B[0,\infty)), \tag{3.1}$$

$$\left|\varphi^{2r\lambda}(x)D^{2r}S_n^{(2r-1)}(f,x)\right|\le C\|\varphi^{2r\lambda}f^{(2r)}\|\quad (f\in w^{2r}(\varphi,[0,\infty)). \tag{3.2}$$

Proof At first let us prove (3.1). We consider the case $\lambda=1$ firstly. Since $\alpha_0^n=1$, $\alpha_1^n=0$, $\alpha_j^n\in\Pi_j$ and $\lambda=1$, $j\ge 2$, we have that for all $x\in[0,\infty)$

$$\begin{aligned}
&\varphi^{2r}(x)D^{2r}S_n^{(2r-1)}(f,x)\\
&=\varphi^{2r}(x)D^{2r}\big(\sum_{j=0}^{2r-1}\alpha_j^n(x)S_{n,j}(f,x)\big)\\
&=\varphi^{2r}(x)S_{n,2r}(f,x)+\sum_{j=2}^{2r-1}\varphi^{2r}(x)\sum_{k=0}^{j}\binom{2r}{k}D^k(\alpha_j^n(x))S_{n,2r+j-k}(f,x)\\
&=:\varphi^{2r}(x)S_{n,2r}(f,x)+S.
\end{aligned} \tag{3.3}$$

We will use the following known estimate for the derivatives of S_n. From formula (9.4.1) in [4] and the procedure of the proof of Theorem 9.4.1 in [4] we can get that

$$|\varphi^{2r}(x)S_{n,2r}(f,x)|\le Cn^r\|f\|$$

and

$$|\varphi^{2r+s}(x)S_{n,2r+s}(f,x)|\le Cn^{\frac{2r+s}{2}}\|f\|. \tag{3.4}$$

To estimate S, we have to consider again the following two cases

Case I First let us consider the above sum S in (3.3) for $x\in E_n^c$: For $S_{n,2r+j-k}$ we use formula (9.4.3) in [4] with $2r+j-k$ instead of m, i.e.,

$$S_{n,2r+j-k}(f,x)=n^{2r+j-k}\sum_{i=0}^{\infty}s_{n,i}(x)\overrightarrow{\Delta}_{\frac{1}{n}}^{2r+j-k}f\left(\frac{i}{n}\right)$$

and with $\left|\overrightarrow{\Delta}_{\frac{1}{n}}^{2r+j-k}f\left(\frac{i}{n}\right)\right|\le C\|f\|$ we have

$$|S_{n,2r+j-k}(f,x)|\le Cn^{2r+j-k}\|f\|\ \text{ for }\ j\ge 2. \tag{3.5}$$

With (2.2), i.e., $|D^k(\alpha_j^n(x))| \le Cn^{-j+k}$ and $\varphi^2(x) \le n^{-1}$ for $x \in E_n^c$, we get for the sum S in (3.3):

$$\begin{aligned}
|S| &\le \sum_{j=2}^{2r-1} \varphi^{2r}(x) \sum_{k=0}^{j} \binom{2r}{k} \left|D^k(\alpha_j^n(x))\right| \cdot \left|S_{n,2r+j-k}(f,x)\right| \\
&\le C \sum_{j=2}^{2r-1} n^{-r} \sum_{k=0}^{j} \binom{2r}{k} n^{-j+k} n^{2r+j-k} \|f\| \\
&\le Cn^r \|f\|. \qquad (3.6)
\end{aligned}$$

Case II Secondly, we have with (3.4) (for $s = j-k$) and (2.4), i.e.,

$$\left|D^k(\alpha_j^n(x))\right| \le Cn^{-(j-k)/2} \varphi^{j-k}(x)$$

for $x \in E_n$:

$$\begin{aligned}
|S| &\le \sum_{j=2}^{2r-1} \varphi^{2r}(x) \sum_{k=0}^{j} \binom{2r}{k} \left|D^k(\alpha_j^n(x))\right| \cdot \left|S_{n,2r+j-k}(f,x)\right| \\
&\le C \sum_{j=2}^{2r-1} \sum_{k=0}^{j} \binom{2r}{k} n^{\frac{-j+k}{2}} \left|\varphi^{2r+j-k} S_{n,2r+j-k}(f,x)\right| \\
&\le C \sum_{j=2}^{2r-1} \sum_{k=0}^{j} \binom{2r}{k} n^{\frac{-j+k}{2}} n^{\frac{2r+j-k}{2}} \|f\| \\
&\le Cn^r \|f\|. \qquad (3.7)
\end{aligned}$$

Combining (3.3), (3.5) and (3.6) for all $x \in [0,\infty)$, we have

$$\left|\varphi^{2r}(x) D^{2r} S_n^{(2r-1)}(f,x)\right| \le Cn^r \|f\|.$$

Next we consider the case $0 \le \lambda < 1$. If $x \in E_n^c$, using (3.5) and $\varphi^{2r\lambda}(x) \le n^{-r\lambda}$, we get

$$|\varphi^{2r\lambda} D^{2r} S_n^{(2r-1)}(f,x)| \le Cn^{-r\lambda} n^{2r} \|f\| \le Cn^r \delta_n^{2r(\lambda-1)}(x) \|f\|.$$

If $x \in E_n$, then $\delta_n(x) \sim \varphi(x)$ and we have

$$\left|\varphi^{2r\lambda}(x) D^{2r} S_n^{(2r-1)}(f,x)\right| = \varphi^{2r(\lambda-1)}(x) \left|\varphi^{2r}(x) D^{2r} S_n^{(2r-1)}(f,x)\right| \le Cn^r \delta_n^{2r(\lambda-1)}(x) \|f\|,$$

which completes the proof of the first inequality (3.1) for all $x \in [0,\infty)$.

Now we come to the proof of the second inequality (3.2). We have as $\alpha_0^n = 1, \alpha_1^n = 0$ and $\alpha_j^n \in \Pi_j$, $j \ge 2$ that for all $x \in [0,1]$

$$\begin{aligned}
&\varphi^{2r\lambda}(x) D^{2r} S_n^{(2r-1)}(f,x) \\
&= \varphi^{2r\lambda}(x) D^{2r} \Big(\sum_{j=0}^{2r-1} \alpha_j^n(x) S_{n,j}(f,x) \Big) \\
&= \varphi^{2r\lambda}(x) S_{n,2r}(f,x) + \sum_{j=2}^{2r-1} \varphi^{2r\lambda}(x) \sum_{i=0}^{j} \binom{2r}{i} D^i\big(\alpha_j^n(x)\big) S_{n,2r+j-i}(f,x) \\
&= \varphi^{2r\lambda}(x) S_{n,2r}(f,x) + S_1. \qquad (3.8)
\end{aligned}$$

From formula (5.3) in [5] we have for $x \in [0,\infty)$

$$\big|\varphi^{2r\lambda}(x)S_{n,2r}(f,x)\big| \le C\|\varphi^{2r\lambda}f^{(2r)}\|. \tag{3.9}$$

To estimate S_1, we need to consider two differenct cases. The first case, for $x \in E_n^c$

$$\begin{aligned}
&|S_{n,2r+j-i}(f,x)|\\
&=\Big|n^{2r+j-i}\sum_{k=0}^{\infty}s_{n,k}(x)\overrightarrow{\Delta}_{\frac{1}{n}}^{2r+j-i}f\big(\frac{k}{n}\big)\Big|\\
&=\Big|n^{2r+j-i}\sum_{k=0}^{\infty}s_{n,k}(x)\sum_{l=0}^{j-i}(-1)^{j-i-l}\binom{j-i}{l}\overrightarrow{\Delta}_{\frac{1}{n}}^{2r}f\big(\frac{k+l}{n}\big)\Big|\\
&\le Cn^{2r+j-i}\bigg(\sum_{k=0}^{\infty}s_{n,k}(x)\Big|\overrightarrow{\Delta}_{\frac{1}{n}}^{2r}f\big(\frac{k}{n}\big)\Big|+\sum_{k=0}^{\infty}s_{n,k}(x)\sum_{l=1}^{j-i}\Big|\overrightarrow{\Delta}_{\frac{1}{n}}^{2r}f\big(\frac{k+l}{n}\big)\Big|\bigg)\\
&=: Cn^{2r+j-i}(I_1+I_2).
\end{aligned} \tag{3.10}$$

Observing[4, p 155, (c)]

$$\begin{aligned}
\Big|\big(\overrightarrow{\Delta}_{\frac{1}{n}}^{2r}f\big)\big(\frac{k}{n}\big)\Big| &\le C\begin{cases} n^{-r+1}\displaystyle\int_0^{\frac{2r}{n}}u^r\big|f^{(2r)}(u)\big|\mathrm{d}u, & k=0,\\ n^{-2r+1}\displaystyle\int_0^{\frac{2r}{n}}\big|f^{(2r)}\big(\frac{k}{n}+u\big)\big|\mathrm{d}u, & k=1,2,\dots,\end{cases}\\
&\le C\begin{cases} n^{-r}\big\|\varphi^{2r\lambda}f^{(2r)}\big\|n^{-r(1-\lambda)}, & k=0,\\ n^{-2r}\big\|\varphi^{2r\lambda}f^{(2r)}\big\|\Big(\dfrac{k}{n}\Big)^{-r\lambda}, & k=1,2,\dots,\end{cases}
\end{aligned} \tag{3.11}$$

we have

$$I_1 \le C\big(n^{-r}n^{-r(1-\lambda)}\big\|\varphi^{2r\lambda}f^{(2r)}\big\|+n^{-2r}\big\|\varphi^{2r\lambda}f^{(2r)}\big\|\sum_{k=1}^{\infty}s_{n,k}(x)\big(\frac{k}{n}\big)^{-r\lambda}\big).$$

By a simple computation, it is easy to get

$$\sum_{k=1}^{\infty}\big(\frac{n}{k}\big)^r s_{n,k}(x)=\sum_{k=1}^{\infty}e^{-nx}\frac{(nx)^{k+r}}{(k+r)!}\cdot\frac{k+1}{k}\cdot\frac{k+2}{k}\cdot\dots\cdot\frac{k+r}{k}\cdot\frac{1}{x^r}\le C\frac{1}{x^r}.$$

Hence for $\lambda \ne 0$ we have

$$\sum_{k=1}^{\infty}\big(\frac{n}{k}\big)^{\lambda r}s_{n,k}(x)\le\bigg(\sum_{k=1}^{\infty}\big(\frac{n}{k}\big)^r s_{n,k}(x)\bigg)^{\lambda}\le Cx^{-r\lambda}$$

and hence

$$\begin{aligned}
I_1 &\le C(n^{-2r}\|\varphi^{2r\lambda}f^{(2r)}\|n^{r\lambda}+n^{-2r}\|\varphi^{2r\lambda}f^{(2r)}\|x^{-r\lambda}\\
&=Cn^{-2r}(n^{r\lambda}+\varphi^{-2r\lambda}(x))\|\varphi^{2r\lambda}f^{(2r)}\|.
\end{aligned} \tag{3.12}$$

For $\lambda = 0$, (3.12) obviously holds too. Thus we get

$$I_1 \le C\big\|\varphi^{2r\lambda}f^{(2r)}\big\|n^{-2r}\big(n^{r\lambda}+\varphi^{-2r\lambda}(x)\big). \tag{3.13}$$

From the procedure of the proof (3.12), we can deduce that

$$I_2 \le Cn^{-2r}\big\|\varphi^{2r\lambda}f^{(2r)}\big\|\varphi^{-2r\lambda}(x). \tag{3.14}$$

By (3.8)–(3.10), (3.13), (3.14) and (2.3) with $|\varphi(x)| \le \frac{1}{\sqrt{n}}$ for $x \in E_n^c$, we have

$$\big|\varphi^{2r\lambda}(x)D^{2r}S_n^{(2r-1)}(f,x)\big| \le C\|\varphi^{2r\lambda}f^{(2r)}\|, \quad x \in E_n^c. \tag{3.15}$$

The secend case, for $x \in E_n$, by (2.16), we have

$$\begin{aligned}
&\varphi^{2r\lambda}(x)\big|S_{n,2r+j-i}(f,x)\big| \\
&= \varphi^{2r\lambda}(x)\big|D^{j-i}S_{n,2r}(f,x)\big| \\
&= \varphi^{2r\lambda}(x)\Big|D^{j-i}n^{2r}\sum_{k=0}^{\infty}s_{n,k}(x)\overrightarrow{\Delta}^{2r}_{\frac{1}{n}}f\big(\frac{k}{n}\big)\Big| \\
&= \varphi^{2r\lambda}(x)\Big|n^{2r}\sum_{k=0}^{\infty}D^{j-i}s_{n,k}(x)\overrightarrow{\Delta}^{2r}_{\frac{1}{n}}f\big(\frac{k}{n}\big)\Big| \\
&\le C\varphi^{2r\lambda}(x)n^{2r}\sum_{k=0}^{\infty}\left\{\sum_{l=0}^{j-i}\Big(\frac{\sqrt{n}}{\varphi(x)}\Big)^{j-i+l}\Big|\frac{k}{n}-x\Big|^{l}\right\}s_{n,k}(x)\Big|\overrightarrow{\Delta}^{2r}_{\frac{1}{n}}f\big(\frac{k}{n}\big)\Big| \\
&\le C\varphi^{2r\lambda}(x)n^{2r}\sum_{l=0}^{j-i}\Big(\frac{\sqrt{n}}{\varphi(x)}\Big)^{j-i+l}\cdot \\
&\quad\left\{\Big(\sum_{k=0}^{\infty}\Big|\frac{k}{n}-x\Big|^{\frac{l}{1-\lambda}}s_{n,k}(x)\Big)^{1-\lambda}\Big(\sum_{k=0}^{\infty}\Big|\overrightarrow{\Delta}^{2r}_{\frac{1}{n}}f\big(\frac{k}{n}\big)\Big|^{\frac{1}{\lambda}}s_{n,k}(x)\Big)^{\lambda}\right\} \\
&= Cn^{2r(1-\lambda)}\sum_{l=0}^{j-i}\Big(\frac{\sqrt{n}}{\varphi(x)}\Big)^{j-i+l}\left\{\Big(\sum_{k=0}^{\infty}\Big|\frac{k}{n}-x\Big|^{\frac{l}{1-\lambda}}s_{n,k}(x)\Big)^{1-\lambda}\cdot\right. \\
&\quad\left.\Big(\sum_{k=0}^{\infty}n^{2r}\varphi^{2r}(x)\Big|\overrightarrow{\Delta}^{2r}_{\frac{1}{n}}f\big(\frac{k}{n}\big)\Big|^{\frac{1}{\lambda}}s_{n,k}(x)\Big)^{\lambda}\right\} \\
&=: Cn^{2r(1-\lambda)}\sum_{l=0}^{j-i}\Big(\frac{\sqrt{n}}{\varphi(x)}\Big)^{j-i+l}\big\{J_1\cdot J_2\big\}.
\end{aligned} \tag{3.16}$$

Noting that

$$\begin{aligned}
\varphi^{2r}(x)s_{n,k}(x) &= x^r e^{-nx}\frac{(nx)^k}{k!} = \frac{(k+1)(k+2)\cdots(k+r)}{n^r}s_{n,k+r}(x) \\
&\le \begin{cases} \dfrac{r!}{n^r}s_{n,r}(x) & k=0, \\ C\left(\dfrac{k}{n}\right)^r s_{n,k+r}(x) & k\ne 0, \end{cases}
\end{aligned}$$

and (3.11) with $\varphi^{2\lambda}\big(\frac{k}{n}\big) \le \varphi^{2\lambda}\big(\frac{k}{n}+y\big)$, $k>0$, $0<y<\frac{2r}{n}$, we have

$$\begin{aligned}
J_2 \le& C\left\{n^r s_{n,r}(x)\Big|\overrightarrow{\Delta}^{2r}_{\frac{1}{n}}f(0)\Big|^{\frac{1}{\lambda}} + n^{2r}\sum_{k=1}^{\infty}s_{n,k+r}(x)\left(\frac{k}{n}\right)^r\Big|\overrightarrow{\Delta}^{2r}_{\frac{1}{n}}f\big(\frac{k}{n}\big)\Big|^{\frac{1}{\lambda}}\right\}^{\lambda} \\
\le& C\left\{n^r s_{n,r}(x)\left(n^{-r+1}\Big|\int_0^{\frac{2r}{n}}u^r f^{(2r)}(u)\mathrm{d}u\Big|\right)^{\frac{1}{\lambda}}+\right. \\
&\left. n^{2r}\sum_{k=1}^{\infty}s_{n,k+r}(x)\left(\frac{k}{n}\right)^r\left(n^{-2r+1}\Big|\int_0^{\frac{2r}{n}}f^{(2r)}\big(\frac{k}{n}+u\big)\mathrm{d}u\Big|\right)^{\frac{1}{\lambda}}\right\}^{\lambda}
\end{aligned}$$

$$\leq C\Big\{n^r\big(n^{-r}n^{-r(1-\lambda)}\big\|\varphi^{2r\lambda}f^{(2r)}\big\|\big)^{\frac{1}{\lambda}}+$$

$$n^{2r}\sum_{k=1}^{\infty}s_{n,k+r}(x)\bigg(n^{-2r+1}\Big|\int_0^{\frac{2r}{n}}\varphi^{2r\lambda}\Big(\frac{k}{n}+u\Big)f^{(2r)}\Big(\frac{k}{n}+u\Big)\mathrm{d}u\Big|\bigg)^{\frac{1}{\lambda}}\Big\}^{\lambda}$$

$$\leq C\Big\{n^r\big(n^{-r}n^{-r(1-\lambda)}\big\|\varphi^{2r\lambda}f^{(2r)}\big\|\big)^{\frac{1}{\lambda}}+n^{2r}\sum_{k=1}^{\infty}s_{n,k+r}(x)\big(n^{-2r}\big\|\varphi^{2r\lambda}f^{(2r)}\big\|\big)^{\frac{1}{\lambda}}\Big\}^{\lambda}$$

$$\leq Cn^{-2r(1-\lambda)}\big\|\varphi^{2r\lambda}f^{(2r)}\big\|. \tag{3.17}$$

By formula (9.4.14) in [4] we choose $q\in N$ such that $2q(1-\lambda)>1$. Then

$$J_1\leq\Big(\sum_{k=0}^{\infty}\big(\frac{k}{n}-x\big)^{2ql}s_{n,k}(x)\Big)^{\frac{1}{2q}}\leq Cn^{-\frac{l}{2}}\varphi^l(x). \tag{3.18}$$

Combining (3.8), (3.9), (3.16)–(3.18) together with (2.4) for $0<\lambda<1$, we have

$$\big|\varphi^{2r\lambda}(x)D^{2r}B_n^{(2r-1)}(f,x)\big|\leq C\big\|\varphi^{2r\lambda}f^{(2r)}\big\|. \tag{3.19}$$

From the above procedure we know that for the case $\lambda=0$ (the case $\lambda=1$ is similar) we need not use Hölder inequality in (3.16) and it is easy to get (3.19).

Combining (3.15) and (3.19) we complete the proof of (3.2). □

Theorem 3.2 *Let $f\in C_B[0,\infty)$, $n\geq 4r, r\in N, 0\leq\lambda\leq 1, 0<\alpha<2r$. Then we have*

$$\big|S_n^{(2r-1)}(f,x)-f(x)\big|=O\Big(\big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\big)^{\alpha}\Big)$$

implies

$$\omega_{\varphi^\lambda}^{2r}(f,t)=O(t^\alpha).$$

Proof The proof of Theorem 3.2 is similar to [6, p145 " $\Longleftarrow$ "] by using Lemma 3.1. The details are omitted. □

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