Journal of Mathematical Research & Exposition
July, 2009, Vol. 29, No. 4, pp. 649–661
DOI:10.3770/j.issn:1000-341X.2009.04.010
Http://jmre.dlut.edu.cn

Possible Spectrums of 3×3 Upper Triangular Operator Matrices

HAI Guo Jun, Alatancang
(Department of Mathematics, Inner Mongolia University, Inner Mongolia 010021, China)
(E-mail: 3695946@163.com)

Abstract Let H_1, H_2 and H_3 be infinite dimensional separable complex Hilbert spaces. We denote by $M_{(D,E,F)}$ a 3×3 upper triangular operator matrix acting on $H_1 \oplus H_2 \oplus H_3$ of the form $M_{(D,E,F)}=\begin{pmatrix} A & D & E \\ 0 & B & F \\ 0 & 0 & C \end{pmatrix}$. For given $A \in \mathcal{B}(H_1)$, $B \in \mathcal{B}(H_2)$ and $C \in \mathcal{B}(H_3)$, the sets $\bigcup_{D,E,F} \sigma_p(M_{(D,E,F)})$, $\bigcup_{D,E,F} \sigma_r(M_{(D,E,F)})$, $\bigcup_{D,E,F} \sigma_c(M_{(D,E,F)})$ and $\bigcup_{D,E,F} \sigma(M_{(D,E,F)})$ are characterized, where $D \in \mathcal{B}(H_2, H_1)$, $E \in \mathcal{B}(H_3, H_1)$, $F \in \mathcal{B}(H_3, H_2)$ and $\sigma(\cdot)$, $\sigma_p(\cdot)$, $\sigma_r(\cdot)$, $\sigma_c(\cdot)$ denote the spectrum, the point spectrum, the residual spectrum and the continuous spectrum, respectively.

Keywords 3×3 upper triangular operator matrices; point spectrum; continuous spectrum; residual spectrum; spectrum.

Document code A
MR(2000) Subject Classification 47A10
Chinese Library Classification O177.1

1. Introduction

Let H_1, H_2 and H_3 be infinite dimensional separable complex Hilbert spaces, and let $\mathcal{B}(H_i, H_j)$ $(i, j = 1, 2, 3)$ denote the Banach space of all bounded linear operators from H_i to H_j, and abbreviate $\mathcal{B}(H_i, H_i)$ to $\mathcal{B}(H_i)$. If $T \in \mathcal{B}(H_i, H_j)$, write T^* for the conjugate of T, $R(T)$ for the range space of T and $N(T)$ for the null space of T. $n(T)$ and $d(T)$ denote, respectively, the dimension of $N(T)$ and $N(T^*)$, i.e., $n(T) = \dim N(T)$, $d(T) = \dim N(T^*)$. For $T \in \mathcal{B}(H_i)$, if $R(T)$ is closed and $d(T) < \infty$, then T is called a lower (right) semi-Fredholm operator and T^* is called an upper (left) semi-Fredholm operator. When $A \in \mathcal{B}(H_1)$, $B \in \mathcal{B}(H_2)$ and $C \in \mathcal{B}(H_3)$ are given, we denote by $M_{(D,E,F)}$ a 3×3 upper triangular operator matrix of the form

$$\begin{pmatrix} A & D & E \\ 0 & B & F \\ 0 & 0 & C \end{pmatrix} : H_1 \oplus H_2 \oplus H_3 \longrightarrow H_1 \oplus H_2 \oplus H_3,$$

Received date: 2007-07-09; **Accepted date**: 2008-07-07
Foundation item: the National Natural Science Foundation of China (No. 10562002); the Specialized Research Foundation for the Doctoral Program of Higher Education (No. 20070126002) and the Scientific Research Foundation for the Returned Overseas Chinese Scholars.

where $D \in \mathcal{B}(H_2, H_1)$, $E \in \mathcal{B}(H_3, H_1)$ and $F \in \mathcal{B}(H_3, H_2)$ are arbitrary. For convenience, when $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$ are given, we denote by M_D a 2×2 upper triangular operator matrix $\begin{pmatrix} A & D \\ 0 & B \end{pmatrix} \in \mathcal{B}(H_1 \oplus H_2)$, where $D \in \mathcal{B}(H_2, H_1)$ is arbitrary.

We denote the complex number by λ, the identity operator by I and complex number field by $\mathbb{C}$. Let X be a Hilbert space. For $T \in \mathcal{B}(X)$, the lower Fredholm spectrum of T is defined as $\sigma_{le}(T) = \{\lambda : T - \lambda I$ is not a lower semi-Fredholm operator$\}$; the resolvent set $\rho(T)$ and the spectrum $\sigma(T)$ of T are defined by $\rho(T) = \{\lambda : N(T - \lambda I) = \{0\}, R(T - \lambda I) = X\}$; $\sigma(T) = \mathbb{C} \backslash \rho(T)$. Furthermore, the spectrum $\sigma(T)$ is classified by two different forms. The one form: the spectrum $\sigma(T)$ is classified to the defect spectrum $\sigma_\delta(T)$ and the approximate point spectrum $\sigma_{ap}(T)$, and we define them by the forms

$\sigma_\delta(T) = \{\lambda : T - \lambda I$ is not surjective$\}$,

$\sigma_{ap}(T) = \{\lambda :$ there exists $x_n \in X, \|x_n\| = 1$ such that $\|(T - \lambda I)x_n\| \to 0 \ (n \to \infty)\}$.

It is not hard to find that $\lambda \notin \sigma_{ap}(T)(\sigma_\delta(T))$ is equivalent to $T - \lambda I$ is left (right) invertible and $\sigma_\delta(T) \cup \sigma_{ap}(T) = \sigma(T)$. The other form: the spectrum $\sigma(T)$ is classified by the point spectrum $\sigma_p(T)$, the residual spectrum $\sigma_r(T)$ and the continuous spectrum $\sigma_c(T)$, and we define them by

$\sigma_p(T) = \{\lambda : T - \lambda I$ is not injective$\}$,

$\sigma_r(T) = \{\lambda : T - \lambda I$ is injective, $\overline{R(T - \lambda I)} \neq X\}$,

$\sigma_c(T) = \{\lambda : T - \lambda I$ is injective, $\overline{R(T - \lambda I)} = X$, and $R(T - \lambda I) \neq X\}$.

We know that operator matrix is a matrix with operators as its entries and the partial operator matrix is a operator matrix, in which some entries are known and others are unknown. In the process of studying the partial operator matrix $\begin{pmatrix} A & D \\ ? & B \end{pmatrix}$, Li[1] induced the definition of the possible spectrum, and called $\bigcup_{X \in \mathcal{B}(H_1, H_2)} \sigma(\begin{pmatrix} A & D \\ X & B \end{pmatrix})$ the possible spectrum of this partial operator matrix. In this paper, for the partial operator matrix $M = \begin{pmatrix} A & ? & ? \\ 0 & B & ? \\ 0 & 0 & C \end{pmatrix}$, $\bigcup_{D,E,F} \sigma(M_{(D,E,F)})$ is called the possible spectrum of M. Similarly, we also define the possible point spectrum $\bigcup_{D,E,F} \sigma_p(M_{(D,E,F)})$, the possible residual spectrum $\bigcup_{D,E,F} \sigma_r(M_{(D,E,F)})$ and the possible continuous spectrum $\bigcup_{D,E,F} \sigma_c(M_{(D,E,F)})$ of M, respectively.

The complementarity problems for the partial operator matrix is very important in operator theory. Recently, this problem, motivated by interpolation theory and control theory, has been studied in a variety of directions by a number of authors, and the spectral complementarity problem is an important direction. The spectral complementarity problem is to study the spectrum of completion of the partial operator matrix, the spectrum distribution and so on. As is known to all, if T is a bounded linear operator on a Hilbert space and has a nontrivial invariant subspace, then T can be decomposed to the form of 2×2 upper triangular operator matrix, so the 2×2 upper triangular operator matrix is studied by a number of authors. For example, when $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$ are given, the intersection of spectrum of M_D was obtained in [2]. After that, many authors studied the intersection of a variety spectra of M_D (see [3-7] and the references therein). Furthermore, the 3×3 operator matrix is studied by numerous authors. Such as, in [8], the author studied the invertibility of 3×3 operator matrix appearing in the linear-quadratic optimal control problem in a Hilbert space. In [9], the author gave the necessary and sufficient condition for $M_{(D,E,F)}$ to be an upper (lower) semi-Fredholm operator for some D, E, F. On this basis, in this paper, we characterize the possible spectrum, the possible point spectrum, the possible residual spectrum and the possible continuous spectrum of M.

2. Preliminaries

We first review some basic knowledge about linear operator and its spectra theory, and next prove some Lemmas and Corollaries.

Lemma 1[3] *There exists $D \in \mathcal{B}(H_2, H_1)$ such that 2×2 operator matrix M_D is left invertible if and only if A is left invertible and*

$$\begin{cases} n(B) \le d(A), & \text{if } R(B) \text{ is closed,} \\ d(A) = \infty, & \text{if } R(B) \text{ is not closed.} \end{cases}$$

Lemma 2[10] *Let X be a linear space, and let X_1 be a linear subspace of X. Then there exists a linear subspace X_2 of X such that $X_1 \cap X_2 = \{0\}$ and $X = X_1 + X_2$.*

Lemma 3[11] *Let X and Y be Banach spaces, $T \in \mathcal{B}(X,Y)$, and let $F \subset Y$ be a finite dimensional subspace. If $R(T) + F$ is closed, then $R(T)$ is closed too.*

Corollary 1 *Let X and Y be Banach spaces, $T \in \mathcal{B}(X,Y)$. If $R(T)$ is not closed, then there exists an infinite dimensional subspace $M \subset \overline{R(T)}$ such that $M \cap R(T) = \{0\}$ and $R(T) + M = \overline{R(T)}$.*

Proof Since $R(T)$ and $\overline{R(T)}$ are linear spaces and $R(T) \subset \overline{R(T)}$, there exists a linear subspace M of $\overline{R(T)}$ such that $M \cap R(T) = \{0\}$ and $R(T) + M = \overline{R(T)}$, by Lemma 2. At that time, M is infinite dimensional. Otherwise, suppose that M is finite dimensional. Since $R(T) + M = \overline{R(T)}$ is closed, $R(T)$ is closed, by Lemma 3, leading to a contradiction. □

Lemma 4 *Let $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$ be given operators, and let $R(A)$ be closed. If there exists $D \in \mathcal{B}(H_2, H_1)$ such that $0 \notin \sigma_p(M_D)$, then $n(B) \le d(A)$.*

Proof Suppose $n(B) > d(A)$. For any $D \in \mathcal{B}(H_2, H_1)$, if $N(B) \cap N(D) \ne \{0\}$, then $M_D(0 \oplus y) = 0$ for any nonzero $y \in N(B) \cap N(D)$; if $N(B) \cap N(D) = \{0\}$, then $\dim DN(B) = \dim N(B) = n(B) > d(A)$. Since $R(A)$ is closed, $DN(B) \cap R(A) \ne \{0\}$. Take $0 \ne z \in DN(B) \cap R(A)$. Then there exist nonzero $x \in H_1$ and $y \in N(B)$ such that $Ax = -Dy = z$, thus $M_D(x \oplus y) = 0$, which means that $0 \in \sigma_p(M_D)$ for every $D \in \mathcal{B}(H_2, H_1)$. It is a contradiction. □

Lemma 5 *Let $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$ be given operators, and let $R(A)$ be closed and $n(B) = d(A) < \infty$. For any $D \in \mathcal{B}(H_2, H_1)$, if $N(B) \cap N(D) = \{0\}$ and $DN(B) \cap R(A) = \{0\}$,* *then $D_3 = P_{R(A)^\perp} D P_{N(B)}$ as an operator from $N(B)$ into $R(A)^\perp$ is invertible, where $P_{R(A)^\perp}$ is the orthogonal projection onto $R(A)^\perp$ and $P_{N(B)}$ is the orthogonal projection onto $N(B)$.*

Proof Denote $n(B) = d(A) = n$. It follows from $N(B) \cap N(D) = \{0\}$ that $\dim DN(B) = n(B) = n < \infty$. Let $\{z_i\}_{i=1}^n$ be an orthogonal basis of $DN(B)$. Since $R(A)$ is closed, z_i has unique decomposition of the form $z_i = x_i + y_i$, $x_i \in R(A)$, $y_i \in R(A)^\perp$. Take arbitrary $\{\alpha_i\}_{i=1}^n$. If $\sum_{i=1}^n \alpha_i y_i = 0$, then $\sum_{i=1}^n \alpha_i z_i \in DN(B) \cap R(A) = \{0\}$, so $\alpha_i = 0$ $(i = 1, , 2, \ldots, n)$. Therefore, there exists a sequence $\{\beta_i\}_{i=1}^n$ such that $\sum_{i=1}^n \beta_i y_i = y$, for every $y \in R(A)^\perp$. However, $y + \sum_{i=1}^n \beta_i x_i = \sum_{i=1}^n \beta_i z_i \in DN(B)$, i.e., there exists $x \in N(B)$ such that $Dx = \sum_{i=1}^n \beta_i z_i$. Hence $D_3 x = y$. Consequently, D_3 is surjective. Also by $n(B) = d(A) < \infty$, D_3 is invertible. □

Lemma 6 *Let $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$ be given operators, and let A be a left invertible operator, $n(B) \le d(A) < \infty$. If there exists $D \in \mathcal{B}(H_2, H_1)$ such that M_D is injective, then*

$$d(M_D) = \begin{cases} d(B), & \text{if } d(A) = n(B), \\ d(A) + d(B) - n(B), & \text{if } R(B), \text{ is closed.} \end{cases}$$

And in the case when $R(B)$ is closed, $R(M_D)$ is closed too.

Proof If $n(B) = d(A) < \infty$, it follows from the injectivity of M_D that $DN(B) \cap R(A) = N(D) \cap N(B) = \{0\}$, and by Lemma 5, $P_{R(A)^\perp} D P_{N(B)}$ as an operator from $N(B)$ into $R(A)^\perp$ is invertible. Since A is left invertible, M_D has the following operator matrix

$$\begin{pmatrix} A_1 & D_1 & D_2 \\ 0 & D_3 & D_4 \\ 0 & 0 & B_1 \end{pmatrix} : H_1 \oplus N(B) \oplus N(B)^\perp \to R(A) \oplus R(A)^\perp \oplus H_2. \tag{2.1}$$

Clearly, A_1 and D_3 are invertible. Hence there exists an invertible operator $V \in \mathcal{B}(H_1 \oplus H_2)$ such that

$$\begin{pmatrix} A_1 & D_1 & D_2 \\ 0 & D_3 & D_4 \\ 0 & 0 & B_1 \end{pmatrix} V = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & D_3 & 0 \\ 0 & 0 & B_1 \end{pmatrix}.$$

Therefore $d(M_D) = d(B_1) = d(B)$.

If $R(B)$ is closed, then B_1 in (2.1) is left invertible. Since A_1 is invertible, there are invertible operators U and V in $\mathcal{B}(H_1 \oplus H_2)$ such that

$$U \begin{pmatrix} A_1 & D_1 & D_2 \\ 0 & D_3 & D_4 \\ 0 & 0 & B_1 \end{pmatrix} V = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & D_3 & 0 \\ 0 & 0 & B_1 \end{pmatrix}. \tag{2.2}$$

Because M_D is injective, D_3 is injective. So $\dim R(D_3) = n(B) < \infty$. Therefore $R(M_D)$ is closed, and it is easy to see that $d(M_D) = d(D_3) + d(B_1) = d(A) + d(B) - n(B)$. □

Lemma 7 *Let H be a Hilbert space and $A \in \mathcal{B}(H)$. Then $\lambda \in \sigma_c(A)$ if and only if $\bar{\lambda} \in \sigma_c(A^*)$.*

Lemma 8 *Let $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$ be given, and let A be left invertible, B be right* *invertible and $d(A) = n(B) = \infty$. Then there exists $D \in \mathcal{B}(H_2, H_1)$ such that $0 \in \sigma_c(M_D)$.*

Proof Suppose that $\{g_i\}_{i=1}^{\infty}$ and $\{f_i\}_{i=1}^{\infty}$ are orthogonal bases of $N(B)$ and $R(A)^{\perp}$. We define an operator $J : N(B) \longrightarrow R(A)^{\perp}$ by $J(g_i) = \frac{1}{i} f_i$ $(i = 1, 2, \ldots)$ and take

$$D = \begin{pmatrix} 0 & 0 \\ J & 0 \end{pmatrix} : N(B) \oplus N(B)^{\perp} \longrightarrow R(A) \oplus R(A)^{\perp}.$$

It is not hard to show that $0 \in \sigma_c(M_D)$. □

3. Main results

Theorem 1 *Let $A \in \mathcal{B}(H_1)$, $B \in \mathcal{B}(H_2)$ and $C \in \mathcal{B}(H_3)$ be given operators. Then*

$$\bigcup_{D,E,F} \sigma(M_{(D,E,F)}) = \sigma(A) \cup \sigma(B) \cup \sigma(C),$$

$$\bigcup_{D,E,F} \sigma_p(M_{(D,E,F)}) = \sigma_p(A) \cup \sigma_p(B) \cup \sigma_p(C).$$

Proof Suppose that there exist D, E, F such that $\lambda \in \sigma_p(M_{(D,E,F)})$. Hence there exists a nonzero vector $x \oplus y \oplus z \in H_1 \oplus H_2 \oplus H_3$ such that

$$\begin{cases} (A - \lambda I)x + Dy + Ez = 0, \\ (B - \lambda I)y + Fz = 0, \\ (C - \lambda I)z = 0. \end{cases}$$

Obviously, if $z \neq 0$, then $\lambda \in \sigma_p(C)$; if $z = 0$, $y \neq 0$, then $\lambda \in \sigma_p(B)$; if $y = z = 0$, $x \neq 0$, then $\lambda \in \sigma_p(A)$, it follows that $\lambda \in \sigma_p(A) \cup \sigma_p(B) \cup \sigma_p(C)$.

Next, suppose that there exist D, E, F such that $\lambda \in \sigma(M_{(D,E,F)})$. To see this, if not, then $\lambda \in \rho(A) \cap \rho(B) \cap \rho(C)$. $A_\lambda \in \mathcal{B}(H_1)$, $B_\lambda \in \mathcal{B}(H_2)$ and $C_\lambda \in \mathcal{B}(H_3)$ denote the inverse of $A - \lambda I$, $B - \lambda I$ and $C - \lambda I$, respectively. It is easy to show that

$$\begin{pmatrix} A_\lambda & -A_\lambda D B_\lambda & -A_\lambda E C_\lambda + A_\lambda D B_\lambda F C_\lambda \\ 0 & B_\lambda & -B_\lambda F C_\lambda \\ 0 & 0 & C_\lambda \end{pmatrix} \in \mathcal{B}(H_1 \oplus H_2 \oplus H_3)$$

is the inverse of $M_{(D,E,F)} - \lambda I$, which is a contradiction. Therefore $\lambda \in \sigma(A) \cup \sigma(B) \cup \sigma(C)$.

Conversely, assume that $\lambda \in \sigma(A) \cup \sigma(B) \cup \sigma(C)$ or $\lambda \in \sigma_p(A) \cup \sigma_p(B) \cup \sigma_p(C)$. Take $D = E = F = 0$, then $\lambda \in \sigma(M_{(D,E,F)})$ or $\lambda \in \sigma_p(M_{(D,E,F)})$. This completes the proof. □

The following two theorems are the main results in this paper.

Theorem 2 *Let $A \in \mathcal{B}(H_1)$, $B \in \mathcal{B}(H_2)$ and $C \in \mathcal{B}(H_3)$ be given operators. Then*

$$\bigcup_{D,E,F} \sigma_r(M_{(D,E,F)}) = \triangle_1 \cup \triangle_2 \cup \triangle_3, \tag{3.1}$$

where

$$\begin{aligned} \triangle_1 =& \{\lambda \in \sigma_{le}(A) : \lambda \notin \sigma_p(A), \max\{d(A - \lambda I), d(B - \lambda I), d(C - \lambda I)\} > 0\}, \\ \triangle_2 =& \{\lambda \in \sigma_{le}(B) : \lambda \notin \sigma_p(A), \lambda \notin \sigma_{le}(A), n(B - \lambda I) \leq d(A - \lambda I), \end{aligned}$$

$$
\begin{aligned}
&n(B-\lambda I)<d(A-\lambda I)+d(B-\lambda I)+d(C-\lambda I)\},\\
\triangle_3=&\{\lambda\notin\sigma_p(A):\lambda\notin\sigma_{le}(A),\lambda\notin\sigma_{le}(B),n(B-\lambda I)\le d(A-\lambda I),\\
&n(B-\lambda I)+n(C-\lambda I)\le d(A-\lambda I)+d(B-\lambda I),\\
&n(B-\lambda I)+n(C-\lambda I)<d(A-\lambda I)+d(B-\lambda I)+d(C-\lambda I)\}.
\end{aligned}
$$

Proof Let $\{g_i^{(1)}\}_{i=1}^{n(B)}$, $\{g_i^{(2)}\}_{i=1}^{n(C)}$, $\{f_i^{(1)}\}_{i=1}^{d(A)}$ and $\{f_i^{(2)}\}_{i=1}^{d(B)}$ be orthogonal bases of $N(B)$, $N(C)$, $R(A)^{\perp}$ and $R(B)^{\perp}$, respectively. If $R(A)$ and $R(B)$ are not closed, then by Corollary 1 there exist infinite dimensional spaces $M\subset\overline{R(A)}$ and $N\subset\overline{R(B)}$ such that $M\cap R(A)=N\cap R(B)=\{0\}$. $\{h_i\}_{i=1}^{\infty}$ and $\{h_i^{(1)}\}_{i=1}^{\infty}$ denote orthogonal bases of M and N, respectively. For convenience, we first show three propositions:

Proposition 1 *Let $A\in\mathcal{B}(H_1)$, $B\in\mathcal{B}(H_2)$ and $C\in\mathcal{B}(H_3)$ be given operators, where A is not a lower semi-Fredholm operator. Then there exist D, E, F such that $0\in\sigma_r(M_{(D,E,F)})$ if and only if A is injective and* $\max\{d(A),d(B),d(C)\}>0$.

Proof Necessity. Suppose that there exist D, E, F such that $0\in\sigma_r(M_{(D,E,F)})$. It is clear that A is injective and $0\in\sigma_p((M_{(D,E,F)})^*)$, thus $0\in\sigma_p(A^*)\cup\sigma_p(B^*)\cup\sigma_p(C^*)$ by Theorem 1. Therefore $\max\{d(A),d(B),d(C)\}>0$.

Sufficiency. Because A is not a lower semi-Fredholm operator, $R(A)$ is not closed or $d(A)=\infty$. If $R(A)$ is not closed, set $F=0$,

$$
\begin{cases}
D(g_i^{(1)})=h_{2i-1}, & i=1,2,\dots,n(B),\\
D(y)=0, & y\in N(B)^{\perp},
\end{cases}
$$

$$
\begin{cases}
E(g_i^{(2)})=h_{2i}, & i=1,2,\dots,n(C),\\
E(y)=0, & y\in N(C)^{\perp}.
\end{cases}
$$

Clearly, $M_{(D,E,F)}$ is injective. Since $\max\{d(A),d(B),d(C)\}>0$, $\overline{R(M_{(D,E,F)})}\ne H_1\oplus H_2\oplus H_3$. Hence $0\in\sigma_r(M_{(D,E,F)})$.

If $d(A)=\infty$, put $F=0$,

$$
\begin{cases}
D(g_i^{(1)})=f_{2i+1}, & i=1,2,\dots,n(B),\\
D(y)=0, & y\in N(B)^{\perp},
\end{cases}
$$

$$
\begin{cases}
E(g_i^{(2)})=f_{2i}, & i=1,2,\dots,n(C),\\
E(y)=0, & y\in N(C)^{\perp}.
\end{cases}
$$

Clearly, $M_{(D,E,F)}$ is injective and $\overline{R(M_{(D,E,F)})}\ne H_1\oplus H_2\oplus H_3$. The proof is completed. □

Proposition 2 *Let $A\in\mathcal{B}(H_1)$, $B\in\mathcal{B}(H_2)$ and $C\in\mathcal{B}(H_3)$ be given, where A is a lower semi-Fredholm operator, B is not a lower semi-Fredholm operator. Then there exist D, E, F such that $0\in\sigma_r(M_{(D,E,F)})$ if and only if A is injective, $d(A)\ge n(B)$ and $d(A)+d(B)+d(C)>n(B)$.*

Proof Necessity. Assume that there exist D, E, F such that $0\in\sigma_r(M_{(D,E,F)})$. Since $M_{(D,E,F)}$ is injective, A and M_D are injective. Because A is a lower semi-Fredholm operator, $n(B)\le d(A)<$

∞ by Lemma 4. If $d(A) > n(B)$, it is obvious that $d(A) + d(B) + d(C) > n(B)$; if $d(A) = n(B)$, then $d(M_D) = d(B)$, by Lemma 6. On the other hand, $\overline{R(M_{(D,E,F)})} \neq H_1 \oplus H_2 \oplus H_3$, i.e., $d(M_D) + d(C) = d(B) + d(C) > 0$, Therefore $d(A) + d(B) + d(C) > n(B)$.

Sufficiency. Because B is not a lower semi-Fredholm operator, $R(B)$ is not closed or $d(B) = \infty$. If $R(B)$ is not closed, set $E = 0$ and

$$\begin{cases} D(g_i^{(1)}) = f_i^{(1)}, & i = 1, 2, \dots, n(B), \\ D(y) = 0, & y \in N(B)^{\perp}, \end{cases} \tag{3.2}$$

$$\begin{cases} F(g_i^{(2)}) = h_i^{(1)}, & i = 1, 2, \dots, n(C), \\ F(y) = 0, & y \in N(C)^{\perp}. \end{cases}$$

Clearly, $M_{(D,E,F)}$ is injective. Because $d(A) \geq n(B)$ and $d(A)+d(B)+d(C) > n(B)$, $\overline{R(M_{(D,E,F)})} \neq H_1 \oplus H_2 \oplus H_3$. Therefore $0 \in \sigma_r(M_{(D,E,F)})$.

If $d(B) = \infty$, define D as (3.2), take $E = 0$ and put

$$\begin{cases} F(g_i^{(2)}) = f_{i+1}^{(2)}, & i = 1, 2, \dots, n(C), \\ F(y) = 0, & y \in N(C)^{\perp}. \end{cases}$$

Clearly, $M_{(D,E,F)}$ is injective and $\overline{R(M_{(D,E,F)})} \neq H_1 \oplus H_2 \oplus H_3$. The proof is completed. □

Proposition 3 *Let $A \in \mathcal{B}(H_1)$, $B \in \mathcal{B}(H_2)$ and $C \in \mathcal{B}(H_3)$ be given, and let A and B be lower semi-Fredholm operators. Then there exist D, E, F such that $0 \in \sigma_r(M_{(D,E,F)})$ if and only if A is injective, $d(A) \geq n(B)$, $d(A) + d(B) \geq n(B) + n(C)$ and $d(A) + d(B) + d(C) > n(B) + n(C)$.*

Proof Necessity. Suppose that there exist D, E, F such that $0 \in \sigma_r(M_{(D,E,F)})$. In the similar way to the proof of Proposition 2, we can prove that A and M_D are injective and $n(B) \leq d(A) < \infty$. Since A and B are lower semi-Fredholm operators, it follows that $R(M_D)$ is closed and $d(M_D) = d(A) + d(B) - n(B) < \infty$, by Lemma 6. From Lemma 4 we obtain that $d(M_D) \geq n(C)$, i.e., $d(A) + d(B) \geq n(B) + n(C)$. If $d(A) + d(B) > n(B) + n(C)$, it is obvious that $d(A) + d(B) + d(C) > n(B) + n(C)$; if $d(A) + d(B) = n(B) + n(C)$, i.e., $d(M_D) = n(C)$, then $d(M_{(D,E,F)}) = d(C)$, by Lemma 6. On the other hand, $\overline{R(M_{(D,E,F)})} \neq H_1 \oplus H_2 \oplus H_3$, $d(C) = d(M_{(D,E,F)}) > 0$, hence $d(A) + d(B) + d(C) > n(B) + n(C)$.

Sufficiency. We define D as (3.2). If $d(B) \geq n(C)$, set $E = 0$,

$$\begin{cases} F(g_i^{(2)}) = f_i^{(2)}, & i = 1, 2, \dots, n(C), \\ F(y) = 0, & y \in N(C)^{\perp}. \end{cases}$$

If $d(B) < n(C)$, since A and B are lower semi-Fredholm, $d(A)$ and $d(B)$ are finite. Also by $d(A) + d(B) \geq n(B) + n(C)$, we obtain $d(A) - n(B) \geq n(C) - d(B)$. Therefore set

$$\begin{cases} E(g_{i+d(B)}^{(2)}) = f_{i+n(B)}^{(1)}, & i = 1, 2, \dots, n(C) - d(B), \\ E(y) = 0, & y \perp \{g_i^{(2)}\}_{i=1+d(B)}^{n(C)}, \end{cases}$$

$$\begin{cases} F(g_i^{(2)}) = f_i^{(2)}, & i = 1, 2, \dots, d(B), \\ F(y) = 0, & y \perp \{g_i^{(2)}\}_{i=1}^{d(B)}. \end{cases}$$

Clearly, $M_{(D,E,F)}$ is injective. Because $d(A)+d(B)\geq n(B)+n(C)$ and $d(A)+d(B)+d(C)>n(B)+n(C)$, $\overline{R(M_{(D,E,F)})}\neq H_1\oplus H_2\oplus H_3$. The proof is completed. □

Now we prove Theorem 2.

The right side of (3.1) includes the left side. Suppose that there exist D, E, F such that $\lambda\in\sigma_r(M_{(D,E,F)})$. If $A-\lambda I$ is not a lower semi-Fredholm operator, then $\lambda\in\triangle_1$ by Proposition 1; if $A-\lambda I$ is a lower semi-Fredholm operator and $B-\lambda I$ is not a lower semi-Fredholm operator, then, by Proposition 2, $\lambda\in\triangle_2$; if $A-\lambda I$ and $B-\lambda I$ are lower semi-Fredholm operators, then $\lambda\in\triangle_3$ by Proposition 3.

The left side of (3.1) includes the right side. If $\lambda\in\triangle_1\cup\triangle_2\cup\triangle_3$, then there exist D, E, F such that $\lambda\in\sigma_r(M_{(D,E,F)})$ by the Propositions above. This ends the proof. □

Therorem 3 *Let $A\in\mathcal{B}(H_1)$, $B\in\mathcal{B}(H_2)$ and $C\in\mathcal{B}(H_3)$ be given. Then*

$$\bigcup_{D,E,F}\sigma_c(M_{(D,E,F)})=\triangle_4\cup\triangle_5\cup\triangle_6\cup\triangle_7. \tag{3.3}$$

where

$$\triangle_4=\{\lambda\notin\sigma_p(A): R(A-\lambda I)\text{ and }R(C-\lambda I)\text{ are not closed, }\overline{R(C-\lambda I)}=H_3\},$$

$$\begin{aligned}\triangle_5=&\{\lambda\notin\sigma_p(A)\cap\sigma_\delta(C): d(B-\lambda I)\leq n(C-\lambda I), R(A-\lambda I),\ R(B-\lambda I)\text{ are not closed}\}\\ &\cup\{\lambda\notin\sigma_p(A)\cap\sigma_\delta(C): d(B-\lambda I)\leq n(C-\lambda I), R(A-\lambda I)\text{ is not closed, }R(B-\lambda I)\\ &\qquad\text{is closed, }n(B-\lambda I)+n(C-\lambda I)\geq d(A-\lambda I)+d(B-\lambda I)\},\end{aligned}$$

$$\begin{aligned}\triangle_6=&\{\lambda\notin\sigma_{ap}(A): R(B-\lambda I),\ R(C-\lambda I)\text{ are not closed,}\\ &\qquad\overline{R(C-\lambda I)}=H_3, d(A-\lambda I)\geq n(B-\lambda I)\}\\ &\cup\{\lambda\notin\sigma_{ap}(A): R(B-\lambda I)\text{ is closed, }R(C-\lambda I)\text{ is not closed, }d(A-\lambda I)\geq n(B-\lambda I),\\ &\qquad\overline{R(C-\lambda I)}=H_3, n(B-\lambda I)+n(C-\lambda I)\leq d(A-\lambda I)+d(B-\lambda I)\},\end{aligned}$$

$$\begin{aligned}\triangle_7=&\{\lambda\notin\sigma_{ap}(A)\cap\sigma_\delta(C): R(B-\lambda I)\text{ is not closed,}\\ &\qquad d(A-\lambda I)\geq n(B-\lambda I), n(C-\lambda I)\geq d(B-\lambda I)\}\\ &\cup\{\lambda\notin\sigma_{ap}(A)\cap\sigma_\delta(C): R(B-\lambda I)\text{ is closed,}\\ &\qquad d(A-\lambda I)\geq n(B-\lambda I), n(C-\lambda I)\geq d(B-\lambda I),\\ &\qquad\max\{d(A-\lambda I), d(B-\lambda I)\}=\max\{n(C-\lambda I), n(B-\lambda I)\}=\infty\}.\end{aligned}$$

Proof Let $\{g_i^{(1)}\}_{i=1}^{n(B)}$, $\{g_i^{(2)}\}_{i=1}^{n(C)}$, $\{f_i^{(1)}\}_{i=1}^{d(A)}$ and $\{f_i^{(2)}\}_{i=1}^{d(B)}$ be orthogonal bases of $N(B)$, $N(C)$, $R(A)^\perp$ and $R(B)^\perp$. Before proving Theorem 3, we first give four propositions:

Proposition 4 *Let $A\in\mathcal{B}(H_1)$, $B\in\mathcal{B}(H_2)$ and $C\in\mathcal{B}(H_3)$ be given operators, and let $R(A)$ and $R(C)$ be not closed. Then there exist D, E, F such that $0\in\sigma_c(M_{(D,E,F)})$ if and only if A is injective and $\overline{R(C)}=H_3$.*

Proof Necessity. Suppose that there exist D, E, F such that $0\in\sigma_c(M_{(D,E,F)})$. Thus $M_{(D,E,F)}$ is injective and $\overline{R(M_{(D,E,F)})}=H_1\oplus H_2\oplus H_3$, hence A is injective and $\overline{R(C)}=H_3$.

Sufficiency. Since $R(A)$ and $R(C)$ are not closed, there exist infinite dimensional spaces $M\subset\overline{R(A)}$ and $N\subset\overline{R(C^*)}=N(C)^\perp$ such that $R(A)\cap M=R(C^*)\cap N=\{0\}$ by Corollary 1. Let $\{h_i^{(1)}\}_{i=1}^\infty$ and $\{h_i^{(2)}\}_{i=1}^\infty$ denote orthogonal bases of M and N. Next, we split the proof into several cases.

Case 1 If $n(B) > d(A)$ and $n(C) > d(B)$, put

$$\begin{cases} D(g_i^{(1)}) = f_i^{(1)}, & i = 1, 2, \dots, d(A), \\ D(g_{i+d(A)}^{(1)}) = h_{2i-1}^{(1)}, & i = 1, 2, \dots, n(B) - d(A), \\ D(y) = 0, & y \in N(B)^{\perp}, \end{cases} \tag{3.4}$$

$$\begin{cases} E(g_{i+d(B)}^{(2)}) = h_{2i}^{(1)}, & i = 1, 2, \dots, n(C) - d(B), \\ E(y) = 0, & y \perp \{g_i^{(1)}\}_{i=d(B)+1}^{n(C)}, \end{cases} \tag{3.5}$$

$$\begin{cases} F(g_i^{(2)}) = f_i^{(2)}, & i = 1, 2, \dots, d(B), \\ F(y) = 0, & y \perp \{g_i^{(2)}\}_{i=1}^{d(B)}. \end{cases} \tag{3.6}$$

Case 2 If $n(B) > d(A)$ and $n(C) < d(B)$, define D as (3.4), set $E = 0$ and

$$\begin{cases} F(g_i^{(2)}) = f_i^{(2)}, & i = 1, 2, \dots, n(C), \\ F(h_{2i}^{(2)}) = f_{n(C)+i}^{(2)}, & i = 1, 2, \dots, d(B) - n(C), \\ F(y) = 0, & y \in N(C)^{\perp} \text{ and } y \perp \{h_{2i}^{(2)}\}_{i=1}^{d(B)-n(C)}. \end{cases} \tag{3.7}$$

Case 3 If $n(B) < d(A)$ and $n(C) > d(B)$, define F as (3.6) and set

$$\begin{cases} D(g_i^{(1)}) = f_i^{(1)}, & i = 1, 2, \dots, n(B), \\ D(y) = 0, & y \in N(B)^{\perp}, \end{cases} \tag{3.8}$$

$$\begin{cases} E(g_{i+d(B)}^{(2)}) = h_i^{(1)}, & i = 1, 2, \dots, n(C) - d(B), \\ E(h_i^{(2)}) = f_{n(B)+i}^{(1)}, & i = 1, 2, \dots, d(A) - n(B), \\ E(y) = 0, & y \perp \{g_i^{(2)}\}_{i=d(B)+1}^{n(C)} \text{ and } y \perp \{h_i^{(2)}\}_{i=1}^{d(A)-n(B)}. \end{cases}$$

Clearly, $M_{(D,E,F)}$ and $(M_{(D,E,F)})^*$ are injective. Since $R(C)$ is not closed, $R(M_{(D,E,F)}) \neq H_1 \oplus H_2 \oplus H_3$. Therefore $0 \in \sigma_c(M_{(D,E,F)})$.

Case 4 If $n(B) = d(A)$ and $n(C) = d(B)$, define D, F as (3.8), (3.6) and take $E = 0$; if $n(B) = d(A)$ and $n(C) > d(B)$, define D, E and F as (3.8), (3.5) and (3.6); If $n(B) = d(A)$ and $n(C) < d(B)$, define D, F as (3.8), (3.7), and take $E = 0$. In the similar way to the above, we obtain $0 \in \sigma_c(M_{(D,E,F)})$.

Case 5 If $n(B) < d(A)$ and $n(C) < d(B)$ or $n(B) \neq d(A)$ and $n(C) = d(B)$, in the similar way to Cases 1 and 4, we can show that there exist D^*, E^*, F^* such that $0 \in \sigma_c((M_{(D,E,F)})^*)$. It follows from Lemma 7 that $0 \in \sigma_c(M_{(D,E,F)})$. The proof is completed. □

Proposition 5 *Let $A \in \mathcal{B}(H_1)$, $B \in \mathcal{B}(H_2)$ and $C \in \mathcal{B}(H_3)$ be given operators, and let $R(A)$ be closed, $R(B)$ and $R(C)$ be not closed. Then there exist D, E, F such that $0 \in \sigma_c(M_{(D,E,F)})$ if and only if A is injective, $\overline{R(C)} = H_3$ and $d(A) \geq n(B)$.*

Proof Necessity. Suppose that there exist D, E, F such that $0 \in \sigma_c(M_{(D,E,F)})$. From the proof of Proposition 4, A is injective and $\overline{R(C)} = H_3$. It follows from the injectivity of $M_{(D,E,F)}$ that M_D is injective, and from the closeness of $R(A)$ and Lemma 4, we can prove that $d(A) \geq n(B)$.

Sufficiency. Since $R(B)$ and $R(C)$ are not closed, there exist infinite dimensional spaces $M \subset \overline{R(B)}$ and $N \subset \overline{R(C^*)} = N(C)^\perp$ such that $R(B) \cap M = R(C^*) \cap N = \{0\}$ by Corollary 1. $\{h_i^{(1)}\}_{i=1}^\infty$ and $\{h_i^{(2)}\}_{i=1}^\infty$ denote orthogonal bases of M and N. If $d(A) = n(B)$ and $d(B) \geq n(C)$, from the proof of Proposition 4, we get that there exist D, E, F such that $0 \in \sigma_c(M_{(D,E,F)})$. If $d(A) > n(B)$ and $d(B) < n(C)$, define D as (3.8) and set

$$\begin{cases} E(h_{2i-1}^{(2)}) = f_{n(B)+i}^{(1)}, & i = 1, 2, \ldots, d(A) - n(B), \\ E(y) = 0, & y \perp \{h_{2i-1}^{(2)}\}_{i=1}^{d(A)-n(B)}, \end{cases} \tag{3.9}$$

$$\begin{cases} F(g_i^{(2)}) = f_i^{(2)}, & i = 1, 2, \ldots, d(B), \\ F(g_{i+d(B)}^{(2)}) = h_i^{(1)}, & i = 1, 2, \ldots, n(C) - d(B)), \\ F(y) = 0, & y \in N(C)^\perp. \end{cases} \tag{3.10}$$

If $d(A) > n(B)$ and $d(B) > n(C)$, define D, E and F as (3.8), (3.9) and (3.7); if $d(A) > n(B)$ and $d(B) = n(C)$, define D, E and F as (3.8), (3.9) and (3.6); if $d(A) = n(B)$ and $d(B) < n(C)$, define D, F as (3.8), (3.10) and take $E = 0$. Clearly, $M_{(D,E,F)}$ and $(M_{(D,E,F)})^*$ are injective. Since $R(C)$ is not closed, $R(M_{(D,E,F)}) \neq H_1 \oplus H_2 \oplus H_3$. The proof is completed. $\square$

Proposition 6 *Let $A \in \mathcal{B}(H_1)$, $B \in \mathcal{B}(H_2)$ and $C \in \mathcal{B}(H_3)$ be given operators, and let $R(A)$ and $R(B)$ be closed, $R(C)$ be not closed. Then there exist D, E, F such that $0 \in \sigma_c(M_{(D,E,F)})$ if and only if A is injective, $\overline{R(C)} = H_3$, $d(A) \geq n(B)$ and $d(A) + d(B) \geq n(B) + n(C)$.*

Proof Necessity. Suppose that there exist D, E, F such that $0 \in \sigma_c(M_{(D,E,F)})$. It follows from the proof of Proposition 5 that A is injective, $\overline{R(C)} = H_3$ and $d(A) \geq n(B)$. Now we will show that $d(A) + d(B) \geq n(B) + n(C)$. Without loss of generality, we suppose that $d(A) < \infty$ and $d(B) < \infty$. By Lemma 6, $R(M_D)$ is closed and $d(M_D) = d(A) + d(B) - n(B)$. Again, from Lemma 4 we obtain that $d(M_D) \geq n(C)$, i.e., $d(A) + d(B) \geq n(C) + n(B)$.

Sufficiency. Since $R(C)$ is not closed, by Corollary 1 there exists an infinite dimensional subspace $N \subset \overline{R(C^*)} = N(C)^\perp$ such that $R(C^*) \cap N = \{0\}$. Let $\{h_i^{(2)}\}_{i=1}^\infty$ be an orthogonal basis of N. If $d(A) > n(B)$ and $d(B) \geq n(C)$ or $d(A) = n(B)$ and $d(B) \geq n(C)$, from the proof of Proposition 5 we can show that there exist D, E, F such that $0 \in \sigma_c(M_{(D,E,F)})$. If $d(A) > n(B)$ and $d(B) < n(C)$, define D and F as (3.8) and (3.6). Since $d(A) + d(B) \geq n(C) + n(B)$, i.e., $d(A) - n(B) \geq n(C) - d(B)$, in the case when $d(A) - n(B) = n(C) - d(B)$, we set

$$\begin{cases} E(g_{d(B)+i}^{(2)}) = f_{n(B)+i}^{(1)}, & i = 1, 2, \ldots, n(C) - d(B), \\ E(y) = 0, & y \perp \{g_i^{(2)}\}_{i=d(B)+1}^{n(C)}. \end{cases}$$

In the case when $d(A) - n(B) > n(C) - d(B)$, denote $k = d(A) + d(B) - n(B) - n(C)$ and set

$$\begin{cases} E(g_{d(B)+i}^{(2)}) = f_{n(B)+i}^{(1)}, & i = 1, 2, \ldots, n(C) - d(B), \\ E(h_i^{(2)}) = f_{d(A)-k+i}^{(1)}, & i = 1, 2, \ldots, k, \\ E(y) = 0, & y \perp \{h_i^{(2)}\}_{i=1}^{k} \text{ and } y \perp \{g_i^{(2)}\}_{i=d(B)+1}^{n(C)}. \end{cases}$$

Clearly, $M_{(D,E,F)}$ and $(M_{(D,E,F)})^*$ are injective. Since $R(C)$ is not closed, it follows that $R(M_{(D,E,F)}) \neq H_1 \oplus H_2 \oplus H_3$. Therefore $0 \in \sigma_c(M_{(D,E,F)})$. The proof is completed. $\square$

Proposition 7 *Let $A \in \mathcal{B}(H_1)$, $B \in \mathcal{B}(H_2)$ and $C \in \mathcal{B}(H_3)$ be given, and let A be left invertible and C be right invertible. Then there exist D, E, F such that $0 \in \sigma_c(M_{(D,E,F)})$ if and only if*

$$\begin{cases} d(B) \le n(C), d(A) \ge n(B), & \text{if } R(B) \text{ is not closed,} \\ \begin{array}{l} d(B) \le n(C), d(A) \ge n(B), \\ \max\{d(A), d(B)\} = \max\{n(C), n(B)\} = \infty, \end{array} & \text{if } R(B) \text{ is closed.} \end{cases}$$

Proof Necessity. Suppose that there exist D, E, F such that $0 \in \sigma_c(M_{(D,E,F)})$, thus M_D is injective. Since A is left invertible, $d(A) \ge n(B)$ by Lemma 4. Again, from $0 \in \sigma_c(M_{(D,E,F)})$ we get that $0 \in \sigma_c((M_{(D,E,F)})^*)$. In the similar way we can prove that $d(C^*) \ge n(B^*)$, i.e., $n(C) \ge d(B)$.

If $R(B)$ is closed, then $\max\{d(A), d(B)\} = \max\{n(C), n(B)\} = \infty$. To see this, if not, suppose that $\max\{d(A), d(B)\} < \infty$ or $\max\{n(C), n(B)\} < \infty$. If $\max\{d(A), d(B)\} < \infty$, then $R(M_D)$ is closed and $d(M_D) = d(A) + d(B) - n(B) < \infty$ by Lemma 6. It follows from Lemma 4 that $n(C) \le d(M_D) < \infty$. Since C is right invertible and M_D is left invertible, $R(M_{(D,E,F)})$ is closed, by Lemma 6. Therefore $0 \notin \sigma_c(M_{(D,E,F)})$; if $\max\{n(B), n(C)\} < \infty$, in the similar way to the proof above, we can show that $0 \notin \sigma_c(M_{(D,E,F)})$. It is a contradiction.

Sufficiency. First assume that $R(B)$ is not closed. By Corollary 1 there exist infinite dimensional spaces $M \subset \overline{R(B)}$ and $N \subset \overline{R(B^*)}$ such that $R(B) \cap M = R(B^*) \cap N = \{0\}$. Let $\{h_i^{(1)}\}_{i=1}^{\infty}$ and $\{h_i^{(2)}\}_{i=1}^{\infty}$ be orthogonal bases of M and N. When $d(A) = n(C) = \infty$, there exists D such that M_D is left invertible by Lemma 1. Since $R(B)$ is not closed, $d(M_D) = \dim H_1 \oplus H_2/R(M_D) \ge \dim H_2/R(B) = \infty$ (where $H_1 \oplus H_2/R(M_D)$ and $H_2/R(B)$ denote quotient spaces), also by Lemma 8 there exist E, F such that $0 \in \sigma_c(M_{(D,E,F)})$. Without loss of generality, assume that $d(A) < \infty$ or $n(C) < \infty$.

If $d(A) = n(B)$ and $d(B) \le n(C)$, in the similar way to the proof of Propositions 4 and 5, we can prove that there exist D, E, F such that $0 \in \sigma_c(M_{(D,E,F)})$. If $d(A) > n(B)$ and $d(B) < n(C)$, define F as (3.10) and set $E = 0$,

$$\begin{cases} D(g_i^{(1)}) = f_i^{(1)}, & i = 1, 2, \ldots, n(B), \\ D(h_i^{(2)}) = f_{n(B)+i}^{(1)}, & i = 1, 2, \ldots, d(A) - n(B), \\ D(y) = 0, & y \in N(B)^{\perp} \text{ and } y \perp \{h_i^{(2)}\}_{i=1}^{d(A)-n(B)}. \end{cases} \tag{3.11}$$

If $d(A) > n(B)$ and $d(B) = n(C)$, define D, F as (3.11), (3.6) and take $E = 0$. Clearly, $M_{(D,E,F)}$ and $(M_{(D,E,F)})^*$ are injective. If $d(A) < \infty$, since $R(B)$ is not closed, therefore M_D is not left invertible, by Lemma 1. Hence $R(M_{(D,E,F)}) \ne H_1 \oplus H_2 \oplus H_3$. Otherwise, suppose that $R(M_{(D,E,F)}) = H_1 \oplus H_2 \oplus H_3$. Then it follows from the injectivity of $M_{(D,E,F)}$ that $M_{(D,E,F)}$ is invertible. It is in contradiction to the fact that M_D is not left invertible; Similarly, if $n(C) < \infty$, we can show that $R(M_{(D,E,F)}) \ne H_1 \oplus H_2 \oplus H_3$. Therefore $0 \in \sigma_c(M_{(D,E,F)})$.

Next assume that $R(B)$ is closed, so $\max\{d(A), d(B)\} = \max\{n(C), n(B)\} = \infty$. If $d(A) = n(B)$ and $d(B) = n(C)$, then $d(A) = n(B) = d(B) = n(C) = \infty$. Set

$$\begin{cases} F(g_i^{(2)}) = f_i^{(2)}, & i = 1, 2, \ldots, \\ F(y) = 0, & y \in N(C)^{\perp}. \end{cases}$$

It is easy to show that $\begin{pmatrix} B & F \\ 0 & C \end{pmatrix}$ is right invertible. Since A is left invertible and $d(A) = n(B) = \infty$, by Lemma 8 there exist D, E such that $0 \in \sigma_c(M_{(D,E,F)})$.

If $d(A) > n(B)$ and $d(B) < n(C)$, then $d(A) = n(C) = \infty$. Define D as (3.8). Clearly, M_D is injective. On the other hand, by the left invertibility of A and the definition of D, we know that M_D has the following decomposition

$$\begin{pmatrix} A_1 & 0 & 0 \\ 0 & D_3 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : H_1 \oplus N(B) \oplus N(B)^{\perp} \to R(A) \oplus R(A)^{\perp} \oplus H_2.$$

It follows from $n(B) < \infty$ that $\dim R(D_3) < \infty$, thus $R(M_D)$ is closed and so M_D is left invertible. Furthermore, since $n(B) < \infty$, $d(M_D) = \infty$. From Lemma 8 we get that there exist E, F such that $0 \in \sigma_c(M_{(D,E,F)})$.

If $d(A) > n(B)$ and $d(B) = n(C)$, then $d(B) = d(A) = n(C) = \infty$. We define D as (3.8). In the similar way to the proof above, we can prove that M_D is left invertible and $d(M_D) = \infty$. By Lemma 8 there exist E, F such that $0 \in \sigma_c(M_{(D,E,F)})$; if $d(A) = n(B)$ and $d(B) < n(C)$, in the similar way to the case when $d(A) > n(B)$ and $d(B) = n(C)$, we can show that there exist D^*, E^*, F^* such that $0 \in \sigma_c((M_{(D,E,F)})^*)$, i.e., $0 \in \sigma_c(M_{(D,E,F)})$. The proof is completed. □

With four propositions above, we now prove Theorem 3.

The right side in (3.3) includes the left side. Suppose that there exist D, E, F such that $\lambda \in \sigma_c(M_{(D,E,F)})$. Clearly, $\lambda \notin \sigma_p(A)$ and $\overline{R(C - \lambda I)} = H_3$. If $R(A - \lambda I)$ and $R(C - \lambda I)$ are not closed, then $\lambda \in \triangle_4$ by Proposition 4; if $R(A - \lambda I)$ and $R(C - \lambda I)$ are closed, then $\lambda \in \triangle_7$ by Proposition 7. if $R(A - \lambda I)$ is closed, $R(C - \lambda I)$ is not closed, then $\lambda \in \triangle_6$ by Propositions 5 and 6; if $R(A - \lambda I)$ is not closed, $R(C - \lambda I)$ is closed, from Lemma 7, Propositions 5, 6 and the conjugation of $M_{(D,E,F)}$ and $(M_{(D,E,F)})^*$, we get that $\lambda \in \triangle_5$.

The left side of (3.3) includes the right side. If $\lambda \in \triangle_4 \cup \triangle_5 \cup \triangle_6 \cup \triangle_7$. from Propositions 4–7, Lemma 7 and the conjugation of $M_{(D,E,F)}$ and $(M_{(D,E,F)})^*$, we know that there exist D, E, F such that $\lambda \in \sigma_c(M_{(D,E,F)})$. This completes the proof. □

Finally, we give an example to illustrate the correctness of our results.

Example Let $H_1 = H_2 = H_3 = \ell_2$. In ℓ_2, let e_i $(i = 1, 2, \cdots)$ denote the element with 1 in the i-th place and zeros elsewhere. For every $x = (x_1, x_2, \ldots) \in \ell_2$, define $A \in \mathcal{B}(\ell_2)$, $B \in \mathcal{B}(\ell_2)$ and $C \in \mathcal{B}(\ell_2)$ by

$$\begin{aligned} Ax &= (x_1, x_2 - x_1, x_3 - x_2, x_4 - x_3, \ldots), \\ Bx &= (0, x_1, x_2, x_3, x_4, x_5, \ldots), \\ Cx &= (x_2, x_3, x_4, x_5, x_6, x_7, \ldots). \end{aligned}$$

It is not hard to show that A and B are injective, $R(C) = \overline{R(A)} = \ell_2 \neq R(A)$ and $d(B) = n(C) = 1$, so $0 \in \triangle_1 \cap \triangle_5$. By Theorems 2 and 3, there exist D_1, E_1, F_1, D_2, E_2 and F_2 such that $0 \in \sigma_r(M_{(D_1,E_1,F_1)}) \cap \sigma_c(M_{(D_2,E_2,F_2)})$. For this, take $D_1 = F_1 = D_2 = E_2 = 0$ and set $F_2 x = E_1 x = (x_1, 0, 0, 0, 0, \ldots)$, for each $x = (x_1, x_2, \ldots) \in \ell_2$. Then $0 \in \sigma_r(M_{(D_1,E_1,F_1)})$ and

$0 \in \sigma_c(M_{(D_2,E_2,F_2)})$.

Remark For C defined above, it is obvious that $0 \in \sigma_p(C) \subset \sigma(C)$. From Theorem 1, there exist D, E, F such that $0 \in \sigma_p(M_{(D,E,F)}) \subset \sigma(M_{(D,E,F)})$. The fact means that the intersection of the possible point spectrum, the possible residual spectrum and the possible continuous spectrum of the partial operator matrix M is not empty.

References

[1] LI Chaokuan. *Spectrum of partial operator matrices* [J]. Chinese Ann. Math. Ser. A., 2000, **21**(5): 529–532.
[2] DU Hongke, JIN Pan. *Perturbation of spectrums of* 2×2 *operator matrices* [J]. Proc. Amer. Math. Soc., 1994, **121**(3): 761–766.
[3] HWANG I S, LEE W Y. *The boundedness below of* 2×2 *upper triangular operator matrices* [J]. Integral Equations Operator Theory, 2001, **39**(3): 267–276.
[4] CAO Xiaohong, GUO Maozheng, MENG Bin. *Semi-Fredholm spectrum and Weyl's theorem for operator matrices* [J]. Acta Math. Sin. (Engl. Ser.), 2006, **22**(1): 169–178.
[5] LI Yuan, DU Hongke. *The intersection of essential approximate point spectra of operator matrices* [J]. J. Math. Anal. Appl., 2006, **323**(2): 1171–1183.
[6] LI Yuan, SUN Xiuhong, DU Hongke. *The intersection of left (right) spectra of* 2×2 *upper triangular operator matrices* [J]. Linear Algebra Appl., 2006, **418**(1): 112–121.
[7] HOU Guolin, Alatancang. *Perturbation of spectra of* 2×2 *upper triangular operator matrices* [J]. J. Systems Sci. Math. Sci., 2006, **26**(3): 257–263. (in Chinese)
[8] HOU Guolin, Alatancang, HUANG Junjie. *Invertibility of an operator appearing in linear quadratic optimal control problems in a Hilbert space* [J]. Acta Math. Sinica (Chin. Ser.), 2007, **50**(2): 473–480. (in Chinese)
[9] CAO Xiaohong. *Weyl's theorem for* 3×3 *upper triangular operator matrices* [J]. Acta Math. Sinica (Chin. Ser.), 2006, **49**(3): 529–538. (in Chinese)
[10] ZEIDLER E. *Applied Functional Analysis. Main Principles and Their Applications* [M]. Springer-Verlag, New York, 1995.
[11] MÜLLER V. *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras* [M]. Birkhäuser Verlag, Basel, 2003.