Journal of Mathematical Research & Exposition
July, 2009, Vol. 29, No. 4, pp. 662–670
DOI:10.3770/j.issn:1000-341X.2009.04.011
Http://jmre.dlut.edu.cn

Integral Representation of Harmonic Function in a Half-Plane

LI Hong Wei[1,2], *DENG Guan Tie*[2]
(1. Mathematics Department, Henan Institute of Education, Henan 450014, China;
2. Sch. Math. Sci. & Lab. Math. Com. Sys., Beijing Normal University, Beijing 100875, China)
(E-mail: hwli218@sohu.com; denggt@bnu.edu.cn)

Abstract In this article, we consider the integral representation of harmonic functions. Using a property of the modified Poisson kernel in a half plane, we prove that a harmonic function $u(z)$ in a half plane with its positive part $u^+(z) = \max\{u(z), 0\}$ satisfying a slowly growing condition can be represented by its integral of a measure on the boundary of the half plan.

Keywords integral representation; harmonic functions; modified Poisson kernel.

Document code A
MR(2000) Subject Classification 31A10; 31A15
Chinese Library Classification O174.5

1. Introduction and main theorem

Let $\alpha \geq 0$ be a real number and let H_α denote the space of all the functions harmonic in the right half plane $\mathbb{C}_+ = \{z = x + iy = re^{i\theta} : x > 0\}$ satisfying

$$I_1 = \int\int_{\mathbb{C}_+} \frac{xu^+(x+iy)\mathrm{d}x\mathrm{d}y}{1 + (x^2 + y^2)^{\alpha+2}} < \infty \tag{1}$$

and

$$I_2 = \sup_{0<\varepsilon<1} \int_{-\infty}^{\infty} \frac{u^+(\varepsilon + iy)dy}{1 + |y|^{2\alpha+2}} < \infty, \tag{2}$$

where $u^+(z) = \max\{u(z), 0\}$ is the positive part of $u(z)$. Suppose $P(z,t) = \frac{1}{\pi}\mathrm{Re}[\frac{1}{z-it}]$ is the Poisson kernel of the right half plane $\mathbb{C}_+$. Suppose m is an integer. A modified Poisson kernel of order m for $z \in \mathbb{C}_+$ is defined by

$$P_m(z,t) = \begin{cases} P(z,t) - \frac{|t|^{m+2}}{\pi}\mathrm{Re}\sum_{k=0}^{m} \frac{i(-iz)^k}{t^{k+1}}, & \text{if } |t| \leq 1, \\ P(z,t) - \frac{1}{\pi}\mathrm{Re}\sum_{k=0}^{m} \frac{i(-iz)^k}{t^{k+1}}, & \text{if } |t| > 1. \end{cases}$$

It is obvious that $P_0(z,t) = P(z,t)$. $P_m(z,t)$ has the following properties: if $z = x + iy =$

Received date: 2007-04-09; **Accepted date**: 2007-07-13
Foundation item: the National Natural Science Foundation of China (No. 10671022); Research Foundation for Doctor Programme (No. 20060027023); Henan Institute of Education Youth Scientific Research Fund (No. 20070107).

$re^{i\theta}, x>0, |z|\geq 1,$

$$x\pi|P_m(z,t)| \leq \begin{cases} (1+3^m m)|z|^{m+1}, & \text{if} \quad |t|\leq 1, \\ 2(3+m)3^m\left(\frac{|z|}{|t|}\right)^{m+1}, & \text{if} \quad 1\leq |t|\leq 2|z|, \\ (2m+4)\left(\frac{|z|}{|t|}\right)^{m+2}, & \text{if} \quad |t|>2|z|. \end{cases} \tag{3}$$

Using this, we prove the following conclusion:

Theorem *Let $\alpha\geq 0$ be a real number and $m\in[2\alpha,2\alpha+1)$ be an integer, $u\in H_\alpha$. Then*

i)

$$I_3=\sup_{0<\varepsilon<1}\int_{-\infty}^{\infty}\frac{|u(it+\varepsilon)|}{1+|t|^{2\alpha+2}}\mathrm{d}t<\infty; \tag{4}$$

ii) There exists a measure μ on $(-\infty,\infty)$ such that

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}|\mu|(t)}{1+|t|^{2\alpha+2}}<\infty; \tag{5}$$

iii) If $\alpha>0$, there exists a polynomial $Q_\alpha(z)=\sum_{k=1}^{m}a_k i(-iz)^k$ of degree $<2\alpha+1$, where a_k $(k=1,2,\dots,m)$ are real numbers, such that

$$u(z)=\mathrm{Re}Q_\alpha(z)+\int_{-\infty}^{\infty}P_m(z,t)\mathrm{d}\mu(t), \quad z=x+iy\in\mathbb{C}_+; \tag{6}$$

iv) If $\alpha=0$, there exists a real number a_1, such that

$$u(z)=a_1x+\int_{-\infty}^{\infty}P(z,t)\mathrm{d}\mu(t), \quad z=x+iy\in\mathbb{C}_+.$$

2. Proof of Theorem

We first prove inequalities in (3). If $|t|\geq 2|z|$,

$$P(z,t)=\frac{1}{\pi}\mathrm{Re}[\frac{1}{z-it}]=\frac{1}{\pi}\mathrm{Re}\sum_{k=0}^{\infty}\frac{i(-iz)^k}{t^{k+1}},$$

$$P_m(z,t)=P(z,t)-\frac{1}{\pi}\mathrm{Re}\sum_{k=0}^{m}\frac{i(-iz)^k}{t^{k+1}}=\frac{1}{\pi}\mathrm{Re}\sum_{k=m+1}^{\infty}\frac{i(-iz)^k}{t^{k+1}}=-\frac{1}{\pi}\mathrm{Im}\sum_{k=m+1}^{\infty}\frac{(-iz)^k}{t^{k+1}}.$$

Since $|\sin k(\theta-\frac{\pi}{2})|\leq k|\sin(\theta-\frac{\pi}{2})|$,

$$|\mathrm{Im}(-iz)^k|=|z|^k|\sin k(\theta-\frac{\pi}{2})|\leq|z|^k k|\sin(\theta-\frac{\pi}{2})|=kx|z|^{k-1},$$

we have

$$x\pi|P_m(z,t)|\leq\sum_{k=m+1}^{\infty}\frac{kx^2|z|^{k-1}}{|t|^{k+1}}.$$

Suppose $S=\sum_{k=m+1}^{\infty}\frac{k|z|^{k-1}}{|t|^{k+1}}$. Then $\frac{|z|}{|t|}S=\sum_{k=m+1}^{\infty}\frac{k|z|^k}{|t|^{k+2}}$,

$$(1-\frac{|z|}{|t|})S=\frac{(m+1)|z|^m}{|t|^{m+2}}+\frac{|z|^{m+1}}{|t|^{m+3}}+\frac{|z|^{m+2}}{|t|^{m+4}}+\cdots$$

$$
\begin{aligned}
&= \frac{(m+1)|z|^m}{|t|^{m+2}} + \frac{|z|^{m+1}}{|t|^{m+2}(|t|-|z|)},\\
S &= \frac{(m+1)|z|^m}{|t|^{m+2}} \cdot \frac{|t|}{|t|-|z|} + \frac{|z|^{m+1}}{|t|^{m+2}(|t|-|z|)} \cdot \frac{|t|}{|t|-|z|}\\
&\le \frac{(m+1)|z|^m}{|t|^{m+2}} \cdot 2 + \frac{|z|^m}{|t|^{m+2}} \cdot 2\\
&= (2m+4)\frac{|z|^m}{|t|^{m+2}},
\end{aligned}
$$

hence

$$x\pi|P_m(z,t)| \le x^2 S \le (2m+4)\frac{|z|^{m+2}}{|t|^{m+2}}.$$

If $1 \le |t| \le 2|z|$,

$$P_m(z,t) = \frac{1}{\pi}\Big[\frac{x}{|z-it|^2} + \mathrm{Im}\sum_{k=0}^{m}\frac{(-iz)^k}{t^{k+1}}\Big],$$

$$\Big|\mathrm{Im}\sum_{k=0}^{m}\frac{(-iz)^k}{t^{k+1}}\Big| \le \sum_{k=0}^{m}\frac{kx|z|^{k-1}}{|t|^{k+1}} = xS_m,$$

$$
\begin{aligned}
S_m &= \frac{|z|^{m-1}}{|t|^{m+1}}\Big[m + \frac{(m-1)|t|}{|z|} + \cdots + \frac{2|t|^{m-2}}{|z|^{m-2}} + \frac{|t|^{m-1}}{|z|^{m-1}}\Big]\\
&\le \frac{|3z|^{m-1}}{|t|^{m+1}}\Big[m + \frac{(m-1)|t|}{|3z|} + \cdots + \frac{2|t|^{m-2}}{|3z|^{m-2}} + \frac{|t|^{m-1}}{|3z|^{m-1}}\Big]\\
&= \frac{|3z|^{m-1}}{|t|^{m+1}}T_m.
\end{aligned}
$$

Similarly to $|t| \ge 2|z|$, $T_m \le 3m$, therefore

$$x\pi|P_m(z,t)| \le 1 + x^2 S_m \le 2(3+m)3^m(\frac{|z|}{|t|})^{m+1}.$$

If $|t| \le 1$,

$$x\pi|P_m(z,t)| \le 1 + x^2 S_m|t|^{m+2} \le (1+3^m m)|z|^{m+1}.$$

Proof of Theorem If $u \in H_\alpha$, $R > 1$, for every $0 < \varepsilon < 1$, applying Nevanlinna formula[1,2] to $u(z+\varepsilon)$, we obtain

$$
\begin{aligned}
u(1+\varepsilon) =& \frac{R^2-1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big(\frac{1}{|\mathrm{Re}^{i\theta}-1|^2} - \frac{1}{|\mathrm{Re}^{-i\theta}+1|^2}\Big)u(\mathrm{Re}^{i\theta}+\varepsilon)\mathrm{d}\theta +\\
&\frac{1}{\pi}\int_{-R}^{R}\Big(\frac{1}{|it-1|^2} - \frac{R^2}{|R^2+it|^2}\Big)u(it+\varepsilon)\mathrm{d}t. \qquad (7)
\end{aligned}
$$

Let $u^-(z) = \max\{-u(z), 0\}$ be the negative part of u,

$$m_+(R,\varepsilon) = \frac{R^2-1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big(\frac{1}{|\mathrm{Re}^{i\theta}-1|^2} - \frac{1}{|\mathrm{Re}^{-i\theta}+1|^2}\Big)u^+(\mathrm{Re}^{i\theta}+\varepsilon)\mathrm{d}\theta;$$

$$m_-(R,\varepsilon) = \frac{R^2-1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big(\frac{1}{|\mathrm{Re}^{i\theta}-1|^2} - \frac{1}{|\mathrm{Re}^{-i\theta}+1|^2}\Big)u^-(\mathrm{Re}^{i\theta}+\varepsilon)\mathrm{d}\theta;$$

$$n_+(R,\varepsilon)=\frac{1}{\pi}\int_{-R}^{R}\Big(\frac{1}{|it-1|^2}-\frac{R^2}{|R^2+it|^2}\Big)u^+(it+\varepsilon)\mathrm{d}t;$$

$$n_-(R,\varepsilon)=\frac{1}{\pi}\int_{-R}^{R}\Big(\frac{1}{|it-1|^2}-\frac{R^2}{|R^2+it|^2}\Big)u^-(it+\varepsilon)\mathrm{d}t.$$

Then if $R>1$, (7) becomes

$$m_-(R,\varepsilon)+n_-(R,\varepsilon)=m_+(R,\varepsilon)+n_+(R,\varepsilon)-u(1+\varepsilon). \tag{8}$$

If $R\geq 2$, $2R>|\mathrm{Re}^{i\theta}-1|\geq\frac{R}{2}$, $2R>|\mathrm{Re}^{-i\theta}+1|>R$,

$$16R^4>|\mathrm{Re}^{i\theta}-1|^2|\mathrm{Re}^{-i\theta}+1|^2\geq\frac{R^4}{4},$$

$$\begin{aligned}m_+(R,\varepsilon)&=\frac{R^2-1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{4R\cos\theta}{|\mathrm{Re}^{i\theta}-1|^2|\mathrm{Re}^{-i\theta}+1|^2}u^+(\mathrm{Re}^{i\theta}+\varepsilon)\mathrm{d}\theta\\&\leq\frac{R^2-1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{4R\cos\theta}{\frac{1}{4}R^4}u^+(\mathrm{Re}^{i\theta}+\varepsilon)\mathrm{d}\theta.\end{aligned}$$

If $|z|=\sqrt{x^2+y^2}\geq 2$, since

$$(\varepsilon+x)^2+y^2\leq 2(\varepsilon^2+x^2)+y^2\leq 2(1+x^2+y^2)\leq 4(x^2+y^2),$$

that is

$$x^2+y^2\geq\frac{1}{4}[(\varepsilon+x)^2+y^2],$$

we have

$$\begin{aligned}\int_2^\infty\frac{m_+(R,\varepsilon)}{R^{2\alpha+1}}\mathrm{d}R&\leq\frac{1}{2\pi}\int_2^\infty\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{R^2-1}{R^{2\alpha+1}}\cdot\frac{4R\cos\theta}{\frac{1}{4}R^4}u^+(\mathrm{Re}^{i\theta}+\varepsilon)\mathrm{d}\theta\mathrm{d}R\\&\leq\frac{8}{\pi}\int_2^\infty\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{R\cos\theta}{R^{2\alpha+3}}u^+(\mathrm{Re}^{i\theta}+\varepsilon)\mathrm{d}\theta\mathrm{d}R\\&=\frac{8}{\pi}\int\int_D\frac{xu^+(x+iy+\varepsilon)}{(x^2+y^2)^{\alpha+2}}\mathrm{d}x\mathrm{d}y\\&\leq\frac{8}{\pi}4^{\alpha+2}\int\int_D\frac{(\varepsilon+x)u^+(\varepsilon+x+iy)}{[(\varepsilon+x)^2+y^2]^{\alpha+2}}\mathrm{d}x\mathrm{d}y\\&\leq\frac{8}{\pi}4^{\alpha+2}I_1,\end{aligned}$$

where $D=\{z=x+iy:x>0,|z|\geq 2\}$. Hence

$$\sup_{0<\varepsilon<1}\int_2^\infty\frac{m_+(R,\varepsilon)}{R^{2\alpha+1}}\mathrm{d}R\leq\frac{8}{\pi}4^{\alpha+2}I_1<\infty. \tag{9}$$

By (9), we have

$$\sup_{0<\varepsilon<1}\liminf_{R\to\infty}m_+(R,\varepsilon)=0. \tag{10}$$

Since

$$n_+(R,\varepsilon)\leq\frac{1}{\pi}\int_{-R}^{R}\frac{u^+(it+\varepsilon)}{t^2+1}\mathrm{d}t, \tag{11}$$

by the Fubini theorem[3], we have

$$\int_2^\infty \frac{1}{R^{2\alpha+1}} n_+(R,\varepsilon)\mathrm{d}R \le \frac{1}{\pi}\int_2^\infty \int_t^\infty \frac{u^+(it+\varepsilon)}{R^{2\alpha+1}(t^2+1)}\mathrm{d}R\mathrm{d}t + \frac{1}{\pi}\int_{-2}^2 \int_2^\infty \frac{u^+(it+\varepsilon)}{R^{2\alpha+1}(t^2+1)}\mathrm{d}R\mathrm{d}t + \frac{1}{\pi}\int_{-\infty}^{-2}\int_{-t}^\infty \frac{u^+(it+\varepsilon)}{R^{2\alpha+1}(t^2+1)}\mathrm{d}R\mathrm{d}t. \tag{12}$$

If $\alpha > 0$,

$$\frac{1}{\pi}\int_2^\infty \int_t^\infty \frac{u^+(it+\varepsilon)}{R^{2\alpha+1}(t^2+1)}\mathrm{d}R\mathrm{d}t = \frac{1}{2\alpha\pi}\int_2^\infty \frac{u^+(it+\varepsilon)}{t^{2\alpha}(t^2+1)}\mathrm{d}t, \tag{13}$$

$$\frac{1}{\pi}\int_{-2}^2 \int_2^\infty \frac{u^+(it+\varepsilon)}{R^{2\alpha+1}(t^2+1)}\mathrm{d}R\mathrm{d}t = \frac{1}{2^{2\alpha+1}\alpha\pi}\int_{-2}^2 \frac{u^+(it+\varepsilon)}{t^2+1}\mathrm{d}t, \tag{14}$$

$$\frac{1}{\pi}\int_{-\infty}^{-2} \int_{-t}^\infty \frac{u^+(it+\varepsilon)}{R^{2\alpha+1}(t^2+1)}\mathrm{d}R\mathrm{d}t = \frac{1}{2\alpha\pi}\int_{-\infty}^{-2} \frac{u^+(it+\varepsilon)}{|t|^{2\alpha}(t^2+1)}\mathrm{d}t. \tag{15}$$

By (2), (12)–(15), we have

$$\sup_{0<\varepsilon<1}\int_2^\infty \frac{1}{R^{2\alpha+1}} n_+(R,\varepsilon)\mathrm{d}R \le \frac{1}{2\alpha\pi} I_2 < \infty, \quad \alpha > 0. \tag{16}$$

It is evident that if $\alpha > 0$,

$$\sup_{0<\varepsilon<1}\int_2^\infty \frac{1}{R^{2\alpha+1}}[-u(1+\varepsilon)]\mathrm{d}R = \frac{1}{2^{2\alpha+1}\alpha}\sup_{0<\varepsilon<1}[-u(1+\varepsilon)] < \infty. \tag{17}$$

(8), (9), (16), (17) imply

$$\sup_{0<\varepsilon<1}\int_2^\infty \frac{1}{R^{2\alpha+1}} n_-(R,\varepsilon)\mathrm{d}R < \infty, \quad \alpha > 0. \tag{18}$$

$$\begin{aligned}&\sup_{0<\varepsilon<1}\frac{1}{8\pi}\int_2^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta}{R^{2\alpha+2}} u^-(R\mathrm{e}^{i\theta}+\varepsilon)\mathrm{d}\theta\mathrm{d}R \\ &\le \sup_{0<\varepsilon<1}\int_2^\infty \frac{1}{R^{2\alpha+1}} m_-(R,\varepsilon)\mathrm{d}R < \infty, \quad \alpha > 0. \end{aligned}\tag{19}$$

If $R > 4$,

$$\begin{aligned}\pi n_-(R,\varepsilon) &\ge \int_{1\le|t|\le\frac{R}{2}} \Big(\frac{1}{1+t^2} - \frac{R^2}{R^4+t^2}\Big) u^-(it+\varepsilon)\mathrm{d}t \\ &\ge \int_{1\le|t|\le\frac{R}{2}} \Big(\frac{1}{1+t^2} - \frac{4}{16t^2+1}\Big) u^-(it+\varepsilon)\mathrm{d}t \\ &\ge \int_{1\le|t|\le\frac{R}{2}} \frac{1}{2(1+t^2)} u^-(it+\varepsilon)\mathrm{d}t. \end{aligned}\tag{20}$$

By (18), (20) and the Fubini theorem[3], we see that if $\alpha > 0$,

$$\begin{aligned}&\sup_{0<\varepsilon<1}\int_1^\infty \frac{1}{R^{2\alpha+1}}\int_1^{\frac{R}{2}} \frac{1}{2(1+t^2)} u^-(it+\varepsilon)\mathrm{d}t\mathrm{d}R = \sup_{0<\varepsilon<1}\frac{1}{2^{2\alpha+2}\alpha}\int_1^\infty \frac{1}{(1+t^2)t^{2\alpha}} u^-(it+\varepsilon)\mathrm{d}t \\ &\le \pi \sup_{0<\varepsilon<1}\int_2^\infty \frac{1}{R^{2\alpha+1}} n_-(R,\varepsilon)\mathrm{d}R < \infty. \end{aligned}\tag{21}$$

(2) and (21) show if $\alpha > 0$,

$$\sup_{0<\varepsilon<1}\int_{-\infty}^\infty \frac{|u(it+\varepsilon)|\mathrm{d}t}{1+|t|^{2\alpha+2}}\mathrm{d}t < \infty.$$

If $\alpha = 0$, (8), (10), (11), (20) imply

$$\sup_{0<\varepsilon<1} \int_{-\infty}^{\infty} \frac{|u(it+\varepsilon)|\mathrm{d}t}{1+|t|^2}\mathrm{d}t < \infty.$$

Thus i) holds. For every positive integer $n \geq 2\alpha$, by the property (3) of the modified Poisson kernel $P_n(z,t)$, we have

$$\Big|x\int_{|t|\leq 1} P_n(z,t)u(it+\varepsilon)\mathrm{d}t\Big| \leq (1+3^n n)|z|^{n+1}\int_{|t|\leq 1}|u(it+\varepsilon)|\mathrm{d}t;$$

$$\Big|x\int_{1\leq|t|\leq 2|z|} P_n(z,t)u(it+\varepsilon)\mathrm{d}t\Big| \leq 2(3+n)3^n|z|^{n+1}\int_{1\leq|t|\leq 2|z|}\frac{|u(it+\varepsilon)|}{|t|^{n+1}}\mathrm{d}t;$$

$$\Big|x\int_{|t|\geq 2|z|} P_n(z,t)u(it+\varepsilon)\mathrm{d}t\Big| \leq (2n+4)|z|^{n+2}\int_{|t|\geq 2|z|}\frac{|u(it+\varepsilon)|}{|t|^{n+2}}\mathrm{d}t$$

hold for $|z|>1, z=x+iy, x>0$. Let

$$u_n(z,\varepsilon) = \int_{-\infty}^{\infty} P_n(z,t)u(it+\varepsilon)\mathrm{d}t.$$

Then

$$xu_n(z,\varepsilon) = o(|z|^{n+2}), \quad |z|\to\infty,\ x>0. \tag{22}$$

For arbitrary $T>2$, put $D_T = \{z = x+iy : x>0, |z|<T\}$. $\overline{D}_T$ indicates the closure of D_T. Suppose $\chi_{[-2T,2T]}(t)$ is the characteristic function of the interval $[-2T,2T]$, $S_n(z,t) = \frac{1}{\pi}\mathrm{Re}\sum_{k=0}^{n}\frac{i(-iz)^k}{t^{k+1}}$. Thus $u_n(z,\varepsilon)$ may be written as

$$\begin{aligned} u_n(z,\varepsilon) = &\int_{|t|\leq 2T} P(z,t)u(it+\varepsilon)\mathrm{d}t - \int_{|t|\leq 1}|t|^{n+2}S_n(z,t)u(it+\varepsilon)\mathrm{d}t - \\ &\int_{1\leq|t|\leq 2T} S_n(z,t)u(it+\varepsilon)\mathrm{d}t + \int_{|t|\geq 2T} P_n(z,t)u(it+\varepsilon)\mathrm{d}t \\ =&X_\varepsilon(z) - Y_\varepsilon(z) - Z_\varepsilon(z) + V_\varepsilon(z). \end{aligned}$$

The function $X_\varepsilon(z)$ is the Poisson integral of $u(it+\varepsilon)\chi_{[-2T,2T]}(t)$, hence it is harmonic in $\overline{D}_T$, with $X_\varepsilon(iy) = u(iy+\varepsilon)$; $S_n(z,t)$ is a harmonic polynomial with $S_n(iy,t)\equiv 0$, so $Y_\varepsilon(z)$, $Z_\varepsilon(z)$ are also harmonic polynomials, with $Y_\varepsilon(iy)=0, \quad Z_\varepsilon(iy)=0$ ($y\in[-T,T]$); Similarly from (3), $V_\varepsilon(z)$ is harmonic in $\overline{D}_T$, with $V_\varepsilon(iy)=0, |y|\leq T$. Thus by the arbitrary of $T>2$, the function $u_n(z,\varepsilon)$ is harmonic in $\overline{\mathbb{C}}_+ = \{z = x+iy = re^{i\theta} : x\geq 0\}$, and

$$\lim_{z\to iy} u_n(z,\varepsilon) = u_n(iy,\varepsilon) = u(iy+\varepsilon), \quad y\in(-\infty,\infty). \tag{23}$$

For real number $\alpha\geq 0$, we denote by $C\,[1+|t|^{2\alpha+2}]$ the space of all continuous functions $G(t)$ on $\mathbb{R}$ for which

$$\lim_{t\to\pm\infty}|G(t)|(1+|t|^{2\alpha+2}) = 0.$$

If its norm is defined by

$$\|G\| = \sup_{t\in\mathbb{R}}|G(t)|(1+|t|^{2\alpha+2}),$$

then $C\,[1+|t|^{2\alpha+2}]$ is a Banach space and

$$P_n(z,t)\in C\,[1+|t|^{2\alpha+2}], \quad n>2\alpha.$$

Let ε_n decrease and tend to 0. Linear functional Λ_n on $C\,[1+|t|^{2\alpha+2}]$ defined by

$$\Lambda_n(G(t)) = \int_{-\infty}^{\infty} G(t)u(\varepsilon_n + it)\mathrm{d}t,$$

satisfies

$$|\Lambda_n(G(t))| \le \Big[\sup_{t\in\mathbb{R}} |G(t)|(1+|t|^{2\alpha+2})\Big] \int_{-\infty}^{\infty} \frac{|u(\varepsilon_n+it)|}{1+|t|^{2\alpha+2}}\mathrm{d}t \le \|G\|\cdot I_3,$$

thus each Λ_n is a bounded linear functional on $C\,[1+|t|^{2\alpha+2}]$, and $\|\Lambda_n\| \le I_3$. By the diagonal process[3], we can construct a subsequence of $u(\varepsilon_n+it)$ (we still denote by $u(\varepsilon_n+it)$ for convenience), such that

$$\Lambda(G) = \lim_{n\to\infty} \Lambda_n(G) = \lim_{n\to\infty} \int_{-\infty}^{\infty} G(t)u(\varepsilon_n+it)\mathrm{d}t, \tag{24}$$

holds for each $G(t) \in C\,[1+|t|^{2\alpha+2}]$ and $\|\Lambda\| \le I_3$. Therefore, Λ is a bounded linear functional on $C\,[1+|t|^{2\alpha+2}]$ and by the Riesz representation theorem[3], there exists a measure μ on $(-\infty,\infty)$, such that for each $G(t) \in C\,[1+|t|^{2\alpha+2}]$,

$$\Lambda(G) = \int_{-\infty}^{\infty} G(t)\mathrm{d}\mu(t), \tag{25}$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}|\mu|(t)}{1+|t|^{2\alpha+2}} = \lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{|u(\varepsilon_n+it)|}{1+|t|^{2\alpha+2}}\mathrm{d}t \le I_3 < \infty,$$

thus (5) holds. Fix a $z \in \mathbb{C}_+$ and put $G(t) = P_{m+1}(z,t)$. Then by (24), (25), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} P_{m+1}(z,t)\mathrm{d}\mu(t) &= \Lambda(G) = \lim_{n\to\infty}\int_{-\infty}^{\infty} P_{m+1}(z,t)\varphi_n(t)\mathrm{d}t \\ &= \lim_{n\to\infty}\int_{-\infty}^{\infty} P_{m+1}(z,t)u(\varepsilon_n+it)\mathrm{d}t = \lim_{n\to\infty} u_{m+1}(z,\varepsilon_n) = u_{m+1}(z).\end{aligned} \tag{26}$$

Let $u_m(z) = \int_{-\infty}^{\infty} P_m(z,t)\mathrm{d}\mu(t)$. Similarly to (22), we see that

$$xu_m(z) = o(|z|^{m+2}),\quad |z|\to\infty,\ x>0. \tag{27}$$

$$\begin{aligned}u_{m+1}(z) =& u_m(z) + \mathrm{Re}\Big[\sum_{k=0}^{m}\int_{-1}^{1}\frac{|t|^{m+2}-|t|^{m+3}}{\pi t^{k+1}}\mathrm{d}\mu(t)\cdot i(-iz)^k\Big] - \\ &\frac{1}{\pi}\mathrm{Re}\Big[\int_{-1}^{1}\frac{|t|^{m+3}}{t^{m+2}}\mathrm{d}\mu(t)\cdot i(-iz)^{m+1}\Big] - \\ &\frac{1}{\pi}\mathrm{Re}\Big[\int_{|t|>1}\frac{1}{t^{m+2}}\mathrm{d}\mu(t)\cdot i(-iz)^{m+1}\Big].\end{aligned}$$

In other words

$$u_{m+1}(z) = u_m(z) + \mathrm{Re}\sum_{k=1}^{m+1} b_k i(-iz)^k. \tag{28}$$

For every $z \in \mathbb{C}_+$, let

$$w_\varepsilon(z) = u(z+\varepsilon) - u_{m+1}(z,\varepsilon).$$

By the Schwarz reflection principle[3], there exists an entire function

$$f_\varepsilon(z) = \sum_{k=0}^{\infty} a_{k,\varepsilon} i(-iz)^k,\quad z\in\mathbb{C}$$

such that $\mathrm{Re} f_\varepsilon(z) = w_\varepsilon(z)$, $x = \mathrm{Re} z > 0$, $\mathrm{Re} f_\varepsilon(iy) \equiv 0$ ($y = \mathrm{Im} z \in (-\infty, \infty)$). Hence $f_\varepsilon(iy) = i\mathrm{Im} f_\varepsilon(iy) = i\sum_{k=0}^{\infty} a_{k,\varepsilon} y^k$. This shows $a_{k,\varepsilon}$ ($k = 0, 1, 2, \ldots$) are real. Therefore

$$\mathrm{Re} f_\varepsilon(-re^{-i\theta}) = -\mathrm{Re} f_\varepsilon(re^{i\theta}) = \sum_{k=0}^{\infty} a_{k,\varepsilon} r^k \sin k(\theta - \frac{\pi}{2}), \quad r > 0, \theta \in (-\infty, \infty). \tag{29}$$

It follows that

$$\begin{aligned} -a_{n,\varepsilon} &= \frac{1}{\pi r^n} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{Re} f_\varepsilon(\mathrm{r}e^{i\theta}) \sin n(\theta - \frac{\pi}{2}) \mathrm{d}\theta \\ &= \frac{2}{\pi r^n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} w_\varepsilon(\mathrm{r}e^{i\theta}) \sin n(\theta - \frac{\pi}{2}) \mathrm{d}\theta, \quad r > 0, n = 1, 2, \ldots . \end{aligned} \tag{30}$$

Since $|\sin n(\theta - \frac{\pi}{2})| \le n|\sin(\theta - \frac{\pi}{2})|$, $n \ge 1$, by (9), (19), (22), (29) and (30), we have

$$|a_{n,\varepsilon}| \le \liminf_{r \to \infty} \frac{2n}{\pi r^n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |w_\varepsilon(\mathrm{r}e^{i\theta})| \cos\theta \mathrm{d}\theta = 0, \quad n \ge m + 3. \tag{31}$$

In addition, it is obvious that $\mathrm{Re}(ia_{0,\varepsilon}) = 0$. If $\alpha > 0$, similarly we have $a_{m+2,\varepsilon} = 0$, so

$$u(z + \varepsilon) = \mathrm{Re} \sum_{k=1}^{m+1} a_{k,\varepsilon} i(-iz)^k + u_{m+1}(z, \varepsilon). \tag{32}$$

If $\alpha = 0$,

$$u(z + \varepsilon) = a_{1,\varepsilon} \mathrm{Re} z + u_1(z, \varepsilon). \tag{33}$$

Since both $u(z + \varepsilon)$ and $u_{m+1}(z, \varepsilon)$ are harmonic in the closed right half plane $\overline{\mathbb{C}}_+$, there exist holomorphic functions $g_\varepsilon(z)$ and $h_\varepsilon(z)$ in the closed right half plane such that $u(z+\varepsilon) = \mathrm{Re} g_\varepsilon(z)$ and $u_{m+1}(z, \varepsilon) = \mathrm{Re} h_\varepsilon(z)$. Let

$$Q_\varepsilon(z) = \sum_{k=1}^{m+1} a_{k,\varepsilon} i(-iz)^k = g_\varepsilon(z) - h_\varepsilon(z).$$

Then

$$Q_\varepsilon(z) = \sum_{k=1}^{m+1} \frac{Q_\varepsilon^{(k)}(1)}{k!} (z - 1)^k = \sum_{k=1}^{m+1} c_{k,\varepsilon} (z - 1)^k.$$

Since $g_\varepsilon(z)$ and $h_\varepsilon(z)$ are holomorphic in the closed right half plane $\overline{\mathbb{C}}_+$, there exists a constant M independent of ε, such that for every $z \in \{z : |z - 1| = \frac{1}{2}\}$, we have $|g_\varepsilon(z) - h_\varepsilon(z)| \le M$. Since

$$Q_\varepsilon^{(k)}(1) = \frac{k!}{2\pi i} \int_{|z-1|=\frac{1}{2}} \frac{Q_\varepsilon(z)}{(z - 1)^{k+1}} \mathrm{d}z = \frac{k!}{2\pi i} \int_{|z-1|=\frac{1}{2}} \frac{g_\varepsilon(z) - h_\varepsilon(z)}{(z - 1)^{k+1}} \mathrm{d}z,$$

we see that

$$|\frac{Q_\varepsilon^{(k)}(1)}{k!}| \le 2^k M.$$

Hence there is a sequence $\{\varepsilon_n : n = 1, 2, \ldots,\}$ of positive numbers with $\varepsilon_n \to 0$ as $n \to \infty$ such that for each $k \in \{1, 2, \ldots, m + 1\}$, $c_{k,\varepsilon_n} \to c_k$, as $n \to \infty$, that is, $Q_{\varepsilon_n}(z) \to Q(z)$ as $n \to \infty$. Take $\varepsilon = \varepsilon_n$ in (32) and let $n \to \infty$, it follows that

$$u(z) = \mathrm{Re} \sum_{k=1}^{m+1} a_k i(-iz)^k + u_{m+1}(z). \tag{34}$$

By (28) and (34), we have

$$u(z) = \mathrm{Re}\sum_{k=1}^{m+1} a_k i(-iz)^k + u_m(z) + \mathrm{Re}\sum_{k=1}^{m+1} b_k i(-iz)^k,$$

that is

$$u(z) = \mathrm{Re}\sum_{k=1}^{m+1} (a_k + b_k) i(-iz)^k + u_m(z). \tag{35}$$

If $\alpha > 0$, by (27), we have the following analogue of (31),

$$a_{m+1} + b_{m+1} = 0.$$

If $\alpha = 0$, using a similar method as in (33), we see that

$$u(z) = (a_1 + b_1)\mathrm{Re}z + u_0(z).$$

This completes the proof of Theorem. □

References

[1] BOAS JR B P. *Entire Functions* [M]. Academic Press, New York, 1954.
[2] LEVIN B YA. *Lectures on Entire Functions* [M]. American Mathematical Society, Providence, RI, 1996.
[3] RUDIN W. *Real and Complex Analysis* [M]. Third edition. McGraw-Hill Book Co., New York, 1987.