Journal of Mathematical Research & Exposition
July, 2009, Vol. 29, No. 4, pp. 671–681
DOI:10.3770/j.issn:1000-341X.2009.04.012
Http://jmre.dlut.edu.cn

Positive Solutions of Singular Fourth Order Four Point Boundary Value Problems with p-Laplacian Operator

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Abstract In this paper, we consider the existence of positive solutions for the singular fourth-order four point boundary value problem with p-Laplacian operator. By using the fixed point theorem of cone expansion and compression, the existence of multiple positive solutions is obtained.

Keywords p-Laplacian; singular boundary value problem; positive solution; cone.

Document code A
MR(2000) Subject Classification 34B15
Chinese Library Classification O175.8

1. Introduction

The nonlinear boundary value problems have been widely studied in recent years. For details, see [1–3] and references therein. Moreover, the singular problems are also considered in [4–5]. In a recent paper[6], by employing the upper and lower solution method, Zhang and Liu established the existence of at least one positive solution for the singular fourth order four point boundary value problem

$$\begin{cases} (\phi_p(u''(t)))'' = f(t, u(t)), \quad t \in (0,1), \\ u(0) = 0, \ \ u(1) = au(\xi), \\ u''(0) = 0, u''(1) = bu''(\eta), \end{cases}$$

where $\phi_p(t) = |t|^{p-2}t$, $p > 1$, $0 < \xi$, $\eta < 1$, $0 \le a$, $b < 1$, $f \in C((0,1) \times (0,+\infty), [0,+\infty))$ may be singular at $t = 0$, $t = 1$, $u = 0$.

However, to the best of our knowledge, the study of multiple positive solutions to the fourth-order four point boundary value problem with p-Laplacian operator has not been found in literature. We intend in this paper to discuss the untouched problem

$$\begin{cases} (\phi_p(u''(t)))'' = f(t, u(t), u''(t)), \quad t \in (0,1), \\ u(0) = 0, \ \ u(1) = au(\xi), \\ u''(0) = 0, u''(1) = bu''(\eta), \end{cases} \tag{1.1}$$

where $\phi_p(t) = |t|^{p-2}t$, $p > 1$, $0 < \xi$, $\eta < 1$, $0 \le a$, $b < 1$, $b_1 = \phi_p(b)$, $f \in C((0,1) \times (0,+\infty) \times (-\infty,0), [0,+\infty))$ may be singular at $t = 0$, $t = 1$, $u = 0$ and $u'' = 0$. According to

Received date: 2007-06-25; **Accepted date**: 2008-03-08
Foundation item: the National Natural Science Foundation of China (No. 10671181).

prior estimate of positive solutions, we construct a cone to overcome the difficulties caused by singularity. Moreover, under some reasonable assumptions on f, the existence of multiple positive solutions is obtained by applying the fixed point theorem of cone expansion and compression.

2. Some preliminaries and lemmas

Defition 2.1 *A function u is said to be a solution of the boundary value problem (1.1), if $u \in C^2[0,1]$ satisfies $\phi_p(u'') \in C^2(0,1)$ and the BVP(1.1). Furthermore, u is said to be a positive solution, if again $u(t) > 0$ for $t \in (0,1)$.*

Now we consider the following linear boundary value problem

$$\begin{cases} (\phi_p(u''(t)))'' = y(t), & t \in (0,1), \\ u(0) = 0, \ \ u(1) = au(\xi), \\ u''(0) = 0, u''(1) = bu''(\eta). \end{cases} \tag{2.1}$$

For BVP(2.1), we have the following lemmas.

Lemma 2.1 *Let $0 < \xi, \eta < 1$, $0 \le a, b < 1$, $\frac{1}{p} + \frac{1}{q} = 1$. If $y \in C[0,1]$ and $y \ge 0$, then BVP(2.1) has a unique solution $u(t) \ge 0$ for $t \in [0,1]$, such that*

$$u(t) = \int_0^1 G(t,s)\phi_q(\int_0^1 H(s,r)y(r)\mathrm{d}r)\mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} s \in [0,\xi]: \begin{cases} \frac{t}{1-a\xi}[(1-s) - a(\xi - s)], & 0 \le t \le s \le 1, \\ \frac{s}{1-a\xi}[(1-t) - a(\xi - t)], & 0 \le s \le t \le 1; \end{cases} \\ s \in [\xi,1]: \begin{cases} \frac{t}{1-a\xi}(1-s), & 0 \le t \le s \le 1, \\ \frac{1}{1-a\xi}[s(1-t) + a\xi(t-s)], & 0 \le s \le t \le 1, \end{cases} \end{cases}$$

and

$$H(t,s) = \begin{cases} s \in [0,\eta]: \begin{cases} \frac{t}{1-b_1\eta}[(1-s) - b_1(\eta - s)], & 0 \le t \le s \le 1, \\ \frac{s}{1-b_1\eta}[(1-t) - b_1(\eta - t)], & 0 \le s \le t \le 1; \end{cases} \\ s \in [\eta,1]: \begin{cases} \frac{t}{1-b_1\eta}(1-s), & 0 \le t \le s \le 1, \\ \frac{1}{1-b_1\eta}[s(1-t) + b_1\eta(t-s)], & 0 \le s \le t \le 1 \end{cases} \end{cases}$$

are the associated Green's function for the problem (2.1).

Lemma 2.2 *The associated Green's function $G(t,s)$ and $H(t,s)$ have the following properties.*

(a) $G(t,s)$, $H(t,s)$ are continuous on $[0,1]\times[0,1]$ and $G(t,s) > 0$, $H(t,s) > 0$ for any $t, s \in (0,1)$.

(b) For any $t, s \in (0,1)$, $G(t,s) \le G(s,s)$, $H(t,s) \le H(s,s)$.

(c) For any $t, s \in (0,1)$, $G(t,s) \ge z(t)G(s,s)$, $H(t,s) \ge z(t)H(s,s)$, where $z(t) = \min\{t, 1-t\}$.

Proof We can easily get the results from Lemma 2.3 in [6].

We also assume

(H_0) $f \in C((0,1)\times(0,+\infty)\times(-\infty,0),\ [0,+\infty))$ and $0 < \int_0^1 H(s,s)f_{rR}(s)\mathrm{d}s < +\infty$, for any $0 < r \leq R$, where $t \in (0,1)$,

$$f_{rR}(t) = \max\left\{f(t,u,v), u \in [f_1(t)r, f_2(t)R], v \in [-R, -g(t)r]\right\},$$

$f_1(t) = \int_0^1 G(t,s)\phi_q(z(s))\mathrm{d}s,\ f_2(t) = \int_0^1 G(t,s)\mathrm{d}s,\ g(t) = \phi_q(z(t)),\ z(t) = \min\{t, 1-t\}$. It is easy to see that $f_1(t)$ and $f_2(t)$ are continuous on $[0,1]$ and $f_1(t) > 0,\ f_2(t) > 0$ for $t \in (0,1)$. We suppose that (H_0) holds throughout the remainder of the paper. Let

$$E = \left\{u \in C^2[0,1], u(0) = 0, u(1) = au(\xi)\right\}.$$

Then E is a Banach space with a norm by $\| u \| = \| u''(t) \|_0 = \max_{t\in[0,1]} |u''(t)|$, $u \in E$. Define

$$P = \{u \in E : u(t) \geq f_1(t) \| u \|, -u'' \geq g(t) \| u \|\}.$$

Then P is a cone in E and $f_1(t) \in P$. In fact, $\| f_1(t) \| = \max_{t\in[0,1]} |g(t)| \leq 1$, so we have $f_1(t) \geq f_1(t) \| f_1(t) \|$, $-f_1''(t) \geq g(t) \| f_1(t) \|$. For $u \in P\setminus\{\theta\}$, define an operator A by

$$Au(t) = \int_0^1 G(t,s)\phi_q(\int_0^1 H(s,r)f(r,u(r),u''(r))\mathrm{d}r)\mathrm{d}s. \tag{2.2}$$

Since $u \in P\setminus\{\theta\}$, we have $u(t) = -\int_0^1 G(t,s)u''(s)\mathrm{d}s$, and

$$u(t) \leq \int_0^1 G(t,s)\mathrm{d}s \| u \| = f_2(t) \| u \|.$$

Consequently,

$$f_1(t) \| u \| \leq u(t) \leq f_2(t) \| u \| \text{ and } g(t) \| u \| \leq -u''(t) \leq \| u \|.$$

Together with (H_0), we get

$$\int_0^1 H(s,r)f(r,u(r),u''(r))\mathrm{d}r < +\infty.$$

So A is well-defined and $A(P\setminus\{\theta\}) \subset E$. Obviously, that u is a positive solution of BVP(1.1) is equivalent to that $Au = u$ in $P\setminus\{\theta\}$ has a fixed point.

Lemma 2.3 *If $u \in P\setminus\{\theta\}$, then we have $Au \in P$, i.e., $A(P\setminus\{\theta\}) \subset P$.*

Proof For any $u \in P\setminus\{\theta\}$, we have

$$Au(t) = \int_0^1 G(t,s)\phi_q(\int_0^1 H(s,r)f(r,u(r),u''(r))\mathrm{d}r)\mathrm{d}s,$$

and

$$\begin{aligned}-(Au)''(t) =&\phi_q(\int_0^1 H(t,s)f(s,u(s),u''(s))\mathrm{d}s) \geq \phi_q(z(t)\int_0^1 H(s,s)f(s,u(s),u''(s))\mathrm{d}s)\\ =&\phi_q(z(t))\phi_q(\int_0^1 H(s,s)f(s,u(s),u''(s))\mathrm{d}s) \geq \phi_q(z(t)) \| Au \| = g(t) \| Au \|.\end{aligned}$$

Since $Au(t) \in E$, we see that

$$Au(t) = -\int_0^1 G(t,s)(Au)''(s))\mathrm{d}s,$$

and

$$Au(t) \geq \int_0^1 G(t,s)\phi_q(z(s))\mathrm{d}s \parallel Au \parallel = f_1(t) \parallel Au \parallel .$$

Therefore $A(P\backslash \{\theta\}) \subset P$. □

Lemma 2.4 *For any $R_2 > R_1 > 0$, $A : \overline{P_{R_2}}\backslash P_{R_1} \longrightarrow P$ is completely continuous, where $P_r = \{u \in P, \parallel u \parallel < r\}\,(r > 0)$.*

Proof For any $u \in \overline{P_{R_2}}\backslash P_{R_1}$, then $f_1(t)R_1 \leq u(t) \leq f_2(t)R_2$ and $g(t)R_1 \leq -u''(t) \leq R_2$. From (2.2), we have

$$-(Au)''(t) = \phi_q(\int_0^1 H(t,s)f(s,u(s),u''(s))\mathrm{d}s),$$

and

$$\parallel Au \parallel \leq \phi_q(\int_0^1 H(s,s)f(s,u(s),u''(s))\mathrm{d}s) \leq \phi_q(\int_0^1 H(s,s)f_{R_1R_2}(s)\mathrm{d}s) \triangleq M.$$

Thus $\parallel Au \parallel \leq M$, which implies that A is bounded on $\overline{P_{R_2}}\backslash P_{R_1}$.

Next, we prove that $\{(Au)''(t), u \in V\}$ is equicontinuous, for all $V \subset \overline{P_{R_2}}\backslash P_{R_1}$. Since $\|Au\| \leq M$ and ϕ_q is uniformly continuous on $[0, \phi_p(M)]$, we only need to show $\{-\phi_p(Au)''(t), u \in V\}$ is equicontinuous on $[0,1]$. Here, we divide our proof into three steps.

Step 1 $\{-\phi_p(Au)''(t), u \in V\}$ is equicontinuous on $[0, \alpha]$.

Let $t \leq \eta$. Then

$$\begin{aligned}-\phi_p(Au)''(t) &= \int_0^1 H(t,s)f(s,u(s),u''(s))\mathrm{d}s \\ &= \int_0^t \frac{s}{1-b_1\eta}[(1-t)-b_1(\eta-t)]f(s,u(s),u''(s))\mathrm{d}s + \\ &\quad \int_t^\eta \frac{t}{1-b_1\eta}[(1-s)-b_1(\eta-s)]f(s,u(s),u''(s))\mathrm{d}s + \\ &\quad \int_\eta^1 \frac{t}{1-b_1\eta}(1-s)f(s,u(s),u''(s))\mathrm{d}s. \end{aligned} \tag{2.3}$$

First, from

$$\int_0^1 H(t,s)f_{R_1R_2}(s)\mathrm{d}s \leq \int_0^1 H(s,s)f_{R_1R_2}(s)\mathrm{d}s = \phi_p(M), \tag{2.4}$$

we can easily get

$$\int_\eta^1 s(1-s)f_{R_1R_2}(s)\mathrm{d}s \leq \phi_p(M), \tag{2.5}$$

$$\int_0^\eta s[(1-s)-b_1(\eta-s)]f_{R_1R_2}(s)\mathrm{d}s \leq \phi_p(M), \tag{2.6}$$

$$\int_0^\eta sf_{R_1R_2}(s)\mathrm{d}s \leq n_1\phi_p(M)\ (n_1 > 0), \tag{2.7}$$

$$\int_\eta^1 (1-s)f_{R_1R_2}(s)\mathrm{d}s \leq n_2\phi_p(M)\ (n_2 > 0). \tag{2.8}$$

Again,

$$
\begin{aligned}
&\int_0^t \frac{s}{1-b_1\eta}[(1-t)-b_1(\eta-t)]f(s,u(s),u''(s))\mathrm{d}s \\
&\quad \le \int_0^t \frac{s}{1-b_1\eta}[(1-s)-b_1(\eta-s)]f(s,u(s),u''(s))\mathrm{d}s \\
&\quad \le \int_0^t \frac{s}{1-b_1\eta}[(1-s)-b_1(\eta-s)]f_{R_1R_2}(s)\mathrm{d}s,
\end{aligned} \tag{2.9}
$$

$$
\int_\eta^1 \frac{t}{1-b_1\eta}(1-s)f(s,u(s),u''(s))\mathrm{d}s \le \frac{t}{1-b_1\eta}\int_\eta^1 (1-s)f_{R_1R_2}(s)\mathrm{d}s. \tag{2.10}
$$

Applying (2.6) and (2.9), we obtain

$$
\lim_{t\longrightarrow 0^+}\int_0^t \frac{s}{1-b_1\eta}[(1-t)-b_1(\eta-t)]f(s,u(s),u''(s))\mathrm{d}s = 0, \tag{2.11}
$$

uniformly with respect to $u \in V$. Employing (2.8) and (2.10), we get

$$
\lim_{t\longrightarrow 0^+}\int_\eta^1 \frac{t}{1-b_1\eta}(1-s)f(s,u(s),u''(s))\mathrm{d}s = 0, \tag{2.12}
$$

uniformly with respect to $u \in V$.

Second, from (2.6), for any $\varepsilon > 0$, there exists $\eta_1 > 0$, such that

$$
\int_0^{\eta_1} \frac{s}{1-b_1\eta}[(1-s)-b_1(\eta-s)]f_{R_1R_2}(s)\mathrm{d}s < \varepsilon.
$$

Let $t < \eta_1$. We have

$$
\begin{aligned}
&\int_t^\eta \frac{t}{1-b_1\eta}[(1-s)-b_1(\eta-s)]f(s,u(s),u''(s))\mathrm{d}s \\
&\quad = \int_t^{\eta_1} \frac{t}{1-b_1\eta}[(1-s)-b_1(\eta-s)]f(s,u(s),u''(s))\mathrm{d}s + \\
&\qquad \int_{\eta_1}^{\eta} \frac{t}{1-b_1\eta}[(1-s)-b_1(\eta-s)]f(s,u(s),u''(s))\mathrm{d}s \\
&\quad \le \int_0^{\eta_1} \frac{s}{1-b_1\eta}[(1-s)-b_1(\eta-s)]f(s,u(s),u''(s))\mathrm{d}s + \\
&\qquad \frac{t}{\eta_1}\int_{\eta_1}^{\eta} \frac{s}{1-b_1\eta}[(1-s)-b_1(\eta-s)]f(s,u(s),u''(s))\mathrm{d}s \\
&\quad \le \int_0^{\eta_1} \frac{s}{1-b_1\eta}[(1-s)-b_1(\eta-s)]f_{R_1R_2}(s)\mathrm{d}s + \frac{t}{\eta_1}\frac{\phi_p(M)}{1-b_1\eta}.
\end{aligned}
$$

Therefore,

$$
\lim_{t\longrightarrow 0^+}\int_t^\eta \frac{t}{1-b_1\eta}[(1-s)-b_1(\eta-s)]f(s,u(s),u''(s))\mathrm{d}s = 0, \tag{2.13}
$$

uniformly with respect to $u \in V$. It follows from (2.11)–(2.13) that

$$
\lim_{t\longrightarrow 0^+} -\phi_p(Au)''(t) = 0,
$$

uniformly with respect to $u \in V$. Then there exists $\alpha > 0$ small enough, without loss of generality, we assume $\alpha < \eta$, such that $\{-\phi_p(Au)''(t), u \in V\}$ is equicontinuous on $[0,\alpha]$.

Step 2 $\{-\phi_p(Au)''(t), u \in V\}$ is equicontinuous on $[\beta, 1]$.

Let $t > \eta$. We obtain

$$\begin{aligned}-\phi_p(Au)''(t) &= \int_0^1 H(t,s)f(s,u(s),u''(s))\mathrm{d}s\\ &= \int_0^\eta \frac{s}{1-b_1\eta}[(1-t)-b_1(\eta-t)]f(s,u(s),u''(s))\mathrm{d}s+\\ &\quad \int_\eta^t \frac{1}{1-b_1\eta}[s(1-t)+b_1\eta(t-s)]f(s,u(s),u''(s))\mathrm{d}s+\\ &\quad \int_t^1 \frac{t}{1-b_1\eta}(1-s)f(s,u(s),u''(s))\mathrm{d}s.\end{aligned}$$

First, notice that

$$\int_t^1 \frac{t}{1-b_1\eta}(1-s)f(s,u(s),u''(s))\mathrm{d}s \le \int_t^1 \frac{s}{1-b_1\eta}(1-s)f_{R_1R_2}(s)\mathrm{d}s.$$

This together with (2.5) guarantees that

$$\lim_{t\longrightarrow 1^-} \int_t^1 \frac{t}{1-b_1\eta}(1-s)f(s,u(s),u''(s))\mathrm{d}s = 0, \tag{2.14}$$

uniformly with respect to $u \in V$.

Secondly, we consider the function

$$\int_0^\eta \frac{s}{1-b_1\eta}[(1-t)-b_1(\eta-t)]f(s,u(s),u''(s))\mathrm{d}s.$$

For any t_1, $t_2 > \eta$, (2.7) implies that

$$\begin{aligned}&|\int_0^\eta \frac{s}{1-b_1\eta}[(1-t_1)-b_1(\eta-t_1)]f(s,u(s),u''(s))\mathrm{d}s-\\ &\quad \int_0^\eta \frac{s}{1-b_1\eta}[(1-t_2)-b_1(\eta-t_2)]f(s,u(s),u''(s))\mathrm{d}s|\\ &= |\int_0^\eta \frac{s}{1-b_1\eta}[(1-b_1)(t_2-t_1)f(s,u(s),u''(s))\mathrm{d}s|\\ &= |\int_0^\eta \frac{s(1-b_1)}{1-b_1\eta}f(s,u(s),u''(s))\mathrm{d}s||t_1-t_2|\\ &\le \frac{1-b_1}{1-b_1\eta}n_1\phi_p(M)|t_1-t_2|.\end{aligned} \tag{2.15}$$

Thirdly, we consider

$$\begin{aligned}&\int_\eta^t \frac{1}{1-b_1\eta}[s(1-t)+b_1\eta(t-s)]f(s,u(s),u''(s))\mathrm{d}s\\ &= \frac{1}{1-b_1\eta}\int_\eta^t s(1-t)f(s,u(s),u''(s)\mathrm{d}s + \frac{b_1\eta}{1-b_1\eta}\int_\eta^t (t-s)f(s,u(s),u''(s)\mathrm{d}s.\end{aligned}$$

For any $\varepsilon > 0$, by (2.5) there exists $\delta > \eta$, such that

$$\int_\delta^1 (1-s)sf_{R_1R_2}(s)\mathrm{d}s < \varepsilon. \tag{2.16}$$

Let $t > \delta$. Since

$$\begin{aligned}
&(1-t)\int_\eta^t sf(s,u(s),u''(s))\mathrm{d}s \\
&\le (1-t)\int_\eta^\delta sf(s,u(s),u''(s))\mathrm{d}s + (1-t)\int_\delta^t sf(s,u(s),u''(s))\mathrm{d}s \\
&\le \frac{(1-t)}{1-\delta}\int_\eta^\delta (1-s)sf(s,u(s),u''(s))\mathrm{d}s + \int_\delta^t (1-s)sf(s,u(s),u''(s))\mathrm{d}s \\
&\le \frac{(1-t)}{1-\delta}\int_\eta^1 (1-s)sf(s,u(s),u''(s))\mathrm{d}s + \int_\delta^1 (1-s)sf(s,u(s),u''(s))\mathrm{d}s \\
&\le \frac{(1-t)}{1-\delta}\int_\eta^1 (1-s)sf_{R_1R_2}(s)\mathrm{d}s + \int_\delta^1 (1-s)sf_{R_1R_2}(s)\mathrm{d}s,
\end{aligned}$$

we have

$$\lim_{t\longrightarrow 1^-}(1-t)\int_\eta^t sf(s,u(s),u''(s))\mathrm{d}s = 0, \tag{2.17}$$

uniformly with respect to $u \in V$. In view of (2.8), there exists $\delta_0 > \eta$, such that

$$\int_{\delta_0}^1 (1-s)f_{R_1R_2}(s)\mathrm{d}s < \varepsilon.$$

Let $t > \delta_0$. We get

$$\int_\eta^t (t-s)f(s,u(s),u''(s))\mathrm{d}s = \int_\eta^{\delta_0}(t-s)f(s,u(s),u''(s))\mathrm{d}s + \int_{\delta_0}^t (t-s)f(s,u(s),u''(s))\mathrm{d}s$$

and

$$\int_{\delta_0}^t (t-s)f(s,u(s),u''(s))\mathrm{d}s \le \int_{\delta_0}^1 (1-s)f_{R_1R_2}(s)\mathrm{d}s < \varepsilon. \tag{2.18}$$

Let $m_1 = \min_{t\in[\eta,\delta_0]} f_1(t)$, $M_1 = \max_{t\in[\eta,\delta_0]} f_2(t)$ and $m_2 = \min_{t\in[\eta,\delta_0]} g(t)$. Then $u(t) \in [m_1R_1, M_1R_2]$, $-u''(t) \in [m_2R_1, R_2]$, so there exists $B > 0$ such that $\max_{t\in[\eta,\delta_0]} |f(t,u(t),u''(t)| \le B$. Thus for any $t_1, t_2 > \eta$, we obtain

$$\begin{aligned}
&|\int_\eta^{\delta_0}(t_1-s)f(s,u(s),u''(s))\mathrm{d}s - \int_\eta^{\delta_0}(t_2-s)f(s,u(s),u''(s))\mathrm{d}s| \\
&= |\int_\eta^{\delta_0}(t_1-t_2)f(s,u(s),u''(s))\mathrm{d}s| \le B|t_1-t_2|.
\end{aligned} \tag{2.19}$$

(2.14), (2.15) and (2.17)–(2.19) guarantee that there exists $\beta > \eta$, such that $\{-\phi_p(Au)''(t), u \in V\}$ is equicontinuous on $[\beta, 1]$.

Step 3 $\{-\phi_p(Au)''(t), u \in V\}$ is equicontinuous on $[\alpha_1, \beta_1](0 < \alpha_1 < \alpha, 1 > \beta_1 > \beta)$. For any $t_1, t_2 \in [\alpha_1, \beta_1]$, we have

$$\begin{aligned}
&|\int_0^1 H(t_1,s)f(s,u(s),u''(s))\mathrm{d}s - \int_0^1 H(t_2,s)f(s,u(s),u''(s))\mathrm{d}s| \\
&\le |\int_0^{\alpha_1} H(t_1,s)f(s,u(s),u''(s))\mathrm{d}s - \int_0^{\alpha_1} H(t_2,s)f(s,u(s),u''(s))\mathrm{d}s| +
\end{aligned}$$

$$|\int_{\alpha_1}^{\beta_1} H(t_1,s)f(s,u(s),u''(s))\mathrm{d}s - \int_{\alpha_1}^{\beta_1} H(t_2,s)f(s,u(s),u''(s))\mathrm{d}s| +$$
$$|\int_{\beta_1}^{1} H(t_1,s)f(s,u(s),u''(s))\mathrm{d}s - \int_{\beta_1}^{1} H(t_2,s)f(s,u(s),u''(s))\mathrm{d}s|.$$

From $\alpha_1 < \alpha < \eta$, we get

$$\begin{aligned}
&|\int_0^{\alpha_1} H(t_1,s)f(s,u(s),u''(s))\mathrm{d}s - \int_0^{\alpha_1} H(t_2,s)f(s,u(s),u''(s))\mathrm{d}s| \\
&\le |\int_0^{\alpha_1} \frac{s}{1-b_1\eta}[(1-t_1)-b_1(\eta-t_1)]f(s,u(s),u''(s))\mathrm{d}s - \\
&\quad \int_0^{\alpha_1} \frac{s}{1-b_1\eta}[(1-t_2)-b_1(\eta-t_2)]f(s,u(s),u''(s))\mathrm{d}s| \\
&\le |\int_0^{\alpha_1} \frac{s}{1-b_1\eta}(1-b_1)f(s,u(s),u''(s))\mathrm{d}s||t_1-t_2| \\
&\le \frac{1-b_1}{1-b_1\eta}|\int_0^{\eta} s f_{R_1R_2}\mathrm{d}s||t_1-t_2|.
\end{aligned} \tag{2.20}$$

From $\beta_1 > \beta > \eta$, we obtain

$$\begin{aligned}
&|\int_{\beta_1}^{1} H(t_1,s)f(s,u(s),u''(s))\mathrm{d}s - \int_{\beta_1}^{1} H(t_2,s)f(s,u(s),u''(s))\mathrm{d}s| \\
&= |\int_{\beta_1}^{1} \frac{t_1}{1-b_1\eta}(1-s)f(s,u(s),u''(s))\mathrm{d}s - \int_{\beta_1}^{1} \frac{t_2}{1-b_1\eta}(1-s)f(s,u(s),u''(s))\mathrm{d}s| \\
&= \frac{1}{1-b_1\eta}|(t_1-t_2)\int_{\beta_1}^{1}(1-s)f(s,u(s),u''(s))\mathrm{d}s| \\
&= \frac{1}{1-b_1\eta}|\int_{\beta_1}^{1}(1-s)f(s,u(s),u''(s))\mathrm{d}s||t_1-t_2| \\
&\le \frac{1}{1-b_1\eta}|\int_{\eta}^{1}(1-s)f_{R_1R_2}\mathrm{d}s||t_1-t_2|.
\end{aligned} \tag{2.21}$$

Since f has no singularity on $[\alpha_1,\beta_1]$, with the same reason as Step 2, $\int_{\alpha_1}^{\beta_1} H(t,s)f(s,u(s),u''(s))\mathrm{d}s$ is equicontinuous on $[\alpha_1,\beta_1]$. Combining (2.20) and (2.21), we know Step 3 holds. From Steps 1 to 3, we see that $\{-\phi_p(Au)''(t), u\in V\}$ is equicontinuous on $[0,1]$. By applying the Ascoli-Arzela theorem, for any $(Au_n)''(t)(u_n\in V)$, there exists $u_0\in C[0,1]$. Without loss of generality, we assume $\|(Au_n)''(t)-u_0\|_0 \longrightarrow 0$. Let $u(t) = -\int_0^1 G(t,s)u_0(s)\mathrm{d}s$. Then $u\in E$, and $\|Au_n-u\| = \|(Au_n)''(t)-u_0\|_0 \longrightarrow 0$. Therefore, AV is relatively compact.

In addition, according to the Lebesgue dominated convergence theorem and

$$\int_0^1 H(s,s)f_{R_1R_2}(s)\mathrm{d}s < +\infty,$$

we can easily get the continuity of A. The proof is completed. □

Lemma 2.5[7] *Suppose that E is a real Banach space and $P\subset E$ is a cone. Let Ω_1, Ω_2 be two bounded sets of E, such that $\theta\in\Omega_1$, $\overline{\Omega_1}\subset\Omega_2$. Let operator $T: P\cap(\overline{\Omega_2}\backslash\Omega_1)\longrightarrow P$ be completely continuous. Suppose one of the following two conditions holds*

(a) $\|Tx\|\le\|x\|$, $\forall x\in P\cap\partial\Omega_1$; $\|Tx\|\ge\|x\|$, $\forall x\in P\cap\partial\Omega_2$;

(b) $\|Tx\| \geq \|x\|, \forall x \in P \cap \partial\Omega_1$*;* $\|Tx\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2$*. Then* T *has at least one fixed point in* $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.

3. The Main Results

We list the following conditions for convenience.

(H_1) There exist $\varepsilon > 0$, $\varphi(t) \in L(0,1)$ such that $f(t,u,v) \geq \varphi(t)$ for $t \in (0,1), u \in (0,\varepsilon], v \in [-\varepsilon, 0)$ and $0 < \int_0^1 H(s,s)\varphi(s)\mathrm{d}s < +\infty$.

(H_2) There exists $R > 0$, such that $\int_0^1 H(s,s) f_{RR}(s)\mathrm{d}s < R^{p-1}$.

(H_3) There exist $N > 0, e > 0, N_1 > 0$ such that $f(t,u,v) \geq N|v|^{p-1}$ for $t \in [c,d], u \in [e,+\infty), -v \in [N_1,+\infty)$, where $N = (z(c)l^{p-1}\int_c^d H(s,s)\mathrm{d}s)^{-1}$, $l = \min_{t\in[c,d]} g(t)$.

(H_4) There exists $k > 0$, such that $f(t,u,v) \geq ku^{p-1}$ for $t \in (0,1), u \in (0,\varepsilon_1], v \in [-\varepsilon_1, 0)$, where $k = (\frac{1}{2}\int_0^1 H(s,s) f_1^{p-1}(s)\mathrm{d}s)^{-1}$.

(H_5) There exist $L > 0, N_2 > 0, e_1 > 0$ such that $f(t,u,v) \geq Lu^{p-1}$ for $t \in [c_1,d_1], u \in [N_2,+\infty), -v \in [e_1,+\infty)$, where $L = (z(c_1)l_1^{p-1}\int_{c_1}^{d_1} H(s,s)\mathrm{d}s)^{-1}$, $l_1 = \min_{t\in[c_1,d_1]} f_1(t)$.

Theorem 3.1 *If* (H_0)*–*(H_3) *are satisfied, then BVP(1.1) has at least two positive solutions.*

Proof $\forall\, u \in \partial P_r$, then $\|u\| = r$, $f_1(t)r \leq u(t) \leq f_2(t)r$ and $g(t)r \leq -u''(t) \leq r$. Suppose r is small enough, guaranteeing $u(t) \in (0,\varepsilon]$, $u'' \in [-\varepsilon, 0)$, $r \leq \min\left\{\phi_q(\frac{1}{2})\phi_q(\int_0^1 H(s,s)\varphi(s)\mathrm{d}s), \frac{R}{2}\right\}$. It follows from (H_1) that

$$\begin{aligned}|(Au)''(\frac{1}{2})| =&\phi_q(\int_0^1 H(\frac{1}{2},s)f(s,u(s),u''(s))\mathrm{d}s)\\ \geq&\phi_q(z(\frac{1}{2}))\phi_q(\int_0^1 H(s,s)\varphi(s)\mathrm{d}s))\\ \geq&\phi_q(\frac{1}{2})\phi_q(\int_0^1 H(s,s)\varphi(s)\mathrm{d}s)) \geq r = \|u\|.\end{aligned}$$

Obviously $\|Au\| \geq \|u\|$.

For any $u \in \partial P_R$, $\|u\| = R$. From (H_2), we get

$$\begin{aligned}|(Au)''(t)| =&\phi_q(\int_0^1 H(t,s)f(s,u(s),u''(s))\mathrm{d}s)\\ \leq&\phi_q(\int_0^1 H(s,s)f(s,u(s),u''(s))\mathrm{d}s)\\ \leq&\phi_q(\int_0^1 H(s,s))f_{RR}(s)\mathrm{d}s)\\ <&\phi_q(R^{p-1}) = R,\end{aligned}$$

which yields $\|Au\| < \|u\|$.

For any $u \in \partial P_{R_1}$, $\|u\| = R_1$. Let R_1 be large enough, such that $R_1 > R$, $\min_{t\in[c,d]} g(t)R_1 > N_1$, and $\min_{t\in[c,d]} f_1(t)R_1 \geq e$. Then for $t \in [c,d]$, we have $-u''(t) \geq g(t)\|u\| \geq \min_{t\in[c,d]} g(t)\|u\|$

and $u(t) \geq f_1(t)\|u\| \geq \min_{t\in[c,d]} f_1(t)\|u\|$. ($H_3$) gives

$$\begin{aligned}-\phi_p((Au)''(c)) \geq& z(c)\int_c^d H(s,s)f(s,u(s),u''(s))\mathrm{d}s\\ \geq& z(c)N\int_c^d H(s,s)|u''(s)|^{p-1}\mathrm{d}s\\ \geq& z(c)N\int_c^d H(s,s)g(s)^{p-1}\mathrm{d}s\|u\|^{p-1}\\ \geq& z(c)l^{p-1}N\int_c^d H(s,s)\mathrm{d}s\|u\|^{p-1},\end{aligned}$$

thus $\|Au\| \geq \|u\|$. In conclusion, the result of Theorem 3.1 follows from the above and Lemma 2.5.

Theorem 3.2 *If (H_0), (H_2), (H_4), (H_5) are satisfied, then BVP(1.1) has at least two positive solutions.*

Proof For any $u \in \partial P_r$, we have $\|u\| = r$, $f_1(t)r \leq u(t) \leq f_2(t)r$ and $g(t)r \leq -u''(t) \leq r$. Let r be small enough ($r < R$), such that $u(t) \in (0, \varepsilon_1]$, $u'' \in [-\varepsilon_1, 0)$. It follows from ($H_4$) that

$$\begin{aligned}-\phi_p((Au)''(\frac{1}{2})) =& \int_0^1 H(\frac{1}{2},s)f(s,u(s),u''(s))\mathrm{d}s\\ \geq& z(\frac{1}{2})k\int_0^1 H(s,s)u(s)^{p-1}\mathrm{d}s\\ \geq& \frac{1}{2}k\int_0^1 H(s,s)f_1^{p-1}(s)\mathrm{d}s\|u\|^{p-1} \geq r^{p-1}.\end{aligned}$$

Obviously, $\|Au\| \geq \|u\|$.

For any $u \in \partial P_R$, from (H_2), we have $\|Au\| < \|u\|$.

For any $u \in \partial P_{R_1}$, we have $\|u\| = R_1$. Let R_1 be large enough, such that $R_1 > R$, $\min_{t\in[c_1,d_1]} f_1(t)R_1 > N_2$, and $\min_{t\in[c_1,d_1]} g(t)R_1 \geq e_1$. Then for $t \in [c_1, d_1]$, we have $u(t) \geq f_1(t)\|u\| \geq \min_{t\in[c_1,d_1]} f_1(t)\|u\|$ and $-u''(t) \geq g(t)\|u\| \geq \min_{t\in[c_1,d_1]} g(t)\|u\|$. ($H_5$) yields

$$\begin{aligned}-\phi_p((Au)''(c_1)) \geq& z(c_1)\int_{c_1}^{d_1} H(s,s)f(s,u(s),u''(s))\mathrm{d}s\\ \geq& z(c_1)L\int_{c_1}^{d_1} H(s,s)|u(s)|^{p-1}\mathrm{d}s\\ \geq& z(c_1)L\int_{c_1}^{d_1} H(s,s)f_1(s)^{p-1}\mathrm{d}s\|u\|^{p-1}\\ \geq& z(c_1)l_1^{p-1}L\int_{c_1}^{d_1} H(s,s)\mathrm{d}s\|u\|^{p-1}.\end{aligned}$$

Thus $\|Au\| \geq \|u\|$. To sum up, by Lemma 2.5, our conclusion follows.

Corollary 3.1 *If (H_0), (H_1), (H_2), (H_5) are satisfied, then BVP(1.1) has at least two positive solutions.*

Corollary 3.2 *If* (H_0), (H_2), (H_3), (H_4) *are satisfied, then BVP(1.1) has at least two positive solutions.*

Remark If (H_0), (H_1), (H_2) or (H_0), (H_2), (H_4) or (H_0), (H_2), (H_3) or (H_0), (H_2), (H_5) are satisfied, then BVP(1.1) has at least one positive solution.

4. Examples

Now we provide two examples as applications of our theorems.

Example 1 Consider the following boundary value problem with p-Laplacian

$$\begin{cases} (|u''(t)|^2u''(t))'' = \frac{1}{35t\sqrt{(1-t)(1-\frac{14}{15}t)}}(\frac{1}{\sqrt{u(t)}} + \frac{1}{\sqrt{-u''(t)}} + (u''(t))^4) \\ u(0) = 0, \quad u(1) = \frac{1}{2}u(\frac{1}{4}) \\ u''(0) = 0, u''(1) = \frac{1}{2}u''(\frac{1}{2}). \end{cases}$$

Let $\varphi(t) = \frac{1}{\sqrt{1-t}}$ and $R = 3$. We can easily find that the conditions of Theorem (3.1) are satisfied, so it has at least two positive solutions.

Example 2 Consider the following boundary value problem with p-Laplacian

$$\begin{cases} (|u''(t)|u''(t))'' = \frac{1}{62t\sqrt{(1-t)(1-\frac{16}{17}t)}}(\frac{1}{\sqrt{u(t)}} + \frac{1}{\sqrt{-u''(t)}} + u^3(t)) \\ u(0) = 0, \quad u(1) = \frac{1}{4}u(\frac{1}{2}) \\ u''(0) = 0, u''(1) = \frac{1}{3}u''(\frac{1}{2}). \end{cases}$$

Let $R = 3$. We can easily find that the conditions of Theorem (3.2) are satisfied, so it has at least two positive solutions.

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