Journal of Mathematical Research & Exposition
July, 2009, Vol. 29, No. 4, pp. 682–686
DOI:10.3770/j.issn:1000-341X.2009.04.013
Http://jmre.dlut.edu.cn

$L(d_1, d_2, \ldots, d_t)$-Number $\lambda(C_n; d_1, d_2, \ldots, d_t)$ of Cycles

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Abstract An $L(d_1, d_2, \ldots, d_t)$-labeling of a graph G is a function f from its vertex set $V(G)$ to the set $\{0, 1, \ldots, k\}$ for some positive integer k such that $|f(x) - f(y)| \geq d_i$, if the distance between vertices x and y in G is equal to i for $i = 1, 2, \ldots, t$. The $L(d_1, d_2, \ldots, d_t)$-number $\lambda(G; d_1, d_2, \ldots, d_t)$ of G is the smallest integer number k such that G has an $L(d_1, d_2, \ldots, d_t)$-labeling with $\max\{f(x)|x \in V(G)\} = k$. In this paper, we obtain the exact values for $\lambda(C_n; 2, 2, 1)$ and $\lambda(C_n; 3, 2, 1)$, and present lower and upper bounds for $\lambda(C_n; 2, \ldots, 2, 1, \ldots, 1)$

Keywords cycle; labeling; $L(d_1, d_2, \ldots, d_t)$-labeling; $\lambda(G; d_1, d_2, \ldots, d_t)$-number.

Document code A
MR(2000) Subject Classification 05C15; 05C78
Chinese Library Classification O157.5

1. Introduction

The channel assignment problem is to assign a channel (nonnegative integer) to each radio transmitter so that interfering transmitters are assigned channels whose separations are not in a set of disallowable separations. Hale[1] formulated this problem into the problems of T-coloring of a graph, which has been extensively studied over the past decades[2−8]. Roberts[9] pointed that we could assign channels to some radio transmitters with different places so that close transmitters would get different channels whose difference is at least 2. Griggs and Yeh[10] first studied the problems of $L(2, 1)$-labeling. An $L(2, 1)$-labeling is a function f from its vertex set $V(G)$ to the set $\{0, 1, \ldots, k\}$ for some integer k such that $|f(x) - f(y)| \geq 2$ if $d(x, y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$. For positive integer numbers k, d_1, d_2, a $k - L(d_1, d_2)$-labeling of a graph G is a function $f : V(G) \to \{0, 1, \ldots, k\}$ such that $|f(x) - f(y)| \geq d_i$ whenever $x, y \in V(G)$ and $d(x, y) = i$ $(i = 1, 2)$. $L(d_1, d_2)$-number of the graph is the smallest integer number k such that k-$L(d_1, d_2)$-labeling exists.

Up to now, there are a lot of results for the $L(d_1, d_2)$-labeling, especially, the $L(2, 1)$-labeling. For example, Griggs and Yeh[10] proved that the $L(2, 1)$-number of a tree is $\triangle + 1$ or $\triangle + 2$, and that the upper bound for the $L(2, 1)$-number of a graph with the largest degree $\triangle$ is at most $\triangle^2 + 2\triangle - 3$. Further they proposed the following conjecture is $\triangle^2$. In addition, they

Received date: 2007-07-18; **Accepted date**: 2008-04-16
Foundation item: the National Natural Science Foundation of China (No. 10531070); the National Basic Research Program of China 973 Program (No. 2006AA11Z209) and the Natural Science Foundation of Shanghai City (No. 06ZR14049).

obtain the exact values for the $L(2,1)$-number for some special graphs such as paths, cycles and wheel graphs. Chang and Kuo[11] proved that for a general graph of maximum degree $\triangle$, an upper bound of $L(2,1)$-number is $\triangle^2 + \triangle$. For more background and information for the $L(d_1, d_2)$-numbers, the readers may refer to an excellent survey[12].

In this survey, Yeh[12] proposed a new notion of $L(d_1, d_2, \ldots, d_t)$-labeling of a graph. An $L(d_1, d_2, \ldots, d_t)$-labeling of a graph G is a function f from its vertex set $V(G)$ to the set $\{0, 1, \ldots, k\}$ for some positive integer k such that $|f(x) - f(y)| \geq d_i$, if the distance between vertices x and y in G is equal to i for $i = 1, 2, \ldots, t$. The $L(d_1, d_2, \ldots, d_t)$-number $\lambda(G; d_1, d_2, \ldots, d_t)$ of G is the smallest integer number k such that G has an $L(d_1, d_2, \ldots, d_t)$-labeling with $\max\{f(x) | x \in V(G)\} = k$. Further, he proposed five problems, one of which was $L(d_1, d_1, \ldots, d_1, d_2, d_2, \ldots, d_2)$-labeling ($d_1 > d_2 \geq 1$).

In this paper, we present the exact values for $\lambda(C_n; 2, 2, 1)$, $\lambda(C_n; 3, 2, 1)$ and give lower and upper bounds for $\lambda(G; 2, 2, \ldots, 2, 1, \ldots, 1)$ (t-fold 2 and t-fold 1).

2. Preliminaries

Denote by C_n a cycle with n vertices $v_1, v_2, \ldots, v_n$.

Proposition 1 *For a graph G, if $\lambda(G; d_1, d_2, \ldots, d_t)$ and $\lambda(G; d_1, d_2, \ldots, d_t, \delta_1, \delta_2, \ldots, \delta_s)$ exist, then $\lambda(G; d_1, d_2, \ldots, d_t) \leq \lambda(G; d_1, d_2, \ldots, d_t, \delta_1, \delta_2, \ldots, \delta_s)$.*

Proof Clearly, it follows from the definition that an $L(G; d_1, d_2, \ldots, d_t, \delta_1, \delta_2, \ldots, \delta_s)$-labeling of G is also an $L(d_1, d_2, \ldots, d_t)$-labeling. Hence the assertion holds.

Proposition 2 *For a graph G, if $\lambda(G; d_1, d_2, \ldots, d_t)$ and $\lambda(G; \delta_1, \delta_2, \ldots, \delta_t)$ exist, and $d_i \leq \delta_i$ $(1 \leq i \leq t)$, then $\lambda(G; d_1, d_2, \ldots, d_t) \leq \lambda(G; \delta_1, \delta_2, \ldots, \delta_t)$.*

Proof Since G has an $L(\delta_1, \delta_2, \ldots, \delta_t)$-labeling, $|f(x) - f(y)| \geq \delta_i$ for $d(x, y) = i (1 \leq i \leq t)$, where $x, y \in V(G)$. By $d_i \leq \delta_i$ $(1 \leq i \leq t)$, we have $|f(x) - f(y)| \geq d_i$. Hence G has an $L(d_1, d_2, \ldots, d_t)$-labeling and $\lambda(G; d_1, d_2, \ldots, d_t) \leq \lambda(G; \delta_1, \delta_2, \ldots, \delta_t)$.

Proposition 3 *Let G be a graph. If the $L(\delta_1, \delta_2, \ldots, \delta_t)$-number exists, then there exists a vertex with labeling 0.*

Proof Suppose the $L(\delta_1, \delta_2, \ldots, \delta_t)$-number exists. Let the vertex v with the smallest labeling value and $f(v) \neq 0$. Now let $g(u) = f(u) - f(v)$ for all u in G. Then it is easy to see that g is a function such that $L(\delta_1, \delta_2, \ldots, \delta_t)$-number exists with $g(v) = 0$.

Proposition 4 *Let G be a graph with the diameter at least $t+1$. If the $L(d_1, d_1, \ldots, d_1, d_2, d_2, \ldots, d_2)$-labelling exists ($t$-fold d_1, and $d_1 > d_2 \geq 1$), then $\lambda(G; d_1, d_1, \ldots, d_1, d_2, d_2, \ldots, d_2,) \geq td_1 + 1$.*

Proof Since the diameter of G is at least $t+1$, there exists a path with vertices $(v_1, v_2, \ldots, v_{t+1}, v_{t+2}, \ldots)$, and $f(v_1) = 0$. Because G has an $L(d_1, d_1, \ldots, d_1, d_2, d_2, \ldots, d_2)$-labeling, the labeling values of v_i $(2 \leq i \leq t+1)$ are different and $|f(v_i) - f(v_j)| \geq d_1$ $(1 \leq i \neq j \leq t+1)$. Hence, among the vertices v_i $(2 \le i \le t+1)$, there is at least one with a labeling value $\ge td_1$; if the maximum of the labeling values of the vertices is td_1, then the labeling values of $v_2, \ldots, v_{t+1}$ are $d_1, 2d_1, \ldots, td_1$. But the distance between v_{t+2} and v_i $(2 \le i \le t+1)$ is not more than t and $f(v_{t+2}) \ne 0$. It is impossible. So

$$\lambda(G; d_1, d_1, \ldots, d_1, d_2, d_2, \ldots, d_2) \ge td_1 + 1.$$

3. Results

Theorem 1 *For $C_n (n \ge 3)$, there are* $\lambda(C_n; 2, 2, 1) = \begin{cases} 4, & n = 3; \\ 8, & n = 5, 9, 13, 17; \\ 7, & n = 6, 10; \\ 6, & \textit{other } n. \end{cases}$

Proof We first show that $\lambda(C_n; 2, 2, 1) \ge 6$ $(n \ge 4)$. By Proposition 4, $\lambda(C_n; 2, 2, 1) \ge 5$ $(n \ge 4)$. If $\lambda(C_n; 2, 2, 1) = 5$ $(n \ge 4)$, by Proposition 3, we can set $f(v_1) = 0$, thus $f(v_2) \ge 2$; if $2 \le f(v_2) \le 4$, with $f(v_3) \ge 2$ and $|f(v_2) - f(v_3)| \ge 2$, then $f(v_3) \in \{2, 4, 5\}$. Hence if $n = 4$, then there are no labeling values for v_4; if $n = 5$, then there are no labeling values for v_4; if $n \ge 6$ with $|f(v_2) - f(v_4)| \ge 2, |f(v_3) - f(v_4)| \ge 2$, then there is only one labeling: $f(v_2) = 3, f(v_3) = 5, f(v_4) = 1$, but, there is no labeling value for v_5. If $f(v_2) = 5$, then $f(v_3) \in \{2, 3\}$, but $f(v_4) \ge 1$. So we have only one labeling, that is, $f(v_2) = 5, f(v_3) = 3, f(v_4) = 1$. In this case, there is no labeling value for v_5 either. Therefore $\lambda(C_n; 2, 2, 1) \ge 6$ $(n \ge 4)$.

Now we can obtain the results by constructing labeling. If $n = 3$, set $f : v_1 v_2 v_3 \to 024$; if $n = 5$, set $f : v_1 v_2 \cdots v_5 \to 02468$; if $n = 9$, set $f : v_1 v_2 \cdots v_9 \to 024681357$; if $n = 13, 17$, the first 9 vertices are valued as $n = 9$, the left vertices are valued as 0246, 02460246; if $n = 6$, set $f : v_1 v_2 \cdots v_6 \to 037146$; if $n = 10$, the first 6 vertices are valued as $n = 6$, the left vertices are valued as 0246.

Finally, we show that $\lambda(C_n; 2, 2, 1) = 6$ for the remaining case. We now construct the following labeling.

If $n \equiv 0 \pmod 4$, set $f : v_1 \cdots v_4 \to 0246$, $f(v_i) = f(v_{i+4})$.

If $n \equiv 1 \pmod 4$ and $n > 17$, set $f : v_1 \cdots v_7 \to 0246135$, $f(v_i) = f(v_{i+7})$, where $i = 1, 2, \ldots, 14$; for the left vertices, the labeling is as $n \equiv 0 \pmod 4$.

If $n \equiv 2 \pmod 4$ and $n > 10$, set $f : v_1 \cdots v_7 \to 0246135$, $f(v_i) = f(v_{i+7})$, where $i = 1, 2, \ldots, 7$; for the left vertices, the labeling is as $n \equiv 0 \pmod 4$.

If $n \equiv 3 \pmod 4$, set $f : v_1 \cdots v_7 \to 0246135$, for the left vertices, the labeling is as $n \equiv 0 \pmod 4$.

By simple calculations, it is easy to see that labeling of the above is $L(2, 2, 1)$-labeling of cycle C_n. □

Theorem 2 *For $C_n (n \ge 3)$, there are (1)* $\lambda(C_n; 3, 2, 1) = \begin{cases} 6, & n = 3; \\ 9, & n = 7. \end{cases}$

(2) $\lambda(C_n; 3, 2, 1) = \begin{cases} 8, & n > 3\ (n \ne 7), \textit{ and is odd}; \\ 7, & n \ge 4, \textit{ and is even}. \end{cases}$

Proof By Proposition 1 and some calculations, it is easy to see that $\lambda(C_n; 3, 2, 1) = 6, 7, 8, 7, 9$, corresponding to $n = 3, 4, \ldots, 7$, respectively.

Now we assume that $n > 7$. If $\lambda(C_n; 3, 2, 1) = 6$, $f(v_1) = 0$; if $f(v_2) = 3$, then $f(v_3) = 6$, $f(v_4) = 1$, $f(v_5) = 4$, but there is also no labeling value for v_6; if $f(v_2) = 4$, there is no labeling value for v_3; if $f(v_2) = 5$ or 6, then $f(v_3) = 2$ or 3, there is no labeling value for v_4. Hence $\lambda(C_n; 3, 2, 1) \geq 7$.

If $n \geq 8$, and is even, set f :

$v_1 \cdots v_4 \to 0725$, $f(v_i) = f(v_{i+4})$ $(i \geq 1)$, if $n \equiv 0 \pmod 4$;

$v_1 \cdots v_6 \to 036147$, $f(v_i) = f(v_{i+6})$ $(i \geq 1)$, if $n \equiv 0 \pmod 6$;

$v_1 \cdots v_6 \to 036147$, $f(v_i) = f(v_{i+6})$ $(1 \leq i \leq n-8)$, $v_{n-1}v_n \to 25$, if $n \equiv 2 \pmod 6$;

$v_1 \cdots v_6 \to 036147$, $f(v_i) = f(v_{i+6})$ $(1 \leq i \leq n-10)$, $v_{n-3} \cdots v_n \to 0527$, if $n \equiv 4 \pmod 6$, and $n \neq 4k$. Hence $\lambda(C_n; 3, 2, 1) = 7$.

If $n \geq 9$, and is odd, we show that $\lambda(C_n; 3, 2, 1) = 8$. In fact, if $\lambda(C_n; 3, 2, 1) = 7$, for the labelings that can be recirculated on C_n are: 0725; 036147; 03614725, the number in each set is even, and each labeling can be removed. So if we label the vertices of C_n by use of these sets, we cannot label the remaining odd vertices of C_n by use of the numbers in $\{0, 1, \ldots, 7\}$. Thus, we obtain $\lambda(C_n; 3, 2, 1) > 7$ when $n \geq 9$, and is odd.

If $n \equiv 3 \pmod 4$, set f: $v_1 \cdots v_7 \to 0741836$, $v_8 \cdots v_{11} \to 0825$, $f(v_i) = f(v_{i+4})$ $(i \geq 8)$; If $n \equiv 1 \pmod 4$, set $f : v_1 \cdots v_5 \to 04826$, $v_8 \cdots v_{11} \to 0826$, $f(v_i) = f(v_{i+4})$ $(i \geq 6)$. Therefore $\lambda(C_n; 3, 2, 1) = 8$. □

Theorem 3 *For $C_n (n \geq 3)$, there are $\lambda(C_n; 2, \ldots, 2, 1, \ldots, 1) \leq 4t$ (t-fold 2 and 1).*

Proof From the proofs of Theorems 1 and 2, we see that the key step in labeling a cycle is how to construct the labeling of C_n $(3 \leq n \leq 4t)$. We will do this.

If $3 \leq n \leq 2t + 1$, set $f(V) \to 024 \cdots (2n - 2)$.

In case of $2t + 2 \leq n \leq 4t$:

(1) If n is odd, set

$$f(V) \to 0(4t)(4t-2) \cdots (4t - 2[\frac{n}{2}] + 2)1(4t-1)(4t-3) \cdots (4t - 2[\frac{n}{2}] + 3);$$

(2) If n is even, set

$$f(V) \to 0(4t)(4t-2) \cdots (4t - 2[\frac{n}{2}] + 4)1(4t-1)(4t-3) \cdots (4t - 2[\frac{n}{2}] + 3).$$

If $n = 4t + 1$, set

$$f(V) \to 0(4t)(4t-2) \cdots (2t+2)(2t-1)(2t-3) \cdots 31(4t-1)(4t-3) \cdots 9(2t+1)24 \cdots (2t).$$

If $n > 4t + 1$, and $n \neq k(2t + 1)$, $k = 2, 3, \ldots$, we separate the vertices of C_n into two parts. The number of the vertices in one part is a multiple of $2t + 1$, which are circularly labeled by $0(4t)(4t-2) \cdots 2$. The number of the vertices in other part is between $2t+2$ and $4t+1$, which are labeled in the same way as the above case of $2t+2 \leq n \leq 4t+1$. So $L(2, \ldots, 2, 1, \ldots, 1)$-labeling exists. □

Theorem 4 *For $C_n(n \geq 3)$, there are* $\lambda(C_n; 2, 1, \ldots, 1) = \begin{cases} 4, & n = 3, 4; \\ 2t+2, & \text{other } n(t-\text{fold } 1). \end{cases}$

Proof If $n \leq 2t+3$, the diameter of the cycle is $[\frac{n}{2}]$, so the labeling values of vertices are different from each other and $\lambda(C_n; 2, 1, \ldots, 1)$ must be more than $n-1$. Next we will show that except for $n = 3, 4$, $\lambda(C_n; 2, 1, \ldots, 1) = n - 1 \leq 2t + 2$.

(1) $n = 3$, set $f(V) \to 024$; $n = 4$, set $f(V) \to 0314$.

(2) If $4 \leq n \leq 2t+3$, we will do the labeling by the following rule: if n is even, set $f(V) \to 024 \cdots (n-2)13 \cdots (n-1)$; if n is odd, set $f(V) \to 024 \cdots (n-1)13 \cdots (n-2)$. So, for C_n, it is obvious that $L(2, 1, \ldots, 1)$-labeling exists.

If $n = k(2t+4)$ $k = 1, 2, \ldots$, set

$$f(V) \to 024 \cdots (2t+2)024 \cdots (2t+2)024 \cdots (2t+2).$$

If $n > 2t+3$ $(n \neq k(2t+4), k = 1, 2, \ldots)$, we separate the vertices of C_n into two parts, the number of the vertices in one part is a multiple of $t+2$, and the number of the vertices in the other part is between $t+3$, and $2t+3$. For the first part, if $t+1$ is even, the vertices are circularly labeled by $024 \cdots (t+1)13 \cdots t$; if $t+1$ is odd, the vertices are circularly labeled by $024 \cdots t13 \cdots (t+1)$. For the other part, the vertices are labeled in the same way as the above. It can be proven that for any n and any cycle, by the above labeling $L(2, 1, \ldots, 1)$-labeling exists. □

Corollary *For $C_n(n \geq 3)$, there are $2t+2 \leq \lambda(C_n; 2, \ldots, 2, 1) \leq 4t$ (t-fold 2).*

Proof By Propositions 1 and 2, the assertion holds.

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