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Solvability of Multi-Point Boundary Value Problem

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Abstract This paper deals with the existence of solutions for the problem

$$\begin{cases} (\phi_p(u'))' = f(t,u,u'), \quad t \in (0,1), \\ u'(0) = 0, \quad u(1) = \sum_{i=1}^{n-2} a_i u(\eta_i), \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s,\ p > 1.\ 0 < \eta_1 < \eta_2 < \cdots < \eta_{n-2} < 1, a_i\ (i = 1,2,\ldots,n-2)$ are non-negative constants and $\sum_{i=1}^{n-2} a_i = 1$. Some known results are improved under some sign and growth conditions. The proof is based on the Brouwer degree theory.

Keywords p-Laplace; multi-point boundary value problem; resonance; degree theory.

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1. Introduction

We consider the existence of solutions for multi-point boundary value problem (BVP)

$$(\phi_p(u'))' = f(t,u,u'), \quad t \in (0,1), \tag{1.1}$$

$$u'(0) = 0, \quad u(1) = \sum_{i=1}^{n-2} a_i u(\eta_i), \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$. $0 < \eta_1 < \eta_2 < \cdots < \eta_{n-2} < 1$, a_i $(i = 1,2,\ldots,n-2)$ are non-negative constants and $\sum_{i=1}^{n-2} a_i = 1$. Eq.(1.1) is widely applied in mechanics and physics[1−3]. When $p = 2$, Eq.(1.1) reduces to $u'' = f(t,u,u')$.

In recent years, p-Laplace equation associated with various boundary value conditions has been studied[4−10]. For example, Carcía-huidobro and Gupta[7] discussed (1.1) with boundary conditions

$$u'(0) = 0, \quad u(1) = u(\eta), \quad \eta \in (0,1)$$

under the following assumptions

(A_1) There are nonnegative functions $d_1(t), d_2(t)$, and $r(t) \in L^1[0,1]$ such that

$$|f(t,u,v)| \le d_1(t)|u|^{p-1} + d_2(t)|v|^{p-1} + r(t), \quad \text{for a.e. } t \in [0,1], u,v \in \mathbb{R};$$

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(A_2) There exists $u_0 > 0$, such that for all $|u| > u_0,\ t \in [0,1]$ and $v \in \mathbb{R}$

$$|f(t,u,v)| \geq \Lambda |u|^{p-1} - A|v|^{p-1} - B,$$

where $\Lambda > 0$, and $A, B \geq 0$ are constants;

(A_3) There is $R > 0$ such that for all $|u| > R$

$$uf(t,u,0) > 0,\ \text{a.e.}\, t \in [0,1], \quad uf(t,u,0) < 0,\ \text{a.e.}\, t \in [0,1]$$

as well as the other conditions.

In this paper, we discuss the solvability of (1.1)–(1.2) and obtain the following result.

Theorem 3.1 *Suppose that $f : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous and has the decomposition*

$$f(t,u,v) = g(t,u,v) + h(t,u,v)$$

which satisfies the following assumptions:

(H_1) There exist $r_1 < 0, r_2 > 0$, such that

$$f(t,r_1,0) \leq 0, \quad f(t,r_2,0) \geq 0, \quad \textit{for all } t \in [0,1];$$

(H_2) $vg(t,u,v) \leq 0$ for all $(t,u) \in [0,1] \times [r_1,r_2], |v| > 1$;

(H_3) $|h(t,u,v)| \leq a(t)|v|^m + b(t)$ for all $(t,u,v) \in [0,1] \times [r_1,r_2] \times \mathbb{R}$, where $a(t), b(t) \in L^1([0,1],\mathbb{R}^+)$.

Then there exists at least one solution for BVP (1.1)–(1.2), provided that

$$p - 1 < m < \left(1 + \frac{1}{\|a\|_1 + \|b\|_1 + r}\right)(p-1), \tag{1.3}$$

where $r = \max\{-r_1, r_2\}$.

Remark 1.1 When $p = 2, n = 3$, BVP (1.1)–(1.2) becomes

$$u'' = f(t,u,u'), \quad t \in (0,1), \tag{1.4}$$

$$u'(0) = 0, \quad u(1) = u(\eta), \quad \eta \in (0,1). \tag{1.5}$$

Feng and Webb in [10] proved that BVP (1.4)–(1.5) has at least a solution under the following assumptions

(B_1) There exists a constant $M \geq 0$ such that

$$uf(t,u,0) > 0, \quad \text{for all } |u| > M, t \in [0,1];$$

(B_2) $vg(t,u,v) \leq 0$ for all $(t,u,v) \in [0,1] \times [-M,M] \times \mathbb{R}$;

(B_3) $|h(t,u,v)| \leq a(t)|u| + b(t)|v| + c(t)|u|^r + d(t)|v|^k + e(t)$ for all $(t,u,v) \in [0,1] \times [-M,M] \times \mathbb{R}$, where $0 \leq r, k < 1, a, b, c, d, e \in L^1[0,1]$ and $\|b\|_1 < \frac{1}{2}$.

It is easy to see that the conditions (A_1)–(A_3) in [7] and (B_1)–(B_3) in [10] are stronger than the ones of Theorem 3.1. To some extent, we improve the results of [7] and [10].

2. Auxiliary results

From now on, we use the classical spaces $C[0,1], C^1[0,1]$ and $L^1[0,1]$. Define the norm in $C[0,1]$ by $\|\cdot\|_\infty$ and in $L^1[0,1]$ by $\|\cdot\|_1$. Moreover, we shall need the following lemmas.

Lemma 2.1[11] *Let $a < b$, $u(t) \in C([a,b],[0,+\infty))$ and $v(t) \in L^1([a,b],[0,+\infty))$. Suppose that there exists a constant $c \geq 0$ and a function $\omega(t)$ such that*

(1) $\int_a^b v(t)\omega(u(t))\mathrm{d}t < +\infty$;

(2) $u(t) \leq c + \int_a^t v(s)\omega(u(s))\mathrm{d}s$, *for all* $t \in [a,b]$.

Then

$$\int_c^{u(t)} \frac{\mathrm{d}s}{\omega(s)} \leq \int_a^t v(s)\mathrm{d}s, \quad \textit{for all } t \in [a,b],$$

where $\omega \in C([0,+\infty),[0,+\infty))$ *is increasing.*

In Lemma 2.1, if the assumption (2) is replaced by

$$u(t) \leq c + \int_t^b v(s)\omega(u(s))\mathrm{d}s, \quad \text{for all } t \in [a,b],$$

then

$$\int_c^{u(t)} \frac{\mathrm{d}s}{\omega(s)} \leq \int_t^b v(s)\mathrm{d}s, \quad \text{for all } t \in [a,b].$$

Consider the auxiliary boundary value problem

$$(\phi_p(\frac{u'}{\lambda}))' = f^*(t,u,u',\lambda), \quad \lambda \in (0,1], \tag{2.1}$$

$$u'(0) = 0, \quad u(1) = \sum_{i=1}^{n-2} a_i u(\eta_i), \tag{2.2}$$

where $0 < \eta_1 < \eta_2 < \cdots < \eta_{n-2} < 1$, a_i $(i = 1,2,\ldots,n-2)$ are non-negative constants and $\sum_{i=1}^{n-2} a_i = 1$, $f^* : [0,1] \times \mathbb{R}^2 \times [0,1] \longrightarrow \mathbb{R}$ is continuous and

$$f^*(t,r,s,1) = f(t,r,s), \quad \text{for all } (t,r,s) \in [0,1] \times \mathbb{R}^2. \tag{2.3}$$

Lemma 2.2 *Suppose (2.3) holds. Furthermore, let $\Omega \subset C^1[0,1]$ be an open bounded set. Assume that*

(C_1) There exists no solution u of BVP (2.1)–(2.2), $0 < \lambda < 1$, such that $u \in \partial\Omega$;

(C_2) The equation

$$F(s) := \sum_{i=1}^{n-2} a_i \int_{\eta_i}^1 \phi_p^{-1}(\int_0^\tau f^*(t,s,0,0)\mathrm{d}t)\mathrm{d}\tau = 0$$

has no solution on $\partial\Omega \cap \mathbb{R}$;

(C_3) The Brouwer degree $\deg_B(F, \Omega \cap \mathbb{R}, 0) \neq 0$.

Then BVP (1.1)–(1.2) has at least one solution in $\bar{\Omega}$.

The proof of Lemma 2.2 is similar to that of Lemma 2.1 in [7], so we omit it.

3. Existence results

Theorem 3.1 *Suppose that the assumptions (H_1)–(H_3) are satisfied. Then there exists at least one solution for BVP (1.1)–(1.2), provided that*

$$p - 1 < m < (1 + \frac{1}{\|a\|_1 + \|b\|_1 + r})(p-1),$$

where $r = \max\{-r_1, r_2\}$.

Proof For all $(t,u,v)\in[0,1]\times\mathbb{R}^2$, define the function $\bar f$ by

$$\bar f(t,u,v)=\begin{cases} f(t,r_2,v), & \text{if } u>r_2,\\ f(t,u,v), & \text{if } r_1\le u\le r_2,\\ f(t,r_1,v), & \text{if } u<r_1.\end{cases}$$

Then, the modified problem corresponding to BVP (1.1)–(1.2) is

$$(\phi_p(u'))'=\bar f(t,u,u'),\quad t\in(0,1), \tag{3.2}$$

$$u'(0)=0,\quad u(1)=\sum_{i=1}^{n-2}a_iu(\eta_i). \tag{3.3}$$

Consider the homotopy problem (2.1)–(2.2), where

$$f^*(t,u,u',\lambda)=\lambda u+(1-\lambda)\bar f(t,u,u').$$

Step 1. Let $u(t)$ be a solution for BVP (2.1)–(2.2). Then we have

$$r_1<u(t)<r_2 \text{ for all } t\in[0,1],\lambda\in(0,1].$$

Otherwise, there exists a point $t_0\in[0,1)$ such that

$$u(t_0)=\min_{t\in[0,1]}u(t)\le r_1 \quad\text{or}\quad u(t_0)=\max_{t\in[0,1]}u(t)\ge r_2.$$

Without loss of generality, we suppose $u(t_0)=\max_{t\in[0,1]}u(t)\ge r_2$ holds, so there are three cases as follows:

Case 1 Let $t_0\in(0,1)$. We have $u'(t_0)=0$ and

$$\begin{aligned} u(t_0)(\phi_p(\frac{u'(t)}{\lambda}))'\mid_{t=t_0}&=u(t_0)f^*(t_0,u(t_0),0,\lambda)\\ &=\lambda(u(t_0))^2+(1-\lambda)u(t_0)f(t_0,r_2,0)>0.\end{aligned}$$

Then, there exists a positive constant $\delta>0$ such that $(\phi_p(\frac{u'(t)}{\lambda}))'>0$, for all $t\in(t_0,t_0+\delta)$. This implies that $\phi_p(\frac{u'(t)}{\lambda}))$ is increasing in $(t_0,t_0+\delta)$. Thus

$$\phi_p(u'(t))>\phi_p(u'(t_0))=\phi_p(0)=0, \text{ for all } t\in(t_0,t_0+\delta).$$

By the monotonicity of ϕ_p, we have $u'(t)>0$, for all $t\in(t_0,t_0+\delta)$. That is, $u(t)$ is increasing in $(t_0,t_0+\delta)$. This is a contradiction.

Case 2 Let $t_0=0$. Then we have

$$u(0)(\phi_p(\frac{u'(t)}{\lambda}))'\mid_{t=0}=\lambda(u(0))^2+(1-\lambda)u(0)f(0,r_2,0)>0.$$

Similar to above process, we can obtain a contradiction.

Case 3 Let $t_0=1$. Combining with the boundary condition (3.3), we know that there exists $\eta\in(0,1)$ such that $u(1)=u(\eta)$. Similar to Case 1, we can obtain a contradiction.

Step 2. We prove that there exists a positive constant M_0 such that $\|u'\|_\infty\le M_0$.

Let $\|u'\|_\infty\le 1$. Then $u'(t)$ has a prior bounds. Otherwise, let $\|u'\|_\infty>1$, that is, there exists a point $t_0\in(0,1]$ such that $|u'(t_0)|=\|u'\|_\infty>1$.

This together with the continuity of $u'(t)$ and $u'(0)=0$ implies that there exists an interval $[\mu,\nu]\subset[0,1],\ t_0\in[\mu,\nu]$ such that $|u'(\mu)|=1$ and $|u'(t)|\geq 1,$ for all $t\in[\mu,\nu]$. Without loss of generality, we assume that $u'(t)\geq 1$ holds, for all $t\in[\mu,\nu]$.

Multiplying (2.1) by $\phi_p(\frac{u'}{\lambda})$ and integrating on both sides of it from μ to t, we obtain

$$\int_\mu^t \phi_p(\frac{u'}{\lambda})(\phi_p(\frac{u'}{\lambda}))'\mathrm{d}s=\int_\mu^t \phi_p(\frac{u'}{\lambda})f^*(s,u,u',\lambda)\mathrm{d}s,$$

that is,

$$\frac{1}{2}\phi_p^2(\frac{u'(t)}{\lambda})-\frac{1}{2}\phi_p^2(\frac{u'(\mu)}{\lambda})=\int_\mu^t \phi_p(\frac{u'}{\lambda})[\lambda u+(1-\lambda)\bar{f}(s,u,u')]\mathrm{d}s.$$

Since $r_1<u(t)<r_2$, for all $t\in[0,1]$, we have

$$\begin{aligned}|\frac{u'(t)}{\lambda}|^{2p-2}=&|\frac{1}{\lambda}|^{2p-2}+2\int_\mu^t \phi_p(\frac{u'}{\lambda})[\lambda u+(1-\lambda)f(s,u,u')]\mathrm{d}s\\ =&|\frac{1}{\lambda}|^{2p-2}+2\int_\mu^t \phi_p(\frac{u'(s)}{\lambda})[\lambda u+(1-\lambda)g(s,u,u')]\mathrm{d}s+\\ &2\int_\mu^t \phi_p(\frac{u'(s)}{\lambda})[(1-\lambda)h(s,u,u')]\mathrm{d}s.\end{aligned}$$

By the assumption (H_2) and $r_1<u(t)<r_2$, for $t\in[0,1]$, we get

$$\int_\mu^t \phi_p(\frac{u'}{\lambda})g(s,u,u')\mathrm{d}s=\int_\mu^t |\frac{u'}{\lambda}|^{p-2}\frac{u'}{\lambda}g(s,u,u')\mathrm{d}s\leq 0,$$

so

$$\begin{aligned}|u'(t)|^{2p-2}&\leq 1+2\int_\mu^t \lambda^{2p-2}\phi_p(|\frac{u'}{\lambda}|)|\lambda u(s)+(1-\lambda)h(s,u,u')|\mathrm{d}s\\ &\leq 1+2\int_\mu^t |u'|^{p-1}(r+a(s)|u'|^m+b(s))\mathrm{d}s,\end{aligned}$$

where $r=\max\{|r_1|,|r_2|\}$. By $|u'(t)|\geq 1$, for $t\in[\mu,\nu]$

$$|u'(t)|^{2p-2}\leq 1+2\int_\mu^t |u'|^{m+p-1}(r+a(s)+b(s))\mathrm{d}s,\quad \text{for all } t\in[\mu,\nu].$$

For convenience, we write

$$z(t)=|u'(t)|^{2p-2},\ \omega(t)=t^{\frac{m+p-1}{2p-2}},\ v(t)=2(a(t)+b(t)+r).$$

Then

$$\int_\mu^\nu v(s)\omega(z(s))\mathrm{d}s<+\infty \quad\text{and}\quad z(t)\leq 1+\int_\mu^t v(s)\omega(z(s))\mathrm{d}s,\quad \text{for all } t\in[\mu,\nu].$$

By Lemma 2.1, we can conclude that

$$\int_1^{z(t)}\frac{ds}{\omega(s)}\leq\int_\mu^t v(s)\mathrm{d}s,\quad \text{for all } t\in[\mu,\nu].$$

So

$$\int_1^{z(t)} s^{-\frac{m+p-1}{2p-2}}\mathrm{d}s\leq\int_\mu^t v(s)\mathrm{d}s\leq 2\int_0^1 (r+a(s)+b(s))\mathrm{d}s$$

$$\leq 2(\|a\|_1 + \|b\|_1 + r) := M_1, \quad \text{for all } t \in [\mu, \nu].$$

By the assumption (H_3) and (1.3), we have

$$\int_1^{+\infty} s^{-\frac{m+p-1}{2p-2}} \mathrm{d}s > 2(\|a\|_1 + \|b\|_1 + r).$$

This inequality implies that there exists a constant M_2 (independent of λ) such that $z(t) \leq M_2$, that is, $|u'(t)| \leq (M_2)^{\frac{1}{2p-2}} := M_3$, for all $t \in [\mu, \nu]$. Put $M_0 = \max\{1, M_3\}$ (independent of λ), then $|u'|_\infty \leq M_0$.

Step 3. By using Lemma 2.2, we prove that BVP (3.2)–(3.3) has at least one solution. Put

$$\Omega = \{u(t) \in C^1[0,1] : r_1 < u(t) < r_2, t \in [0,1]; \|u'\|_\infty < M_0 + 1\}.$$

It is clear that the assumption (C_1) of Lemma 2.2 is satisfied.

By the assumption (H_1) and (3.1), we deduce

$$f^*(t, r_1, 0, 0) = \bar{f}(t, r_1, 0) \leq 0, \quad f^*(t, r_2, 0, 0) = \bar{f}(t, r_2, 0) \geq 0.$$

Combining with the monotonicity of ϕ_p yields

$$F(r_1) = \sum_{i=1}^{n-2} a_i \int_{\eta_i}^1 \phi_p^{-1}(\int_0^\tau f^*(t, r_1, 0, 0)\mathrm{d}t)\mathrm{d}\tau) \leq 0,$$

$$F(r_2) = \sum_{i=1}^{n-2} a_i \int_{\eta_i}^1 \phi_p^{-1}(\int_0^\tau f^*(t, r_2, 0, 0)\mathrm{d}t)\mathrm{d}\tau) \geq 0.$$

If $F(r_1) \cdot F(r_2) = 0$, we can conclude that BVP (1.1)–(1.2) has at least one solution r_1 or r_2. Otherwise, $F(r_1)F(r_2) < 0$, which implies the assumption (C_2) of Lemma 2.2 holds. By the property of Brouwer degree, we have $\deg_B(F, \Omega \cap \mathbb{R}, 0) = 1$. So the assumption ($C_3$) of Lemma 2.2 is satisfied. By Lemma 2.2, we prove that BVP (3.2)–(3.3) has at least one solution $u(t)$ satisfying $r_1 < u(t) < r_2$, for all $t \in [0,1]$. This implies that BVP (1.1)–(1.2) has at least one solution. The proof is completed. □

Theorem 3.2 *Let assumptions (H_1)–(H_3) be satisfied. Furthermore, suppose the following inequality*

(H_4) $f(t, u_1, v_1) > f(t, u_2, v_2)$, *for all* $u_1, u_2, v_1, v_2 \in \mathbb{R}, u_1 > u_2, v_1 \leq v_2$

holds. Then there exists a unique solution for BVP (1.1)–(1.2).

Proof We have proved that BVP (1.1)–(1.2) has at least one solution in Theorem 3.1. Next, we will obtain the uniqueness of the solution for BVP (1.1)–(1.2) by (H_4).

Assume to the contrary that there exist two different solutions $x(t), y(t)$ of BVP (1.1)–(1.2). Let $z(t) = x(t) - y(t)$. By the condition (1.2), there exists some $t_0 \in [0,1)$ such that $z(t_0) = \max_{t\in[0,1]} z(t) > 0$.

Case 1 If $t_0 \in (0,1)$, then $z'(t_0) = 0, z(t_0) > 0$. By the continuity of $z'(t), z(t)$, there exists an interval $[t_0, t_1]$ such that $z(s) > 0, z'(s) \leq 0$, for all $s \in [t_0, t_1]$.

Since

$$(\phi_p(x') - \phi_p(y'))' = f(t, x, x') - f(t, y, y'),$$

by (H_4), we have

$$\int_{t_0}^{s}(\phi_p(x'(t))-\phi_p(y'(t)))'\mathrm{d}t=\int_{t_0}^{s}(f(t,x(t),x'(t))-f(t,y(t),y'(t)))\mathrm{d}t>0, \quad \text{for all} \quad s\in[t_0,t_1].$$

Combining with the monotonicity of ϕ_p, we have $z'(s)=x'(s)-y'(s)>0$, for all $s\in[t_0,t_1]$. This is a contradiction.

Case 2 If $t_0=0$, then $z'(0)=0, z(0)>0$. By the continuity of $z'(t), z(t)$, there exists an interval $[0,t_2]$ such that $z(s)>0, z'(s)\le 0$, for all $s\in[0,t_2]$. Similar to above process, we obtain $z'(s)=x'(s)-y'(s)>0$, for all $s\in[0,t_2]$. This is a contradiction.

Combining with the two cases, we deduce that BVP (1.1)–(1.2) has a unique solution.

Especially, let $\eta_i\to 1$, $i=1,2,\ldots,n-2$. We can obtain the following result. □

Corollary 3.1 *Suppose the assumptions (H_1)–(H_4) in Theorem 3.2 hold. Then the following Neumann BVP*

$$(\phi_p(u'))'=f(t,u,u'), \quad t\in(0,1),$$
$$u'(0)=u'(1)=0$$

has at least one solution.

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