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Energy Decaying and Blow-Up of Solution for a Kirchhoff Equation with Strong Damping

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Abstract The initial boundary value problem for a Kirchhoff equation with Lipschitz type continuous coefficient is studied on bounded domain. Under some conditions, the energy decaying and blow-up of solution are discussed. By refining method, the exponent decay estimates of the energy function and the estimates of the life span of blow-up solutions are given.

Keywords strong damping; Kirchhoff equation; blow-up; energy decaying; life span.

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1. Introduction

We are concerned with the blow up of solutions of the initial boundary value problem for the following Kirchhoff equation with Lipschitz type continuous coefficient and strong damping:

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u - \omega\Delta u_t = f(u), \tag{1.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \tag{1.3}$$

where $\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ and $\Omega \in R^N$, $N \geq 1$ is a bounded domain with a smooth boundary $\partial\Omega$. $f(u) = |u|^{p-2}u$, $p > 2$ is a nonlinear function and $M(s) = m_0 + bs^\gamma$ a positive local Lipschitz function. Here, $m_0 > 0$, $b \geq 0$, $\gamma \geq 1$, $s \geq 0$, ω are constants.

When $M \equiv 1$, the equation (1.1) becomes a nonlinear wave equation which has been extensively studied and several results concerning existence and blowing-up have been established[1−3].

On the contrary, when M is not a constant function, for the case that $\omega = 0$, the equation (1.1), as a special case, becomes the Kirchhoff equation which has been introduced in order to describe the nonlinear vibrations of an elastic string. Ono etc. studied this case and some results concerning existence and blowing-up were obtained[4−6].

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In this paper we shall deal with the energy decaying and the blow up behavior of solutions for problem (1.1)–(1.3) for the case that M is not a constant function and $\omega \neq 0$. We derive the blow up properties of solutions of this problem with negative and positive initial energy by the method different from the references [4]–[6].

The content of this paper is organized as follows. In Section 2, we give some assumptions and lemmas. In Section 3, we first define an energy function $E(t)$ and show that it is a non-increasing function of t. Then we obtain the exponent decay estimates of the energy function. In Section 4, we study the blow-up properties of solutions even for positive initial energy. Estimates for the blow-up time T^* (life-span) are also given.

2. Assumptions and preliminaries

In this section, we shall introduce some preliminaries needed in the proof of our result. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms.

Lemma 2.1 (Sobolev-Poincaré inequality[7]) *If $2 \leq p \leq \frac{2N}{N-2}$, $u \in H_0^1(\Omega)$, then $\|u\| \leq B\|\nabla u\|_2$ holds with some constant B, where $\|\cdot\|_p$ denotes the norm of $L^p(\Omega)$.*

Lemma 2.2[8] *Suppose that $\delta > 0$ and $B(t)$ is a nonnegative $C^2(0,\infty)$ function such that*

$$B''(t) - 4(\delta+1)B'(t) + 4(\delta+1)B(t) \geq 0. \tag{2.1}$$

If

$$B'(0) > r_2 B(0) + K_0, \tag{2.2}$$

then we have $\forall t > 0$, $B'(t) > K_0$. Here, K_0 is a constant and $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$ the smallest positive root of the equation $r^2 - 4(\delta+1)r + 4(\delta+1) = 0$.

Lemma 2.3[8] *If $J(t)$ is a non-increasing function on $[t_0,\infty)$, $t_0 \geq 0$ such that*

$$J'(t)^2 \geq a + bJ(t)^{2+\frac{1}{\delta}}, \quad \forall t_0 \geq 0, \tag{2.3}$$

where $a > 0, b \in R$, then there exists a finite time $T^\star$ such that $\lim_{t\to T^{\star-}} J(t) = 0$. Moreover, for the case that $b < 0$, $J(t_0) < \min\{1, \sqrt{\frac{a}{-b}}\}$, an upper bound of $T^\star$ is $t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}} - J(t_0)}$. If $b = 0$, we have $T^\star \leq t_0 + \frac{J(t_0)}{\sqrt{a}}$. If $b > 0$, we have $T^\star \leq \frac{J(t_0)}{\sqrt{a}}$ or $T^\star \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}}\{1 - [1 + cJ(t_0)]^{\frac{-1}{2\delta}}\}$. Here, $c = (\frac{b}{a})^{\frac{\delta}{2+\delta}}$ is a constant.

Lemma 2.4[9] *Suppose that $\phi(t)$ is a non-increasing function on $[0,T]$, $T > 1$. If $\phi(t) \geq 0$ and $\phi(t)^{1+r} \leq \omega_0 (\phi(t) - \phi(t+1))$, where $\omega_0 > 0$, $r \geq 0$ are constants, then we have*

(i) $\phi(t) \leq \left(\phi(0)^{-r} + \omega_0^{-1} r \max\{t-1, 0\}\right)^{-\frac{1}{r}}$ for $r > 0$;

(ii) $\phi(t) \leq \phi(0) e^{-\omega_1 \max\{t-1,0\}}$ for $r = 0$, where $\omega_1 = \ln \frac{\omega_0}{\omega_0 - 1}$, $\omega_0 > 1$.

Now, we put forward two assumptions as follows.

(A1) $f(0) = 0$ and $\forall u, v \in R, \exists k_1$ such that $|f(u) - f(v)| \leq k_1|u - v|(|u|^{p-2} + |v|^{p-2})$ and $2 < p \leq \frac{2(N-1)}{N-2}$ $(\infty, N \leq 2)$.

(A2) $\forall s \in R, \exists \delta > 0$ *such that* $sf(s) \geq (2+4\delta)F(s)$, *where* $F(s) = \int_0^s f(r)\mathrm{d}r$ *and* $\forall s \geq 0, (2\delta+1)M^\star(s) - (M(s)+2\delta m_0)s \geq 0$, *here,* $M^\star(s) = \int_0^s M(r)\mathrm{d}r$.

3. Exponent decay estimates of the energy function

In this section, we shall discuss the decay estimates of the energy of problem (1.1)–(1.3) with $f(u) = |u|^{p-2}u$. For simplicity, we only consider the situation $\omega = 1$. As for the local existence of solution for this problem, simulating the method put forward in [9], we can easily prove it by using the contraction mapping principle. We omit it here.

Assume that $u(t) \in H_0^1(\Omega)$. Let

$$I_1(t) = m_0\|\nabla u(t)\|_2^2 - \|u(t)\|_p^p, \tag{3.1}$$

$$I_2(t) = m_0\|\nabla u(t)\|_2^2 + b\|\nabla u(t)\|_2^{2(\gamma+1)} - \|u(t)\|_p^p, \tag{3.2}$$

$$J(t) = \frac{1}{2}m_0\|\nabla u(t)\|_2^2 + \frac{b}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} - \frac{1}{p}\|u(t)\|_p^p. \tag{3.3}$$

Now we define the energy of the solution u of (1.1)–(1.3) by

$$E(t) = \frac{1}{2}\|u(t)\|_2^2 + J(t). \tag{3.4}$$

For simplicity, we choose $m_0 = b = 1$. After some simple computation, we have $E'(t) = -\|\nabla u_t\|_2^2 < 0$. That is to say, $E(t)$ is a non-increasing function on $[0,\infty)$. Moreover, we have the following lemma.

Lemma 3.1 *Suppose that u is the solution of (1.1)–(1.3) and (A1) holds. If* $u_0 \in H_0^1(\Omega)\bigcap H^2(\Omega)$, $u_1 \in L^2(\Omega)$, $I_1(u_0) > 0$ *and*

$$\alpha = B_1^p\Big(\frac{2p}{p-2}E(0)\Big)^{\frac{p-2}{2}} < 1, \tag{3.5}$$

then $\forall t \geq 0$, $I_2(t) > 0$.

Proof Since $I_1(u_0) > 0$, it follows from the continuity of $u(t)$ that

$$I_1(t) > 0, \tag{3.6}$$

for some interval near $t = 0$. Let $t_{\max} > 0$ be a maximal time (possibly $t_{\max} = T$), when (3.6) holds on $[0, t_{\max})$. From (3.1) and (3.3), we have

$$J(t) \geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p \geq \frac{2p}{p-2}\|\nabla u\|_2^2 + \frac{1}{p}I_1(t). \tag{3.7}$$

By the definition of $E(t)$, we get

$$\|\nabla u\|_2^2 \leq \frac{2p}{p-2}J(t) \leq \frac{2p}{p-2}E(t) \leq \frac{2p}{p-2}E(0). \tag{3.8}$$

Then, from the Poincaré inequality and (3.5), we obtain

$$\|\nabla u\|_p^p \leq B_1^p\|\nabla u\|_2^p \leq \alpha\|\nabla u\|_2^2 < \|\nabla u\|_2^2, \quad t \in [0, t_{\max}). \tag{3.9}$$

Thus, $I_1(t) > 0$, $t \in [0, t_{\max})$. This implies that we can take $t_{\max} = T$. But, from (3.1) and (3.2), we see that $I_2(t) \geq I_1(t)$, $t \in [0,T]$. Therefore, we have $I_2(t) > 0$, $\forall t \in [0,T]$.

Next, we want to show that $t=\infty$. Multiplying (1.1) by $-2\Delta u$, and integrating it over Ω, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|\Delta u\|_2^2-2\int_\Omega u_t\Delta u\mathrm{d}x\Big\}+2M\left(\|\nabla u\|_2^2\right)\|\Delta u\|_2^2\\&\le 2\|\nabla u_t\|_2^2-2\int_\Omega |u|^{p-2}u\Delta u\mathrm{d}x.\end{aligned}\tag{3.10}$$

Since $2E'(t)=-2\|\nabla u_t\|_2^2$, multiplying (3.10) by $\varepsilon, 0<\varepsilon\le 1$ and adding them together gives

$$\frac{\mathrm{d}}{\mathrm{d}t}E^\star(t)+2(1-\varepsilon)\|\nabla u_t\|_2^2+2\varepsilon M\left(\|\nabla u\|_2^2\right)\|\Delta u\|_2^2\le -2\varepsilon\int_\Omega |u|^{p-2}u\Delta u\mathrm{d}x,\tag{3.11}$$

where

$$E^\star(t)=2E(t)-2\varepsilon\int_\Omega u_t\Delta u\mathrm{d}x+\varepsilon\|\Delta u\|_2^2.\tag{3.12}$$

By Young's inequality, we get $|2\varepsilon\int_\Omega u_t\Delta u\mathrm{d}x|\le 2\varepsilon\|u_t\|_2^2+\frac{\varepsilon}{2}\|\Delta u\|_2^2$. Hence, choosing $\varepsilon=\frac{2}{5}$, we see that

$$E^\star(t)\ge\frac{1}{5}\left(\|u_t\|_2^2+\|\Delta u\|_2^2\right).\tag{3.13}$$

Moreover, we note that

$$\begin{aligned}2\Big|\int_\Omega |u|^{p-2}u\Delta u\mathrm{d}x\Big|&\le 2(p-1)\int_\Omega |u|^{p-2}|\nabla u|^2\mathrm{d}x\\&\le 2(p-1)\|u\|_{(p-2)\theta_1}^{p-2}\|\nabla u\|_{2\theta_2}^2,\end{aligned}\tag{3.14}$$

where $\theta_1+\theta_2=\theta_1\theta_2$. So we put $\theta_1=1$ and $\theta_2=\infty$, if $N=1$; $\theta_1=1+\varepsilon_1$, $\forall\varepsilon_1>0$, if $N=2$; and $\theta_1=\frac{N}{2},\theta_2=\frac{N}{N-2}$, if $N\ge 3$.

Then, by the Poincaré inequality, (3.8) and (3.12), we have

$$2\Big|\int_\Omega |u|^{p-2}u\Delta u\mathrm{d}x\Big|\le 2B_1^p(p-1)\|\nabla u_t\|_2^{p-2}\|\Delta u\|_2^2\le c_1E^\star(t),\tag{3.15}$$

where $c_1=10B_1^p(p-1)\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}}$.

Substituting (3.15) into (3.11), and then integrating it over $(0,t)$, we obtain

$$E^\star(t)+\frac{4}{5}\int_0^t\|\Delta u(s)\|_2^2\mathrm{d}s\le E^\star(0)+\int_0^t c_1E^\star(s)\mathrm{d}s.\tag{3.16}$$

By Gronwall's Lemma, we deduce $E^\star(t)\le E^\star(0)\exp(c_1t)$, $\forall t\ge 0$. Therefore by continuity principle, we have $T=\infty$.

Taking $\eta=1-\alpha$ in (3.9), we have $\|u(t)\|_p^p\le(1-\eta)\|\nabla u_t\|_2^2$, $\forall t\in[0,\infty)$.

Next, we declare the exponent decay estimates of the energy function as follows.

Theorem 3.2 *Assume that $I_1(u_0)>0$ and (3.5) holds. Then we have*

$$E(t)\le E(0)e^{-\tau_1 t},\tag{3.17}$$

where τ_1 is a constant.

Proof By integrating $E'(t)=-\|\nabla u_t\|_2^2$ over $[t,t+1]$, we get

$$E(t)-E(t+1)\equiv D(t)^2,\tag{3.18}$$

where $D(t)^2 = \int_t^{t+1} \|\nabla u_t\|_2^2 \mathrm{d}t$. Thus, there exist $t_1 \in [t, t+\frac{1}{4}]$, $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|\nabla u_t(t_i)\|_2^2 = 4D(t)^2, \quad i = 1, 2. \tag{3.19}$$

Next, Multiplying (1.1) by u and then integrating it over $\Omega \times [t_1, t_2]$, we get

$$\begin{aligned}&\int_{t_1}^{t_2} \{\|\nabla u\|_2^2 + \|\nabla u\|_2^{2(\gamma+1)} - \|u\|_p^p\} \mathrm{d}t \\ &= -\int_{t_1}^{t_2} \int_\Omega u_{tt} u \mathrm{d}x\mathrm{d}t - \int_{t_1}^{t_2} \int_\Omega \nabla u_t \cdot \nabla u \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.20}$$

Then, by Hölder inequality and Young's inequality, from (3.2), we obtain

$$\Big| \int_{t_1}^{t_2} \int_\Omega \nabla u_t \cdot \nabla u \mathrm{d}x\mathrm{d}t \Big| \leq \int_{t_1}^{t_2} \|\nabla u_t\|_2 \|\nabla u\|_2 \mathrm{d}t. \tag{3.21}$$

By Hölder inequality and Poincaré inequality, integrating (3.21) by parts, we have

$$\Big| \int_{t_1}^{t_2} \int_\Omega u_{tt} u \mathrm{d}x\mathrm{d}t \Big| \leq B_1^2 \sum_{i=1}^{2} \|\nabla u_t(t_i)\|_2 \|\nabla u(t_i)\|_2 + B_1^2 \int_t^{t+1} \|\nabla u_t\|_2^2 \mathrm{d}t. \tag{3.22}$$

So, we deduce

$$\begin{aligned}\int_{t_1}^{t_2} I_2(t)\mathrm{d}t \leq & B_1^2 \sum_{i=1}^{2} \|\nabla u_t(t_i)\|_2 \|\nabla u(t_i)\|_2 + \\ & B_1^2 \int_t^{t+1} \|\nabla u_t\|_2^2 \mathrm{d}t + \int_{t_1}^{t_2} \|\nabla u_t\|_2 \|\nabla u\|_2 \mathrm{d}t.\end{aligned} \tag{3.23}$$

Furthermore, by (3.19) and (3.8), we have

$$\|\nabla u_t(t_i)\|_2 \|\nabla u(t_i)\|_2 \leq c_2 D(t) \sup_{t_1 \leq s \leq t_2} E(s)^{\frac{1}{2}}, \tag{3.24}$$

and

$$\int_{t_1}^{t_2} \|\nabla u_t\|_2 \|\nabla u\|_2 \mathrm{d}t \leq \frac{c_2}{2} D(t) \sup_{t_1 \leq s \leq t_2} E(s)^{\frac{1}{2}}, \tag{3.25}$$

where $c_2 = 2(\frac{2p}{p-2})^{1/2}$. Thus, we get

$$\int_{t_1}^{t_2} I_2(t)\mathrm{d}t \leq c_3 D(t) \sup_{t_1 \leq s \leq t_2} E(s)^{\frac{1}{2}} + B_1^2 D(t)^2, \tag{3.26}$$

where $c_3 = (2B_1^2 + \frac{1}{2})c_2$. On the other hand, by the definition of $E(t)$, we have

$$E(t) \leq \frac{1}{2}\|u_t\|_2^2 + c_4 \|\nabla u\|_2^2 + c_5 I_2(t), \tag{3.27}$$

where $c_4 = \frac{1}{2} - \frac{1}{p}$, $c_5 = \frac{1}{p} + \frac{1}{2(\gamma+1)}$. Integrating (3.27) over (t_1, t_2), we get

$$\int_{t_1}^{t_2} E(t)\mathrm{d}t \leq \frac{1}{2} \int_{t_1}^{t_2} \|u_t\|_2^2 \mathrm{d}t + c_5 \int_{t_1}^{t_2} I_2(t)\mathrm{d}t \tag{3.28}$$

and

$$\int_{t_1}^{t_2} E(t)\mathrm{d}t \leq \frac{B_1^2}{2} D(t)^2 + c_5 \int_{t_1}^{t_2} I_2(t)\mathrm{d}t. \tag{3.29}$$

Next, Multiplying (1.1) (take $f(u)=|u|^{p-2}u$) by u_t and then integrating it over $[t,t_2]\times\Omega$, we have

$$E(t)=E(t_2)+\int_t^{t_2}\|\nabla u(t)\|_2^2\mathrm{d}t. \tag{3.30}$$

Since $t_2-t_1\geq\frac{1}{2}$, we get $E(t_2)\leq 2\int_{t_1}^{t_2}E(t)\mathrm{d}t$. Thus, we have

$$E(t)\leq 2\int_{t_1}^{t_2}E(t)\mathrm{d}t+\int_t^{t+1}\|\nabla u(t)\|_2^2\mathrm{d}t=2\int_{t_1}^{t_2}E(t)\mathrm{d}t+D(t)^2. \tag{3.31}$$

So,

$$E(t)\leq c_6D(t)^2+c_7D(t)\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}, \tag{3.32}$$

where $c_6=(2c_5+1)B_1^2+1$, $c_7=2c_5c_3$.

By Young's inequality, we have

$$E(t)\leq c_8D(t)^2, \tag{3.33}$$

where c_8 is a positive constant. Thus, $\forall t\geq 0$, $E(t)\leq c_9[E(t)-E(t+1)]$, here, $c_9=\max\{c_8,1\}$.

Then, by Lemma 2.4, we obtain

$$E(t)\leq E(0)^{-\tau_1 t},\quad t\in[0,\infty), \tag{3.34}$$

where $\tau_1=\ln\frac{c_9}{c_9-1}$. The proof is completed. □

4. Blow-up of solutions and the life-span

In this section, we shall discuss the blow-up phenomena of problem (1.1)–(1.3). For simplicity, we only consider the situation $m_0=1,b=1$.

Definition *u is called the blow up solution of (1.1)–(1.3), if $\exists T^\star<\infty$ such that*

$$\lim_{t\to T^{\star-}}\Big(\int_\Omega|\nabla u|^2\mathrm{d}x\Big)^{-1}=0.$$

We define the energy function of the solution u of (1.1)–(1.3) by

$$E(t)=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}M^\star\left(\|\nabla u(t)\|_2^2\right)-\int_\Omega F\left(u(t)\right)\mathrm{d}x,\quad t\geq 0. \tag{4.1}$$

Then we have

$$E(t)=E(0)-\int_0^t\|\nabla u_t(t)\|_2^2\mathrm{d}t. \tag{4.2}$$

Now, let u be a solution of (1.1)–(1.3) and define

$$a(t)=\int_\Omega u^2\mathrm{d}x+\int_0^t\int_\Omega|\nabla u|^2\mathrm{d}x\mathrm{d}t,\quad t\geq 0. \tag{4.3}$$

Then we have

$$a'(t)=2\int_\Omega uu_t\mathrm{d}x+\|\nabla u\|_2^2. \tag{4.4}$$

$$a''(t)=2\|u_t\|_2^2-2M\left(\|\nabla u\|_2^2\right)\|\nabla u\|_2^2+2\int_\Omega f(u)u\mathrm{d}x. \tag{4.5}$$

By using Hölder inequality and Young's inequality, we easily obtain the following lemma.

Lemma 4.1 *Suppose that (A1)–(A2) hold. Then we have*

$$a''(t) - 4(\delta + 1) \int_{\Omega} u_t^2 \mathrm{d}x \geq (-4 - 8\delta)E(0) + (4 + 8\delta) \int_0^t \|\nabla u_t\|_2^2 \mathrm{d}t. \tag{4.6}$$

Next, we consider three different cases on the sign of the initial energy $E(0)$.

(i) If $E(0) < 0$, then by Lemma 4.1, we have $a'(t) \geq a'(0) - 4(1 + 2\delta)E(0)t$, $t \geq 0$. Thus $\forall t \geq t^\star$, we have $a'(t) > \|\nabla u_0\|_2^2$, where

$$t^\star = \max\Big\{\frac{a'(0) - \|\nabla u_0\|_2^2}{4(1 + 2\delta)E(0)}, \quad 0\Big\}. \tag{4.7}$$

(ii) If $E(0) = 0$, then $\forall t \geq 0$, we have $a''(t) \geq 0$. Moreover, if $a'(0) > \|\nabla u_0\|_2^2$, then $\forall t \geq 0$, we have $a'(t) > \|\nabla u_0\|_2^2$.

(iii) For the case that $E(0) > 0$, we first note that

$$2\int_0^t \int_{\Omega} \nabla u \cdot \nabla u_t \mathrm{d}x\mathrm{d}t = \|\nabla u(t)\|_2^2 - \|\nabla u_0\|_2^2. \tag{4.8}$$

By using Hölder inequality and Young's inequality, we have

$$\|\nabla u(t)\|_2^2 \leq \|\nabla u_0\|_2^2 + \int_0^t \|\nabla u(t)\|_2^2 \mathrm{d}t + \int_0^t \|\nabla u_t(t)\|_2^2 \mathrm{d}t. \tag{4.9}$$

By using Hölder inequality and Young's inequality in (4.4) and by (4.9), we obtain

$$a'(t) \leq a(t) + \|\nabla u_0\|_2^2 + \|u_t\|_2^2 + \int_0^t \|\nabla u_t(t)\|_2^2 \mathrm{d}t. \tag{4.10}$$

Thus, we have $a''(t) - 4(\delta+1)a'(t) + 4(\delta+1)a(t) + K_1 \geq 0$, here, $K_1 = (4+8\delta)E(0) + 4(\delta+1)\|\nabla u_0\|_2^2$.

Let

$$b(t) = a(t) + \frac{K_1}{4(\delta + 1)}, \quad t > 0. \tag{4.11}$$

Then $b(t)$ satisfies (2.1). By (2.2), we see that if

$$a'(0) > r_2\Big[a(0) + \frac{K_1}{4(\delta + 1)}\Big] + \|\nabla u_0\|_2^2, \tag{4.12}$$

then $a'(t) > \|\nabla u_0\|_2^2$, $t > 0$.

Consequently, we have

Lemma 4.2 *Assume that (A1)–(A2) hold and that either one of the following conditions is satisfied:*

(i) $E(0) < 0$,

(ii) $E(0) = 0$ *and* $a'(0) > \|\nabla u_0\|_2^2$,

(iii) $E(0) > 0$ *and (4.12) holds.*

Then $a'(t) > \|\nabla u_0\|_2^2$, $\forall t > t_0$. *where* $t_0 = t^\star$ *is given by (4.7) in case (i) and* $t_0 = 0$ *in cases (ii) and (iii).*

Now, we find the estimate for the life span of $a(t)$. Let

$$J(t) = \big(a(t) + (T_1 - t)\|\nabla u_0\|_2^2\big)^{-\delta}, \quad t \in [0, T_1], \tag{4.13}$$

where $T_1 > 0$ is a certain constant which will be specified later.

Then we have $J'(t) = -\delta J(t)^{1+\frac{1}{\delta}} \left(a'(t) - \|\nabla u_0\|_2^2\right)$ and

$$J''(t) = -\delta J(t)^{1+\frac{2}{\delta}} V(t). \tag{4.14}$$

where

$$V(t) = a''(t) \left(a(t) + (T_1 - t)\|\nabla u_0\|_2^2\right) - (1+\delta) \left(a'(t) - \|\nabla u_0\|_2^2\right)^2. \tag{4.15}$$

For simplicity of calculation, we denote that $P = \int_\Omega u^2 \mathrm{d}x$, $Q = \int_0^t \|\nabla u(t)\|_2^2 \mathrm{d}t$, $R = \int_\Omega u_t^2 \mathrm{d}x$ and $S = \int_0^t \|\nabla u_t(t)\|_2^2 \mathrm{d}t$. Thus, we get

$$a'(t) \le 2\left(\sqrt{RP} + \sqrt{QS}\right) + \|\nabla u_0\|_2^2. \tag{4.16}$$

By Lemma 4.1, we have

$$a''(t) \ge (-4 - 8\delta)E(0) + 4(1+\delta)(R+S). \tag{4.17}$$

Then, we obtain

$$\begin{aligned} V(t) \ge & [(-4-8\delta)E(0) + 4(1+\delta)(R+S)] \left(a(t) + (T_1 - t)\|\nabla u_0\|_2^2\right) - \\ & 4(1+\delta)\left(\sqrt{RP} + \sqrt{QS}\right)^2. \end{aligned}$$

By (4.13), we have

$$\begin{aligned} V(t) \ge & (-4-8\delta)E(0)J(t)^{-\frac{1}{\delta}} + 4(1+\delta)(R+S)(T_1 - t)\|\nabla u_0\|_2^2 + \\ & 4(1+\delta)\left[((R+S)(P+Q) - \left(\sqrt{RP} + \sqrt{QS}\right)^2\right]. \end{aligned}$$

By Schwarz inequality, we get

$$V(t) \ge (-4-8\delta)E(0)J(t)^{-\frac{1}{\delta}}, \quad t \ge t_0. \tag{4.18}$$

Therefore, we get

$$J''(t) \le \delta(4+8\delta)E(0)J(t)^{1+\frac{1}{\delta}}, \quad t \ge t_0. \tag{4.19}$$

Note that by Lemma 4.2, $J'(t) < 0$, $t > t_0$. Multiplying (4.19) by $J'(t)$ and integrating it from t_0 to t, we have $J'(t)^2 \ge \alpha + \beta J(t)^{2+\frac{1}{\delta}}$, $t \ge t_0$. Where

$$\alpha = \delta^2 J(t_0)^{2+\frac{1}{\delta}} \left[(a'(t_0) - \|\nabla u_0\|_2^2)^2 - 8E(0)J(t_0)^{-\frac{1}{\delta}}\right], \tag{4.20}$$

$$\beta = 8\delta^2 E(0). \tag{4.21}$$

We observe that $\alpha > 0$ if and only if $E(0) < \frac{\left(a'(t_0) - \|\nabla u_0\|_2^2\right)^2}{8\left[a(t_0) + (T_1 - t_0)\|\nabla u_0\|_2^2\right]}$.

Then by Lemma 2.3, there exists a finite time $T^\star$ such that $\lim_{t\to T^{\star}-} J(t) = 0$ and the upper bounds of $T^\star$ are estimated respectively according to the sign of $E(0)$. This will imply that $\lim_{t\to T^\star -}\{\int_\Omega u^2 \mathrm{d}x + \int_0^t \|\nabla u\|_2^2 \mathrm{d}t\}^{-1} = 0$. Thus by Poincaré inequality, we deduce

$$\lim_{t\to T^\star -} \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{-1} = 0. \tag{4.21}$$

So, we have our results as follows.

Theorem 4.3 *Assume that (A1)–(A2) hold and that either one of the following conditions is satisfied:*

(i) $E(0) < 0$,

(ii) $E(0) = 0$, *and* $a'(0) > \|\nabla u_0\|_2^2$,

(iii) $0 < E(0) < \frac{\left(a'(t_0)-\|\nabla u_0\|_2^2\right)^2}{8\left[a(t_0)+(T_1-t_0)\|\nabla u_0\|_2^2\right]}$ *and (4.12) holds.*

Then the solution u *blows up at finite time* $T^\star$. *And in case (i), we have* $T^\star \leq t_0 - \frac{J(t_0)}{J'(t_0)}$. *Furthermore, if* $J(t_0) < \min\{1, \sqrt{\frac{\alpha}{-\beta}}\}$, *then we have* $T^\star \leq t_0 + \sqrt{\frac{1}{-\beta}} \ln \frac{\sqrt{\frac{\alpha}{-\beta}}}{\sqrt{\frac{\alpha}{-\beta}} - J(t_0)}$. *In case (ii),* $T^\star \leq t_0 - \frac{J(t_0)}{J'(t_0)}$ *or* $T^\star \leq t_0 + \frac{J(t_0)}{\sqrt{\alpha}}$. *In case (iii),* $T^\star \leq \frac{J(t_0)}{\sqrt{\alpha}}$ *or* $T^\star \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{\alpha}}\{1 - [1 + cJ(t_0)]^{\frac{-1}{2\delta}}\}$, *where* $c = (\frac{\beta}{\alpha})^{\frac{\delta}{2+\delta}}$, *here* α *and* β *are in (4.20) and (4.21), respectively.*

Remark The choice of T_1 in (4.13) is possible under some conditions.

(i) In the case $E(0) = 0$, we can choose $T_1 \geq \frac{\|u_0\|_2^2}{\delta^2 B_1^2 \|u_1\|_2^2}$. In particular, we choose $T_1 = \frac{\|u_0\|_2^2}{\delta^2 B_1^2 \|u_1\|_2^2}$, then we get $T^\star \leq \frac{\|u_0\|_2^2}{\delta^2 B_1^2 \|u_1\|_2^2}$.

(ii) In the case $E(0) < 0$, we can choose T_1 as in (i) if $\int_\Omega u_0 u_1 \mathrm{d}x > 0$ or $T_1 \geq t^\star - \frac{J(t^\star)}{J'(t^\star)}$ if $\int_\Omega u_0 u_1 \mathrm{d}x \leq 0$.

(iii) For the case $E(0) > 0$. Under the condition $E(0) < \min\{k_1, k_2\}$, here

$$k_1 = \frac{(1+\delta)[a'(0) - r_2 a(0) - (r_2+1)\|\nabla u_0\|_2^2]}{r_2(1+2\delta)}, \quad k_2 = \frac{[4(\int_\Omega u_0 u_1 \mathrm{d}x)^2 - 1][\delta - \|\nabla u_0\|_2^2]}{8\delta \|\nabla u_0\|_2^2},$$

if $\|\nabla u_0\|_2^2 < \delta$, T_1 is chosen to satisfy $k_3 \leq T_1 \leq T_4$, here

$$k_3 = \frac{\|u_0\|_2^2}{\delta - \|\nabla u_0\|_2^2}, \; k_4 = \frac{4(\int_\Omega u_0 u_1 \mathrm{d}x)^2 - 8E(0)\|u_0\|_2^2 - 1}{8E(0)\|\nabla u_0\|_2^2}.$$

Therefore we have $T \leq T^\star \leq \frac{k_3}{\sqrt{4(\int_\Omega u_0 u_1 \mathrm{d}x)^2 - 8E(0) k_3}}$.

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